Stochastic Resonance in multistable systems

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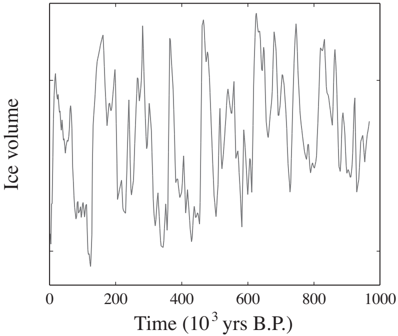
Evidence of climatic variability :



Preferred frequency

$$\frac{1}{\omega} \sim 10^5 \text{ years}$$

eccentricity of earth's orbit ?

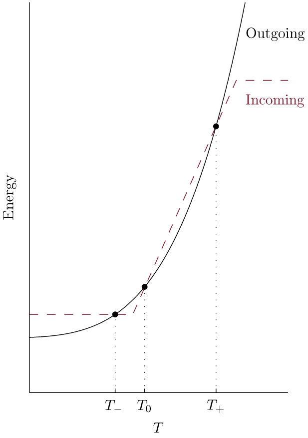
$$\epsilon \sin \omega t \qquad \epsilon \sim 10^{-3}$$

Need for a mechanism of amplification of weak signals in the presence of noise.

Classical setting of Stochastic Resonance

In absence of periodic forcing

$$C\frac{dT}{dt} = \qquad \text{Incoming radiation} - \text{Outgoing radiation} + \text{Stochastic fluctuations}$$



or equivalently,

$$C\frac{dT}{dt} = -\frac{\partial U}{\partial T} + F\left(T\right) \quad \text{(one variable system)} \tag{1}$$

- U : kinetic potential, possessing two wells (stable states) separated by a maximum (intermediate unstable state).
- Stochastic fluctuations : white noise

$$\begin{aligned} < F\left(t\right) > &= 0 \\ < F\left(t\right)\left(t'\right) > &= q^2\delta\left(t-t'\right) \end{aligned}$$

→ Fokker Planck equation for the probability masses around the two stable states T_- (state 1) and T_+ (state 2).

Steady state solution expressed entirely in terms of U :

$$P_s \sim \exp\left(-\frac{2}{q^2}U\right) \tag{2}$$

Phenomenological theory of Kramers : diffusion over a potential barrier (q^2 small)

Mapping the problem into a discrete process

$$\text{state } 1 \underset{k_{21}}{\overset{k_{12}}{\rightleftarrows}} \text{state } 2 \tag{3}$$

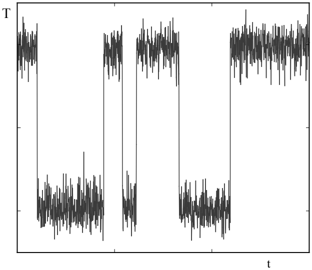
k's : "rate constants"

$$\frac{dP_1}{dt} = -k_{12}P_1 + k_{21}P_2 \qquad \text{with} \qquad P_1 + P_2 = 1 \tag{4}$$

$$k_{12} \sim \mathrm{e}^{-\frac{2}{q^2}\Delta U}, \qquad \Delta U = \qquad \underbrace{U \text{ (unstable state) } - U \text{ (reference state)}}_{\text{potential barrier}}$$

Time scale of transitions between states 1 and 2

$$< \tau > \sim \frac{1}{k_{12}} \qquad \text{long time scale of order } 10^5 \text{ years}$$



Presence of a periodic forcing $\varepsilon \sin \omega t$

$$\left.\begin{array}{ll} \varepsilon \sim & 10^{-3} \\ \omega \sim & 10^{-5} \text{ years} \end{array}\right\} \text{eccentricity of earth}$$

Adiabatic approximation

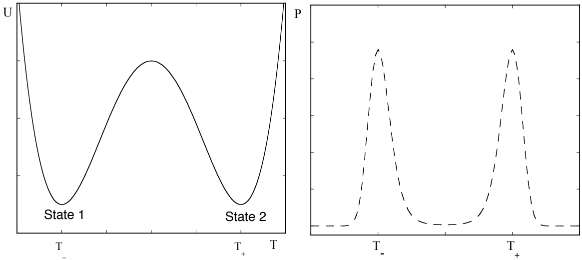
$$P_i\,(t) = P_i^{(o)} + \varepsilon R_i \sin\left(\omega t + \varphi\right) \qquad i = 1, 2$$

R_i : amplitude of response

Results

R_i appreciable if

- $P_1^{(0)} \sim P_2^{(0)} \sim 0.5$ COEXISTANCE



- and $1/\omega \geq <\tau>$

$$R_i \sim \frac{2\varepsilon}{q^2}\left[1+(\tau\omega)^2\right]^{-1/2}$$

$$\varphi = -\arctan(\tau\omega)$$

Typically $\varepsilon R \sim 20\%$

Preferred frequency ω

SR in multistable systems

(1 variable systems, additive periodic forcing of small amplitude)

Again, mapping into a discrete state process

$$\text{state } 1 \underset{k_{21}}{\overset{k_{12}}{\rightleftarrows}} \text{state } 2 \underset{k_{32}}{\overset{k_{23}}{\rightleftarrows}} \text{state } 3 \cdots \text{state } n-1 \underset{k_{n,n-1}}{\overset{k_{n-1,n}}{\rightleftarrows}} \text{state } n$$

$$\frac{dP_i\left(t\right)}{dt} = \sum_{j=1}^{n} M_{ij}\left(t\right) P_j\left(t\right) \qquad i = 1, \cdots n \tag{5}$$

$$M = \begin{pmatrix} -k_{12}\left(t\right) & k_{21}\left(t\right) & 0 & \cdots & 0 \\ k_{12} & -\left(k_{21}+k_{23}\right) & k_{32} & \cdots & 0 \\ 0 & k_{23} & -\left(k_{32}+k_{34}\right) & & 0 \\ \vdots & & & & \vdots \\ \cdots & \cdots & \cdots & & -k_{n,n-1} \end{pmatrix}$$

Rate constants

$$
\begin{aligned}
U &= U^{(o)} - \varepsilon x \sin\omega t \\
k_{i,i\pm1}(t) &= k^{(o)}_{i,i\pm1} \exp\left[\frac{2\varepsilon}{q^2}\Delta x(i,i\pm1)\sin\omega t\right] \\
k^{(o)}_{i,i\pm1} &\sim \exp\left[-\frac{2}{q^2}\Delta U_o(i,i\pm1)\right]
\end{aligned}
$$

Linear response

$$\begin{cases} k_{i,i\pm1} = & k^{(o)}_{i,i\pm1} + \varepsilon\Delta_{i,i\pm1}\sin\omega t \\ \Delta_{i,i\pm1} = & \frac{2}{q^2} k^{(o)}_{i,i\pm1}\Delta x\,(i, i\pm1) \end{cases}$$

$$\begin{cases} M\,(t) = & M_0 + \varepsilon\Delta\sin\omega t \\ \mathbf{P}\,(t) = & \mathbf{P}_0 + \varepsilon\delta\mathbf{P}\,(t) \\ \sum_{i=1}^{n} P_i = & 1 \\ \sum_{i=1}^{n} \delta P_i = & 0 \end{cases}$$

- M_0, Δ : tridiagonal matrices
- $\mathbf{P}_0$: invariant P with $\varepsilon = 0$
- $\delta\mathbf{P}$: induced response

$$\frac{d\delta\mathbf{P}\left(t\right)}{dt} = M_0\delta\mathbf{P} + \varepsilon\sin\omega t\Delta\mathbf{P}_0$$

Long time solution : $\delta\mathbf{P}\left(t\right) = \varepsilon\left(\mathbf{A}\cos\omega t + \mathbf{B}\sin\omega t\right)$

$$\begin{cases} \delta P_i\left(t\right) = & R_i\sin\left(\omega t + \varphi_i\right) \\ R_i = & \varepsilon\left(A_i^2 + B_i^2\right)^{1/2} \\ \varphi_i = & \arctan\left(\frac{A_i}{B_i}\right) \end{cases}$$

Let λ_k and u_k be eigenvalues and eigenvectors of M_0.
Expanding $\Delta\mathbf{P}_0$ in the basis of $\mathbf{u}_k$

$$\Delta\mathbf{P}_0 = \sum_{k=1}^{n}\gamma_k\mathbf{u}_k$$

$$\begin{cases} \mathbf{A} = & -\sum_{k=1}^{n}\frac{\omega}{\lambda_k^2 + \omega^2}\gamma_k\mathbf{u}_k \\ \mathbf{B} = & -\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_k^2 + \omega^2}\gamma_k\mathbf{u}_k \end{cases}$$

Simplification :

all k's $\sim$ identical (one of the prerequisites of classical SR)

$$k_{12} = k_{21} = \cdots k_0$$

$$M_0 = k_0 \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & & \cdots & 0 & 0 \\ \vdots & & \ddots & & \dots & \vdots \\ \cdots & \cdots & \cdots & & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} \lambda_k &= -2k_0 \left(1 - \cos \frac{(k-1)\,\pi}{n}\right) \qquad k = 1, \cdots n \\ u_1^i &= 1 \\ u_k^i &= \cos \left[\frac{(k-1)\,(2i-1)\,\pi}{2n}\right] \qquad i = 1, \cdots n \quad k = 2, \cdots n \end{aligned}$$

Toy model

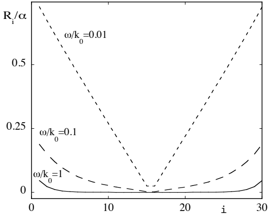
$$
\begin{aligned}
U_0(x) &= -\cos x & \quad 0 \le x \le 2\pi n \\
\pi, 3\pi, 5\pi, \cdots & & \quad \text{stable states} \\
2\pi, 4\pi, 6\pi, \cdots & & \quad \text{unstable states}
\end{aligned}
$$

$$
\begin{aligned}
A_i &= \frac{1}{N^2}\,\frac{4\pi k_0}{n}\,\frac{2}{q^2}\sum_{k\ \text{even}} \cos\frac{(k-1)\pi}{2n}\cos\frac{(2i-1)(k-1)\pi}{2n}\,\frac{\omega}{\lambda_k^2+\omega^2} \\
B_i &= \frac{1}{N^2}\,\frac{4\pi k_0}{n}\,\frac{2}{q^2}\sum_{k\ \text{even}} \cos\frac{(k-1)\pi}{2n}\cos\frac{(2i-1)(k-1)\pi}{2n}\,\frac{\lambda_k}{\lambda_k^2+\omega^2} \\
N &: \quad \text{norm of } \mathbf{u}_k
\end{aligned}
$$

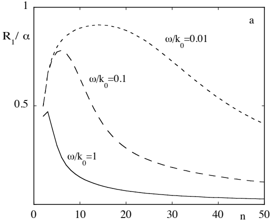
$$
\left.
\begin{aligned}
R_i &= \varepsilon\left(A_i^2+B_i^2\right)^{1/2} \\
\varphi_i &= \arctan\frac{A_i}{B_i}
\end{aligned}
\right\}
$$

Optimal response near boundaries

$\alpha = \frac{2\varepsilon\Delta x}{q^2}$



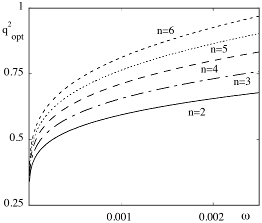
Response maximized for some n depending on ω/k_0



Is there an optimal $q^2_{\rm opt}$?

example : $n = 6$

optimal q^2 increases with n



$$R_1 = \frac{2\varepsilon\pi}{3q^2}\left\{\frac{(\omega/k_0)^4 + 15\,(\omega/k_0)^2 + 25}{\left[(\omega/k_0)^4 + 14\ (\omega/k_0)^2 +\ 1\right]\left[(\omega/k_0)^2 + 4\right]}\right\}^{1/2}$$

$\omega/k_0 \equiv \omega\tau$

Conclusions

- Extension of classical SR for an arbitrary number of simultaneously stable states for systems involving one variable
- Amplitude and phase of response of a stable state have been determined as a function of its location
- Existence of an optimal q^2
- Optimal number of intermediate stable states for which response is maximized

Extension of this work :

- Multivariate systems
- non potential systems
- more complex communication geometries of stable states