STOCHASTIC RESONANCE IN MULTISTABLE SYSTEMS

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Evidence of climatic variability:

Preferred frequency

\[ \frac{1}{\omega} \sim 10^5 \text{ years} \]

Eccentricity of earth’s orbit?

\[ \epsilon \sin \omega t \quad \epsilon \sim 10^{-3} \]

Need for a mechanism of amplification of weak signals in the presence of noise.
In absence of periodic forcing

\[
C \frac{dT}{dt} = \text{Incoming radiation} - \text{Outgoing radiation} + \text{Stochastic fluctuations}
\]

or equivalently,

\[
C \frac{dT}{dt} = -\frac{\partial U}{\partial T} + F(T) \quad \text{(one variable system)}
\]

\(\triangleright\) \(U\) : kinetic potential, possessing two wells (stable states) separated by a maximum (intermediate unstable state).

\(\triangleright\) Stochastic fluctuations : white noise

\[
< F(t) > = 0
\]

\[
< F(t) (t') > = q^2 \delta (t - t')
\]

\(\rightarrow\) Fokker Planck equation for the probability masses around the two stable states \(T_-\) (state 1) and \(T_+\) (state 2).

Steady state solution expressed entirely in terms of \(U\) :

\[
P_s \sim \exp \left( -\frac{2}{q^2} U \right)
\]
Phenomenological theory of Kramers: diffusion over a potential barrier ($q^2$ small)

Mapping the problem into a discrete process

$$\text{state 1} \xleftrightarrow{k_{12}} \text{state 2}$$

$k$'s: "rate constants"

$$\frac{dP_1}{dt} = -k_{12}P_1 + k_{21}P_2 \quad \text{with} \quad P_1 + P_2 = 1$$

$$k_{12} \sim e^{-\frac{2}{q^2} \Delta U}, \quad \Delta U = U \text{ (unstable state)} - U \text{ (reference state)}$$

Potential barrier

Time scale of transitions between states 1 and 2

$$\langle \tau \rangle \sim \frac{1}{k_{12}} \quad \text{long time scale of order } 10^5 \text{ years}$$
Classical setting of Stochastic Resonance

Presence of a periodic forcing $\varepsilon \sin \omega t$

\[
\varepsilon \sim 10^{-3} \\
\omega \sim 10^{-5} \text{ years}
\]

eccentricity of earth

Adiabatic approximation

\[
P_i(t) = P_i^{(o)} + \varepsilon R_i \sin (\omega t + \varphi) \quad i = 1, 2
\]

$R_i$ : amplitude of response
Results

$R_i$ appreciable if

\[ P_1^{(0)} \sim P_2^{(0)} \sim 0.5 \text{ COEXISTANCE} \]

\[ \frac{1}{\omega} \geq <\tau> \]

\[ R_i \sim \frac{2\varepsilon}{q^2} \left[ 1 + (\tau \omega)^2 \right]^{-1/2} \]

\[ \varphi = -\arctan(\tau \omega) \]

Typically $\varepsilon R \sim 20\%$
SR in multistable systems

(1 variable systems, additive periodic forcing of small amplitude)

Again, mapping into a discrete state process

\[
\text{state } 1 \xrightleftharpoons[k_{12}(t)]{k_{21}(t)} \text{state } 2 \xrightleftharpoons[k_{23}(t)]{k_{32}(t)} \text{state } 3 \cdots \text{state } n - 1 \xrightleftharpoons[k_{n-1,n}(t)]{k_{n,n-1}(t)} \text{state } n
\]

\[
\frac{dP_i(t)}{dt} = \sum_{j=1}^{n} M_{ij}(t) P_j(t) \quad i = 1, \cdots, n
\]

(5)

\[
M = \begin{pmatrix}
-k_{12}(t) & k_{21}(t) & 0 & \cdots & 0 \\
k_{12} & -(k_{21} + k_{23}) & k_{32} & \cdots & 0 \\
0 & k_{23} & -(k_{32} + k_{34}) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & -k_{n,n-1}
\end{pmatrix}
\]
Rate constants

\[
U = U^{(o)} - \varepsilon x \sin \omega t \\

k_{i,i\pm 1}(t) = k_{i,i\pm 1}^{(o)} \exp \left[ \frac{2\varepsilon}{q^2} \Delta x (i, i \pm 1) \sin \omega t \right] \\

k_{i,i\pm 1}^{(o)} \sim \exp \left[ -\frac{2}{q^2} \Delta U_{o} (i, i \pm 1) \right]
\]
Linear response

\[ k_{i,i \pm 1} = k_{i,i \pm 1}^{(o)} + \varepsilon \Delta_{i,i \pm 1} \sin \omega t \]
\[ \Delta_{i,i \pm 1} = \frac{2}{q^2} k_{i,i \pm 1}^{(o)} \Delta x (i, i \pm 1) \]

\[ M(t) = M_0 + \varepsilon \Delta \sin \omega t \]
\[ \mathbf{P}(t) = \mathbf{P}_0 + \varepsilon \delta \mathbf{P}(t) \]

- \( M_0, \Delta \): tridiagonal matrices
- \( \mathbf{P}_0 \): invariant \( P \) with \( \varepsilon = 0 \)
- \( \delta \mathbf{P} \): induced response

\[
\begin{align*}
\sum_{i=1}^{n} P_i & = 1 \\
\sum_{i=1}^{n} \delta P_i & = 0
\end{align*}
\]
\[
\frac{d\delta \mathbf{P}(t)}{dt} = M_0 \delta \mathbf{P} + \varepsilon \sin \omega t \Delta \mathbf{P}_0
\]

Long time solution: \( \delta \mathbf{P}(t) = \varepsilon (A \cos \omega t + B \sin \omega t) \)

\[
\begin{align*}
\delta P_i(t) &= R_i \sin (\omega t + \varphi_i) \\
R_i &= \varepsilon \left( A_i^2 + B_i^2 \right)^{1/2} \\
\varphi_i &= \arctan \left( \frac{A_i}{B_i} \right)
\end{align*}
\]

Let \( \lambda_k \) and \( u_k \) be eigenvalues and eigenvectors of \( M_0 \).
Expanding \( \Delta \mathbf{P}_0 \) in the basis of \( u_k \)

\[
\Delta \mathbf{P}_0 = \sum_{k=1}^{n} \gamma_k u_k
\]

\[
\begin{align*}
A &= -\sum_{k=1}^{n} \frac{\lambda_k^2 + \omega^2 \gamma_k u_k}{} \\
B &= -\sum_{k=1}^{n} \frac{\lambda_k}{\lambda_k^2 + \omega^2 \gamma_k u_k}
\end{align*}
\]
Simplification:

all $k$’s $\sim$ identical (one of the prerequisites of classical SR)

\[
k_{12} = k_{21} = \cdots k_0
\]

\[
M_0 = k_0 \begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \cdots & 1 & -1
\end{bmatrix}
\]

\[
\lambda_k = -2k_0 \left( 1 - \cos \left( \frac{(k - 1) \pi}{n} \right) \right) \quad k = 1, \cdots n
\]

\[
u_{i_1} = 1
\]

\[
u_{i_k} = \cos \left[ \frac{(k - 1)(2i - 1) \pi}{2n} \right] \quad i = 1, \cdots n \quad k = 2, \cdots n
\]
Toy model

\[ U_0 (x) = -\cos x \quad \text{for} \quad 0 \leq x \leq 2\pi n \]
\[ \pi, 3\pi, 5\pi, \cdots \quad \text{stable states} \]
\[ 2\pi, 4\pi, 6\pi, \cdots \quad \text{unstable states} \]

\[ A_i = \frac{1}{N^2} \frac{4\pi k_0}{n} \frac{2}{q^2} \sum_{k \text{ even}} \cos \frac{(k - 1) \pi}{2n} \cos \frac{(2i - 1)(k - 1) \pi}{2n} \frac{\omega}{\lambda_k^2 + \omega^2} \]
\[ B_i = \frac{1}{N^2} \frac{4\pi k_0}{n} \frac{2}{q^2} \sum_{k \text{ even}} \cos \frac{(k - 1) \pi}{2n} \cos \frac{(2i - 1)(k - 1) \pi}{2n} \frac{\lambda_k}{\lambda_k^2 + \omega^2} \]
\[ N : \quad \text{norm of } u_k \]

\[ R_i = \varepsilon \left( A_i^2 + B_i^2 \right)^{1/2} \]
\[ \varphi_i = \arctan \frac{A_i}{B_i} \]
\[ \alpha = \frac{2\varepsilon \Delta x}{q^2} \]

Is there an optimal \( q_{\text{opt}}^2 \)?

Example: \( n = 6 \)

\[ R_1 = \frac{2\varepsilon \pi}{3q^2} \left\{ \frac{(\omega/k_0)^4 + 15(\omega/k_0)^2 + 25}{[(\omega/k_0)^4 + 14(\omega/k_0)^2 + 1][\omega/k_0^2 + 4]} \right\}^{1/2} \]

\( \omega/k_0 \equiv \omega \tau \)
Conclusions

- Extension of classical SR for an arbitrary number of simultaneously stable states for systems involving one variable
- Amplitude and phase of response of a stable state have been determined as a function of its location
- Existence of an optimal $q^2$
- Optimal number of intermediate stable states for which response is maximized

Extension of this work:
- Multivariate systems
- Non potential systems
- More complex communication geometries of stable states