Atmospheric and climatic variability : some generic dynamical and probabilistic features

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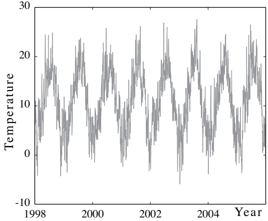
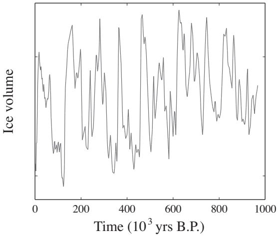
Ubiquity of variability in atmospheric and climate dynamics over a wide range of time and space scales



Evolution of daily temperature at Uccle, Brussels



Evolution of continental ice over the last million years

Limitations to reliable predictions

Objectives

- Present a unifying view of the issues of variability and predictability in the light of nonlinear dynamics, stochastic processes and complex systems research
- Bring out universal trends
- Multilevel approach with emphasis on the intertwinning between the deterministic and the probabilistic levels of description

Outline

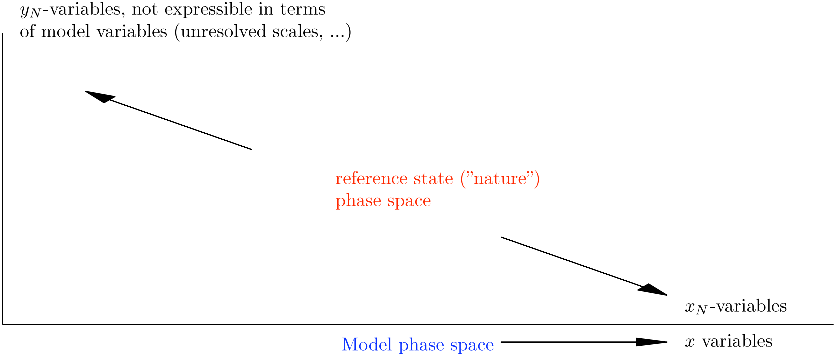
1. Error dynamics
2. Extremes
3. Transitions between states
4. Thermodynamic signatures
5. Conclusions

I. Error Dynamics

Error propagation and amplification as one the main signatures of complexity of atmospheric dynamics.

- sensitivity to initial conditions : amplification of small initial errors.
- sensitivity to parameters : amplification of model errors.

Formulation



Focus on case where model and "nature" variables span the same phase space

Model :

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mu\right) \qquad \mathbf{f}: \ \text{nonlinear}$$

with

$$\mathbf{x} \equiv \left(x_1, \cdots x_n\right), \qquad \mathbf{f} \equiv \left(f_1, \cdots f_n\right)$$

Nature :

$$\begin{aligned} \frac{d\mathbf{x}_N}{dt} &= \mathbf{f}_N\left(\mathbf{x}_N, \mu_N\right) \\ &= \mathbf{f}\left(\mathbf{x}_N, \mu_N\right) + \eta \mathbf{g}\left(\mathbf{x}_N, \mu_N\right) \end{aligned}$$

with

$$\mathbf{x}_N \equiv \left(x_{N1}, \cdots x_{Nn}\right), \quad \mathbf{f}_N \equiv \left(f_{N1}, \cdots f_{Nn}\right), \quad \mathbf{g} = \left(g_1, \cdots g_n\right)$$

Error :

$$\mathbf{u}\left(t\right) = \mathbf{x}\left(t\right) - \mathbf{x}_N\left(t\right)$$

Set $\mu = \mu_N + \delta\mu$

focussing on the limits of $|\mathbf{u}(0)|$ small (NOT satisfied in operational weather prediction models) and $|\delta\mu/\mu_N| << 1$ with $\eta = \mathcal{O}(\delta\mu) \equiv \gamma\delta\mu$. This will allow us to sort out some universal features.

Linearized evolution of the error in these limits :

$$\frac{d\mathbf{u}}{dt} = J\ .\ \mathbf{u} + \boldsymbol{\Phi}\delta\mu$$

$$\text{with} \qquad \underbrace{J = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_N}_{\text{Jacobian matrix}} \qquad \underbrace{\boldsymbol{\Phi} = \left(\frac{\partial \mathbf{f}}{\partial \mu}\right)_N - \gamma \mathbf{g}_N}_{\text{model error source term}}$$

Formal solution :

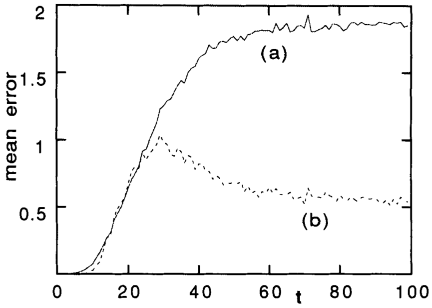
$$\mathbf{u}(t) = M(t,0)\ .\ \mathbf{u}(0) + \int_0^t dt' M(t,t')\ .\ \boldsymbol{\Phi}(t')$$

where the fundamental matrix $M(t, t_0)$ satisfies

$$\frac{dM(t,t_0)}{dt} = J\ .\ M(t,t_0)$$

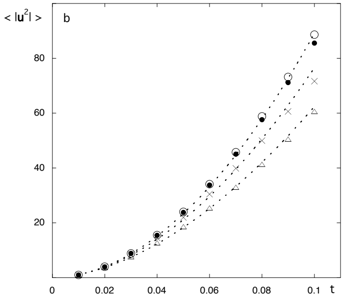
Behavior of mean quadratic error $< u^2\,(t) >$

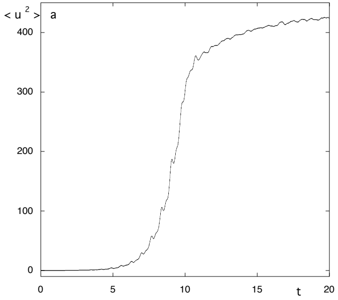
- In absence of model error ($\delta\mu = 0$) and for systems giving rise to chaotic dynamics, logistic-like growth



- Initial stage of exponential growth driven by largest Lyapunov exponent ;
- Intermediate explosive stage ;
- Final saturation level of the order of the size of the attractor.

- In absence of initial error $(u(0) = 0)$





  - Mean quadratic model errors start at zero level and increase as t^2 with a state-dependent proportionality coefficient depending on the magnitude of the error in the parameters, averaged over the attractor of the reference system. Instability of motion and the largest Lyapunov exponent play no crucial role ;
  - For longer times, logistic like growth. Finite saturation level, practically independent of the quality of the model.

In presence of both initial and model errors :

- Time t^* for which total mean quadratic error attains a minimum ;
- time $\bar{t}$ for which there is, typically, crossover between the growth of the two types of error, i.e., initial errors dominate for very short times $(t < \bar{t})$ and model errors take over subsequently $(t > \bar{t})$.

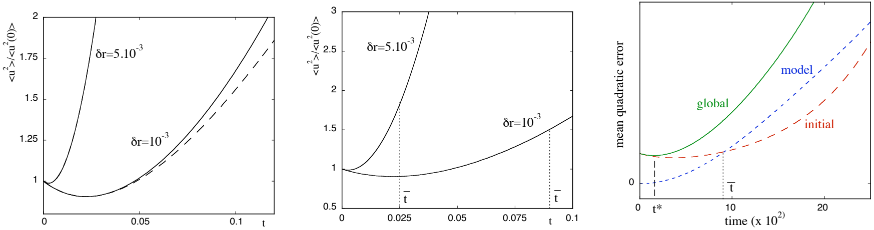
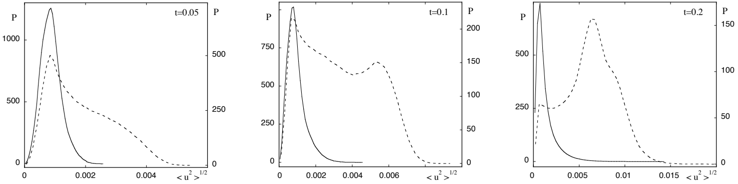


Illustration on a low-order model

Some further aspects :

- High variability of individual errors around the mean : unexpected transient evolution of the error probability distributions



  Complicates further task of prediction.

- Can errors be controlled ?
  Growing trend : model error source terms by stochastic forcings to be added to the model equations.
  Main results
  For short time scales :
  - Control in the form of additive white noise deteriorates model performance as far as mean error growth is concerned ;
  - Variability can be enhanced, and brought closer to natural variability.

  For long time scales :
  - Situation subtler. Performance system-dependent.

II. Extremes

Importance of extremes in nature, technology and society.

Extremes typically imply rare events and large deviations and thus pose the problem of prediction on a new basis.

Key quantity for the purposes of prediction : waiting time between events of a given intensity x.

1. Classical statistical approach

i.i.d.r.v's.

Cumulative pdf of process $F(x)$.

pdf of the time of first exceedence of x.

$$r_k(x) = F^{k-1}(x)(1 - F(x)) \quad k = 1, 2, \cdots$$

Smooth exponential distribution

$$T(x) = < k > \quad = \quad \frac{1}{1 - F(x)}$$

$$D(x) = < \delta k^2 >^{1/2} \quad = \quad \frac{F^{1/2}(x)}{1 - F(x)}$$

Variability comparable to mean for x large. Failure of simple recipes !

2. Beyond the statistical approach :

Build a theory of extremes for deterministic dynamical systems showing complex behavior.

Evolution law

$$\begin{aligned}
X_{n+1} &= f(X_n, \mu) \qquad a \le X \le b \\
\text{"Cell" } C &: \quad a \le X \le x \\
\bar{C} &: \quad x < X \le b \\
\text{Then} & \\
r_k &= \text{Prob } \left(\bar{C}, k | C, 1; \ldots C, k-1\right) \\
&= \frac{1}{F(x)} \int_a^x dX_0 \rho(X_0) \prod_{j=1}^{k-1} \Theta\left(x - f^{(j)}(X_0)\right)\left(1 - \Theta\left(x - f^{(k)}(X_0)\right)\right)
\end{aligned}$$

X_0 : initial state

ρ : invariant probability

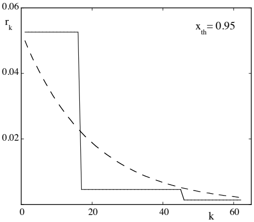
Θ : Heaviside function

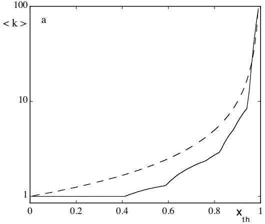
Fundamental differences from classical theory
New element : r_k is no longer a smooth function

A. Uniform quasi-periodic motion

$$X_{n+1} = a + X_n \mod 1$$
$$a : \text{irrational}$$

$$\rho(X) = 1$$
$$F(X) = X$$

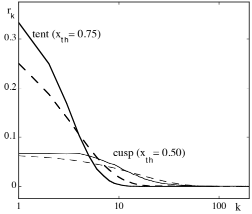




B. Chaotic dynamics

a. tent map (fully developed chaos)

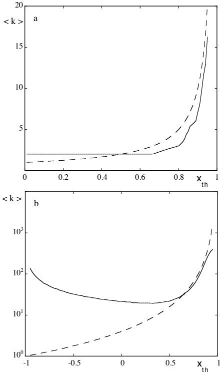
$$\begin{cases} \rho(X) = & 1 \\ F(X) = & X \end{cases}$$



b. cusp map (intermittent chaos)

$$\begin{cases} \rho(X) = & \dfrac{1-X}{2} \\ F(X) = & 1-\rho^2(X) \end{cases}$$

tent map



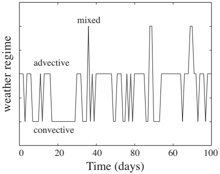
cusp map

III. Transitions between states

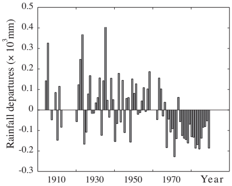
So far variability around a given state, represented by an attractor embedded in phase space.

Evidence of multiple states/attractors in atmospheric and climate dynamics :

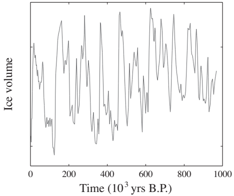
- Atmospheric regimes (zonal flaws, blocking)



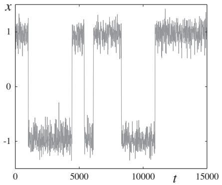
- Onset of droughts



- Transitions between glacial and interglacial climates



Basic mechanism : fluctuation-induced transitions



Small-scale diffusion-like process around a given state interrupted intermittently by large deviations leading to another state.

Role of external forcings

Generic mechanisms :

- Weak periodic forcing of period comparable to transition time between states (e.g. astronomical forcing)
  $\rightarrow$ enhancement of the rate of transitions between states through the mechanism of stochastic resonance.
- Parameters varying slowly in time (anthropogenic effects, secular variations of solar constant, ...)

A case study

Minimal model giving rise to bistable behavior :

$$\begin{aligned} \frac{dx}{dt} &= \lambda(t)\,x - x^3 + F(t) \\ \lambda &= \lambda_0 + \varepsilon t \qquad \varepsilon << 1 \end{aligned}$$

$F(t)$: Gaussian white noise
reference steady states :$x_{\pm} = \pm\lambda_0^{1/2}$

Fokker-Planck equation :

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[(\lambda_0 + \varepsilon t)\,x - x^3\right] P + \frac{q^2}{2}\frac{\partial^2 P}{\partial x^2}$$

Mapping into a discrete 2-state process for the probability masses N in the 2 states $x_+(t)$, $x_-(t)$.

Setting

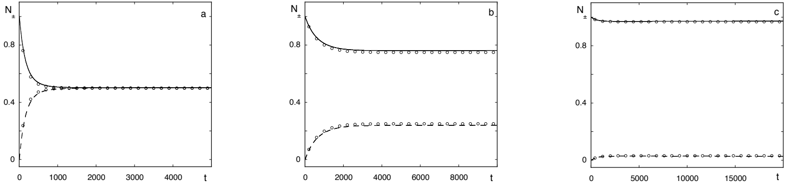
$$N_{\pm}(t) = \frac{1}{2} \pm \delta N(t)$$

yields in the adiabatic approximation

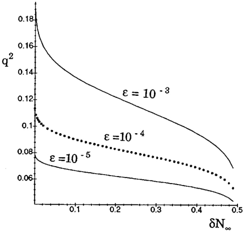
$$\delta N(t) = \delta N(0)\exp\left\{-\frac{\sqrt{2}}{\pi}\frac{q^2}{\varepsilon}\left[\exp\left(-\frac{\lambda_0^2}{2q^2}\right) - \exp\left(-\frac{(\lambda_0+\varepsilon t)^2}{2q^2}\right)\right]\right\}$$

$q^2 = 0.12$ $q^2 = 0.08$ $q^2 = 0.06$

$\lambda_0 = 1$
$\varepsilon = 10^{-4}$

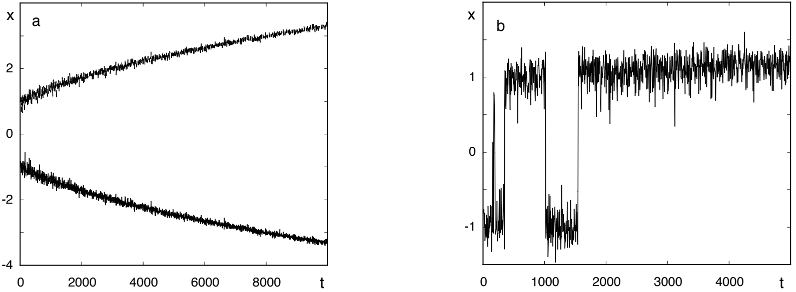


Quenching of transitions, system locks into a frozen state.



Sensitivity of the asymptotic value of the excess probability mass δN_∞ on the noise strength q^2 and on the rate of the ramp ε

Stochastic simulations (breakdown of ergodicity)



$\varepsilon = 10^{-3}$, $q^2 = 0.08$ $\qquad$ $\varepsilon = 10^{-4}$, $q^2 = 0.12$

IV. Thermodynamic Signatures

Extent to which thermodynamics allows to characterize the complexity of atmospheric and climate dynamics in an intrinsic manner and in a universal way.

- Full success in thermodynamic equilibrium (extremal properties of entropy,...).
- Weak form of universality, limited to near equilibrium phenomena in terms of the dissipation associated to the process of interest (minimum entropy production theorem).
- Claims about global organizing principles in atmospheric and climate physics-related problems valid arbitrarily far from thermodynamic equilibrium : the maximum dissipation "principle".

Our thesis : such principles are not to be expected, at least as long as a macroscopic level of description is adopted.

Lessons from low-order models, where one can write out explicitly an entropy balance equation and identify and compute entropy production – the principal quantity measuring dissipation – unequivocally.

Lorenz's 3-mode thermal convection model

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(-x+y) \\
\frac{dy}{dt} &= rx - y - xz \\
\frac{dz}{dt} &= bz - xy
\end{aligned}
$$

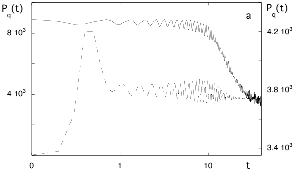
Thermal dissipation

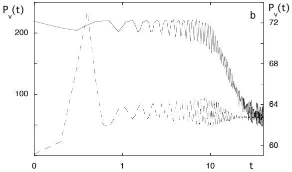
$$P_q(t) \propto \frac{3}{\sqrt{2}} y^2(t) + 4\sqrt{2} z^2(t)$$

viscous dissipation

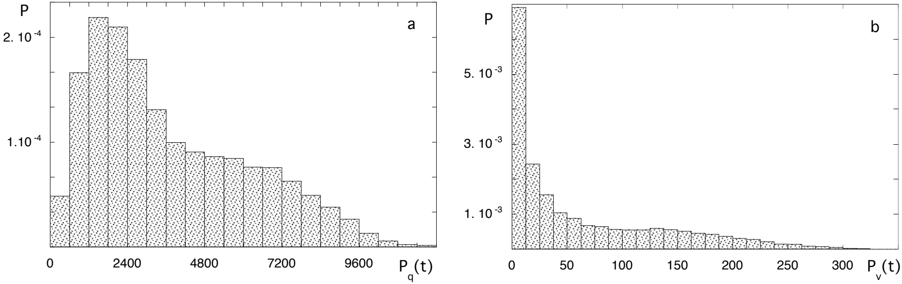
$$P_v(t) \propto x^2(t)$$

Time dependence of $P_q(t)$ and $P_v(t)$





Probability density of P_q and P_v on the attractors



3-mode quasi-geostrophic limit of the shallow water model

$$a_i \frac{dy_i}{dt} = b_{ij} y_i y_j - \sum_{k=1}^{3} c_k y_k + F_i$$
$$(i, j, k = 1, 2, 3)$$

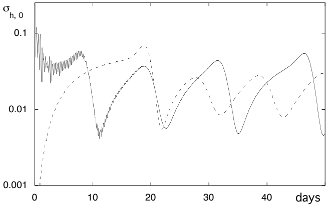
y_i : fourier-like expansion coefficients of vorticity, height and forcing.

Viscous dissipation

$$\sigma_h = 2\Psi_{xy}^2 + \frac{1}{2}\left(\Psi_{xx} - \Psi_{yy}\right)$$

Ψ : stream function.

Time dependence of σ



In both cases :depending on initial state dissipation may increase as well as decrease, prior to the stabilization on the final attracting state.

No variational property whereby dissipation would attain an extremum in the regime of stable operation.

Global climate as a system of maximum dissipation ?

Paltridge's 2 box model

$$\begin{aligned} \frac{dT_1}{dt} &= F - \sigma T_1^4 - \bar{J} \\ \frac{dT_2}{dt} &= \bar{J} - \sigma T_2^4 \end{aligned}$$

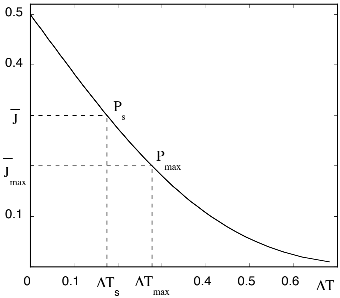
$$\begin{cases} T_1: & \text{equatorial temperature} \\ T_2: & \text{polar temperature} \\ F: & \text{solar energy} \\ \sigma T_i^4: & \text{infrared radiant energy} \\ \bar{J}: & \text{heat flux} \end{cases}$$

single steady-state solution

$$\Delta T_s = T_{1s} - T_{2s} = \left(1 - \bar{J}\right)^{1/4} - \bar{J}^{1/4}$$

Thermodynamic entropy production uniquely defined

$$P_s = \frac{\bar{J}\Delta T_s}{T_{1s}T_{2s}}$$



Attracting states depend on $\bar{J}$

No preference for $P_{\max}$ whatsoever !

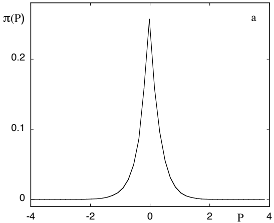
Effect of stochastic perturbations

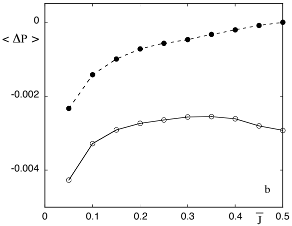
$$
\begin{aligned}
J &= \bar{J} + \delta J(t) \\
\delta J(t) &= f(t)\, G(T_1, T_2)
\end{aligned}
$$

$$
\begin{aligned}
f(t) &: \quad \text{noise process} \\
G(T_1, T_2) &: \quad \text{coupling function}
\end{aligned}
$$

Fokker-Planck equation analysis complemented with stochastic simulations yields

$\bar{J} = 0.3,\ q^2 = 10^{-3}$





For given parameter values fluctuations tend to reduce dissipation : entropy production associated with deterministic (macroscopic) behavior is a maximum with respect to the entropy production associated with fluctuating paths around deterministic ones.

BUT

- There is no mechanism allowing the system (keeping parameters fixed) to approach, through successive perturbations, the regime of maximum entropy production with an appreciable probability.

$\rightarrow$ Maximum dissipation principle invalidated. No organizing principle operating in nature in the form of an extremal property involving entropy production.

Irreversible thermodynamics may still be promising in some augmented form (information on probabilistic structure ?). Traditional tools (equilibrium systems) are not sufficient.

V. Conclusions

- Nonlinear dynamics and complex systems approaches shed new light on the long-standing problems of variability and predictability of atmospheric and climatic fields.
- Generic results established in well-defined limits as reference for understanding and testing results from operational numerical prediction models.
- Variability around mean often comparable to mean itself, especially when extremes, threshold crossings and transitions between states are concerned : Need for a probabilistic approach.
- Search for global thermodynamics-inspired organizing principles remains a challenging task, despite the failure of current attempts based on the maximum dissipation "principle".