Physical vs. Numerical Dispersion in Nonhydrostatic Ocean Modeling

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“... focused examination and testing of ocean model resolution must be a prime goal of ocean climate model development over the next decades.”
- Griffies [2000]

“... models are imperfect tools. Furthermore, discussion and comprehension of the results from complex models depend on the results from idealized models.”
- Philander [2009]
Abstract

Many popular ocean models second-order accurate, inducing numerical dispersion generated from odd-order terms in the truncation error.

Internal waves are (often) a dynamical balance between nonlinearity and nonhydrostasy (physical dispersion).

Numerical dispersion mimics physical dispersion due to nonhydrostasy.

To lowest order, the ratio of numerical to physical dispersion is

\[ \Gamma = K \lambda^2 \]

\( K \) is typically an \( O(1) \) constant
\( \lambda \equiv \frac{\Delta x}{h_1} \) is the grid leptic ratio, or lepticity
\( \Delta x \) is the horizontal grid spacing
\( h_1 \) is the upper layer (pycnocline) depth

We derive this relationship for simple models (KdV equation), and show that it holds in a real ocean model (SUNTANS – Fringer et. al. 2006).

To ensure relative dominance of physical over numerical dispersive effects:

\[ \Gamma \ll 1 \quad \text{↔} \quad \lambda \approx O(0.1) \quad \text{↔} \quad \Delta x < h_1 \]
Ocean Modeling: Range of Scales

Most ocean models are traditionally **hydrostatic**.

The hydrostatic approximation is valid when

*Horizontal scale >> Vertical scale*

Nonhydrostatic models compute motions for which

*Horizontal scale ~ Vertical scale*

This requires computationally expensive solution of a 3D elliptic equation.
Internal Waves

Internal waves are (often) a dynamical balance between:

- **Nonlinearity** & **Nonhydrostasis**
- (physical dispersion)

\[ \delta = \frac{a}{H} = \text{wave amplitude} \div \text{depth} \]

\[ \mathcal{E} = \frac{H}{L} = \text{aspect ratio} \]
The KdV equation

When modeling solitary waves, the behavior of a fully nonhydrostatic ocean model can be well approximated with the KdV (Korteweg and de-Vries, 1895) equation:

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -g \nabla \eta - g \nabla r - \nabla q
\]

The well-known solution to the KdV equation is:

\[
\xi(x,t) = -a \ \text{sech}^2 \left( \frac{x-ct}{L_0} \right)
\]

\[
L_0 = \sqrt{\frac{4}{3a} \frac{\varepsilon^2}{\delta}} = \frac{4}{3a} \text{ dispersion} = \frac{4}{3a} \text{ nonlinearity}
\]
Comparison of KdV to the nonhydrostatic model SUNTANS

KdV model uses coefficients that account for continuous stratification (Liu et al. 2004).

Since SUNTANS is well-approximated by KdV, we can analyze the numerical properties of the KdV equation to quantify the dispersive error in more complex models.
**Numerical Discretization of KdV Eq.**

- SUNTANS and many ocean models discretize the equations with second-order accuracy in time and space. (e.g. POM, Blumberg and Mellor, 1987; MICOM, Bleck et al., 1992; MOM, Pacanowski and Griffes, 1999).

- A second-order accurate discretization of the KdV equation using “leap-frog” (i.e. POM) is given by

\[
\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2} \delta \xi\right) \frac{\partial \xi}{\partial x} + \frac{\epsilon^2}{6} \frac{\partial^3 \xi}{\partial x^3} = 0
\]

\[
\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta t} + \left(1 - \frac{3}{2} \delta \xi^n\right) \frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} + \frac{\epsilon^2}{6} \frac{1}{2} \frac{\Delta x^2}{\Delta x} \frac{\xi_{i+2}^n - \xi_{i+1}^n + \xi_{i-1}^n - \frac{1}{2} \xi_{i-2}^n}{\Delta x^3} = 0
\]

- Taylor series expansions can be used to determine the truncation error, or modified equivalent PDE:

\[
\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} = \frac{\partial \xi}{\partial x}\left|_i^n\right. + \frac{\Delta x^2}{6} \frac{\partial^3 \xi}{\partial x^3}\left|_i^n\right. + \frac{\Delta x^4}{120} \frac{\partial^5 \xi}{\partial x^5}\left|_i^n\right. + \frac{\Delta x^6}{5040} \frac{\partial^7 \xi}{\partial x^7}\left|_i^n\right. + O(\Delta x^8)
\]
Modified equivalent KdV equation

The discrete KdV equation produces a modified equivalent PDE (Hirt 1968) which introduces new terms due to discretization errors:

\[
\text{KdV:} \quad \frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2} \delta \xi \right) \frac{\partial \xi}{\partial x} + \frac{\varepsilon^2}{6} \frac{\partial^3 \xi}{\partial x^3} = 0
\]

\[
\text{Modified KdV:} \quad \frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2} \delta \xi \right) \frac{\partial \xi}{\partial x} + (1 + \Gamma) \frac{\varepsilon^2}{6} \frac{\partial^3 \xi}{\partial x^3} = O \left( \Delta x^4, \Delta t^4, \delta \Delta x^2, \varepsilon^2 \Delta x^2 \right)
\]

The numerical discretization of the first-order derivative produces numerical dispersion. Errors in the nonlinear term are smaller by \( \sim \) a factor \( \delta \).

\[
\Gamma = \frac{\text{numerical dispersion}}{\text{physical dispersion}} = K \left( \frac{\Delta x'}{\varepsilon} \right)^2 = K \left( \frac{\Delta x}{h_1} \right)^2 = K \lambda^2
\]

\( K \) is typically an \( O(1) \) constant

\( \lambda \equiv \frac{\Delta x}{h_1} \) is the grid leptic ratio, or lepticity

Scotti & Mitran (2008)

\( \Delta x \) is the horizontal grid spacing

\( h_1 \) is the upper layer (pycnocline) depth
Numerical dispersion in hydrostatic and nonhydrostatic ocean modeling

- "Nonhydrostatic" models possess both physical & numerical dispersion:

\[
\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2} \delta \xi \right) \frac{\partial \xi}{\partial x} + (1 + \Gamma) \frac{\varepsilon^2}{6} \frac{\partial^3 \xi}{\partial x^3} = 0
\]

- "Hydrostatic" models possess only numerical dispersion:

\[
\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2} \delta \xi \right) \frac{\partial \xi}{\partial x} + \Gamma \frac{\varepsilon^2}{6} \frac{\partial^3 \xi}{\partial x^3} = 0
\]
Hydrostatic vs. Nonhydrostatic

Ocean Model (SUNTANS)

\[ \Delta x = \frac{h_1}{4} \]

Hydrostatic model dispersion (numerical):

\[ \Gamma = 0.005 \]

Nonhydrostatic model dispersion (physical+numerical):

\[ 1 + \Gamma = 1.005 \]

Numerical dispersion is 200 times smaller than physical dispersion.
Hydrostatic vs. Nonhydrostatic Ocean Model (SUNTANS)

\[ \Delta x = 8h_1 \]

Hydrostatic model dispersion (numerical):
\[ \Gamma = 5 \]

Nonhydrostatic model dispersion (physical+numerical):
\[ 1 + \Gamma = 6 \]

Numerical dispersion is 5 times larger than physical dispersion.
Hydrostatic and nonhydrostatic models produce the same "numerical solitary-like waves" for large $\lambda$. Hydrostatic models produce sharp fronts due to small numerical dispersion. Nonhydrostatic models converge to the correct solitary wave for small $\lambda$. 

**Effects of $\lambda = \Delta x / h_1$ (grid resolution)**
Hydrostatic vs. Nonhydrostatic Modeled soliton widths

KdV equation

Ocean Model (SUNTANS)

\[
\frac{L}{L_0} = \sqrt{1 + \Gamma}
\]

\[
\frac{L_h}{L_0} = \sqrt{\Gamma}
\]

\[
\Gamma = K \lambda^2
\]

\[
\frac{L}{L_0} = \sqrt{1 + K \lambda^2}
\]

\[
\frac{L_h}{L_0} = \lambda \sqrt{K}
\]
Conclusions

- To resolve nonhydrostatic effects in internal gravity waves, the grid lepticity must satisfy \( \lambda = \Delta x / h_1 \approx O(0.1) \).

- Large \( \lambda \) leads to excessive numerical dispersion and hydrostatic and nonhydrostatic models produce the same (incorrect) results.

- This analysis assumes second-order accuracy. Third-order accurate models (in both time and space) would not produce (lowest order) numerical dispersion and provide more accurate results.

- This condition \( \Delta x < h_1 \) may be a significant additional resolution requirement beyond the current-state-of-the art in ocean modeling of internal waves.
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