Inverse problem on the reconstruction of the vertical heat exchange coefficient in a model of World Ocean hydrodynamics

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1. Mathematical model of the ocean hydrothermodynamics

Consider the spherical coordinate system (λ,θ,r) $\in D\subset R^3$, $z=R-r$, ($\lambda\in[0,2\pi]$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$), S_R is the sphere of the radius R, $\Omega\equiv\Gamma_S\subset\Gamma\equiv\partial D$ is a part of this sphere, the ocean velocity vector is presented in the coordinate form $\mathbf{U}=(u,v,w)\equiv(\mathbf{u},w)$, where $\mathbf{u}=(u,v)$ is the 'horizontal' velocity vector and w is the 'vertical' velocity. The sea level will be specified by the equation $z=\xi(\lambda,\theta,t)$, where $(\lambda,\theta,R)\in\Omega$, t is time, $t\in[0,\bar{t}\,]$, $\bar{t}<\infty$, further $f(u)=l+u\sin\theta/(r\cos\theta)$; $n\equiv 1/r, m\equiv 1/(r\cos\theta)$, $l=2\omega\sin\theta$, where ω is the Earth angular velocity.

In $D\times(0,\bar{t}\,)$ we will write the system of the nonlinear hydrothermodynamic equations for the functions u,v,ξ,T,S,q:

$$\begin{cases}\dfrac{d\mathbf{u}}{dt}+\begin{bmatrix}0 & -f\\ f & 0\end{bmatrix}\mathbf{u}-g\,\mathrm{grad}\,\xi+A_u\mathbf{u}+(A_k)^2\mathbf{u}\\ \quad =\mathbf{f}-\dfrac{1}{\rho_0}\mathrm{grad}\,P_a-\dfrac{g}{\rho_0}\mathrm{grad}\displaystyle\int_0^z\rho_1(T,S)dz'\\ \dfrac{\partial\xi}{\partial t}-m\dfrac{\partial}{\partial x}\left(\displaystyle\int_0^H\Theta(z)udz\right)-m\dfrac{\partial}{\partial y}\left(\displaystyle\int_0^H\Theta(z)\dfrac{n}{m}vdz\right)=f_3,\\ \dfrac{dT}{dt}+A_T(q,T)=f_T,\quad \dfrac{dS}{dt}+A_SS=f_S,\end{cases}$$

where: d/dt is the "total derivative"; $\Theta(z)\equiv r(z)/R$; $\mathbf{f}=(f_1,f_2),f_3$, f_T, f_S are the functions of the 'internal' sources; $\rho=\rho_0+\rho_1$ is the water density, $g,\rho_0=const>0$; T,S are the functions of water temperature and salinity; P_a is a given function, $0<z<H(\lambda,\Theta)<<R$ and

$$A_TT\equiv-(\mu_T\Delta T+\frac{1}{r^2}\frac{\partial}{\partial z}\nu_Tr^2\frac{\partial T}{\partial z})\ \text{ in } D,\ \ A_T(q,T)\equiv A_TT-\frac{1}{r^2}\frac{\partial}{\partial z}r^2q\ \text{ in } D_0,$$

$$A_T(q,T)\equiv A_TT-\frac{1}{r^2}\frac{\partial}{\partial z}r^2q\ \text{ and } q=\tilde{\nu}_T\frac{\partial T}{\partial z}\ \text{ in } D\setminus D_0,$$

where D_0 is a subdomain of D, ν_T is a given nonnegative "background" turbulent exchange (thermal diffusivity, vertical heat exchange) coefficient.

Boundary conditions on the "sea surface"

Boundary conditions on the "sea surface" $\Gamma_S\equiv\Omega$ at $z=0$:

$$\begin{cases}\left(\displaystyle\int_0^H\Theta\bar{u}dz\right)\bar{n}+\beta_0m_{op}\sqrt{gH}\,\xi=m_{op}\sqrt{gH}\,d_s\ \text{on}\,\partial\Omega,\\ U_n^{(-)}u-\nu\dfrac{\partial u}{\partial z}-k_{33}\dfrac{\partial}{\partial z}A_ku=\tau_x^{(a)}\rho_0,\ U_n^{(-)}v-\nu\dfrac{\partial v}{\partial z}-k_{33}\dfrac{\partial}{\partial z}A_kv=\tau_y^{(a)}\rho_0,\\ A_ku=0,\quad A_kv=0,\quad U_n^{(-)}T-\nu_T\dfrac{\partial T}{\partial z}+\gamma_T(T-T_a)=Q+U_n^{(-)}d_T,\\ U_n^{(-)}S-\nu_S\dfrac{\partial S}{\partial z}+\gamma_S(S-S_a)=Q_S+U_n^{(-)}d_S,\end{cases}$$

where $U_n=\vec{U}\cdot\vec{N},\vec{U}=(u,v,w)\equiv(\vec{u},w)\equiv(\mathbf{u},w),\vec{N}=(n_1,n_2,n_3)\equiv(\vec{n},n_3)$, $U_n^{(-)}=(|U_n|-U_n)/2$, $Q\equiv q\cdot n_3$ is the vertical "turbulent heat flux" at the sea surface and $Q=0$ at $z=H(\lambda,\Theta)$. The functions $q,Q,\tilde{\nu}_T$ will be the äadditional unknowns" accordingly in $D,\Omega,D_1\equiv D\setminus D_0$. (On others parts of Γ some boundary conditions are also imposed.)

To solve the inverse problem we use the splitting method and method of "fictitious controls" (Agoshkov V.I., EGU-2011)

Let us introduce U_c:

$$U_c\equiv\frac{1}{r^2}\frac{\partial}{\partial z}r^2q\equiv\frac{1}{r^2}\frac{\partial}{\partial z}r^2(m_0q+(1-m_0)\tilde{\nu}_T\frac{\partial T}{\partial z})$$

– the "fictitious control" ("fictitious force"), m_0 is the characteristic function of D_0. Then further the "T - equation" will be rewritten as

$$\frac{dT}{dt}+A_TT=f_T+U_c.$$

2. Approximation by splitting method: Problem I

Step 1. We consider the system:

$$\begin{cases}T_t+(\bar{U},\mathbf{Grad})T-\mathbf{Div}(\hat{a}_T\cdot\mathbf{Grad}\,T)=f_T+U_c\ \text{in}\ D\times(t_{j-1},t_j),\\ T=T_{j-1}\ \text{for}\ t=t_{j-1}\ \text{in}\ D,\\ -\nu_T\dfrac{\partial T}{\partial z}=Q_0\ \text{on}\ \Gamma_S\times(t_{j-1},t_j),\\ \dfrac{\partial T}{\partial N_T}=0\ \text{on}\ \Gamma_{w,c}\times(t_{j-1},t_j),\\ \bar{U}_n^{(-)}T+\dfrac{\partial T}{\partial N_T}=\bar{U}_n^{(-)}d_T+Q_T\ \text{on}\ \Gamma_{w,op}\times(t_{j-1},t_j),\\ \dfrac{\partial T}{\partial N_T}=0\ \text{on}\ \Gamma_H\times(t_{j-1},t_j),\\ T_j\equiv T\ \text{on}\ D\times(t_{j-1},t_j),\end{cases}$$

where $\Gamma_w=\Gamma_{w,c}\cup\Gamma_{w,op}$ is the "vertical lateral boundary", Γ_H is "the ocean bottom", $Q_0\equiv Q+\bar{U}_n^{(-)}d_T-\bar{U}_n^{(-)}T-\gamma_T(T-T_a)$, U_c,Q_0 are "fictitious controls".

Step 2.

$$\begin{cases}S_t+(\bar{U},\mathbf{Grad})S-\mathbf{Div}(\hat{a}_S\cdot\mathbf{Grad}\,S)=f_S\ \text{in}\ D\times(t_{j-1},t_j),\\ S=S_{j-1}\ \text{at}\ t=t_{j-1}\ \text{in}\ D,\\ \bar{U}_n^{(-)}S-\nu_S\dfrac{\partial S}{\partial z}+\gamma_S(S-S_a)=Q_S+\bar{U}_n^{(-)}d_S\ \text{on}\ \Gamma_S\times(t_{j-1},t_j),\\ \dfrac{\partial S}{\partial N_S}=0\ \text{on}\ \Gamma_{w,c}\times(t_{j-1},t_j),\\ \bar{U}_n^{(-)}S+\dfrac{\partial S}{\partial N_S}=\bar{U}_n^{(-)}d_S+Q_S\ \text{on}\ \Gamma_{w,op}\times(t_{j-1},t_j),\\ \dfrac{\partial S}{\partial N_S}=0\ \text{on}\ \Gamma_H\times(t_{j-1},t_j),\\ S_j\equiv S\ \text{on}\ D\times(t_{j-1},t_j).\end{cases}$$

Step 3.

$$\frac{d\bar{u}}{dt}+\begin{bmatrix}0 & -f\\ f & 0\end{bmatrix}\bar{u}-g\,\mathrm{grad}\,\xi+A_u\bar{u}+(A_k)^2\bar{u}=$$

$$=\bar{f}-\frac{1}{\rho_0}\mathrm{grad}\left(P_a+g\int_0^z\rho_1(\bar{T},\bar{S})dz'\right)\ \text{ in }\ D\times(t_{j-1},t_j),$$

$$\xi_t-\mathrm{div}\left(\int_0^H\Theta\underline{u}dz\right)=f_3\ \text{in}\ \Omega\times(t_{j-1},t_j),$$

$$\bar{u}=\bar{u}_{j-1},\xi=\xi_{j-1}\ \text{at}\ t=t_{j-1},$$

$$\left(\int_0^H\Theta\bar{u}dz\right)\cdot n+\beta_0m_{op}\sqrt{gH}\,\xi=m_{op}\sqrt{gH}\,d_s\ \text{on}\ \partial\Omega\times(t_{j-1},t_j)$$

3. Method of "fictitious controls"

Assume $U_c,Q_0,\tilde{\nu}_T$ to be unknown. Suppose the only function obtained by processing observation data is the function T_{obs} on $\bar{\Omega}\equiv\Omega\cup\partial\Omega$ and in D at $t\in(t_{j-1},t_j)$. Let the physical meaning of this function be an approximation to the sea temperature function T.

The statement of the inverse problem is the following: find $\phi\equiv(u,v,\xi,T,S)$ - the solution of Problem I and $q,Q,\tilde{\nu}_T$ such that the relation $T=T_{obs}$, is satisfied in $(D\cup\Omega)\times(t_{j-1},t_j)$.

A case is admitted when T_{obs} exists only in some subset from $(\Omega\cup D)\times(t_{j-1},t_j)$. But for simplicity we consider here the case when T_{obs} is given in $(\Omega\cup D)\times(t_{j-1},t_j)$.

The inverse problem stated above can be solved by the following steps.

First step: Solve the problem: find T,U_c,Q_0 s.t.

$$\frac{dT}{dt}+A_TT=f_T+U_c,\ \ T=T_{obs}\ \text{ in }\ (D\cup\Omega)\times(t_{j-1},t_j).$$

Second step: Solve the problems

$$\frac{1}{r^2}\frac{\partial}{\partial z}r^2q=U_c\ \text{ in }\ D\times(t_{j-1},t_j),\ \ q=0\ \text{ at }\ z=H,$$

$$\frac{1}{r^2}\frac{\partial}{\partial z}r^2(\tilde{\nu}_T\frac{\partial T}{\partial z})=\frac{1}{r^2}\frac{\partial}{\partial z}r^2q\ \text{ in }\ (D\setminus D_0)\times(t_{j-1},t_j),$$

$$Q=Q_0-\bar{U}_n^{(-)}d_T+\bar{U}_n^{(-)}T+\gamma_T(T-T_a)$$

where $T=T_{obs}$ in $(D\cup\Omega)\times(t_{j-1},t_j)$.

4. Variational approach for solving the inverse problem – variational data assimilation problem

Let us introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha\equiv\Im_\alpha(Q_0,U_c,\phi)=\frac{1}{2}\int_0^{\bar{t}}\int_\Omega\alpha|Q_0|^2d\Omega dt+\frac{1}{2}\int_0^{\bar{t}}\int_D\alpha|U_c|^2dDdt+\Im_0(\phi)=\sum_{j=1}^J\int_{t_{j-1}}^{t_j}\Im_\alpha^{(j)}dt,$$

where

$$\Im_0(\phi)\equiv\Im_0(T)=\frac{1}{2}\int_0^{\bar{t}}\int_\Omega|T-T_{obs}|^2d\Omega dt+\frac{1}{2}\int_0^{\bar{t}}\int_D|T-T_{obs}|^2dDdt,$$

$$\Im_\alpha^{(j)}=\frac{1}{2}\int_{t_{j-1}}^{t_j}\int_\Omega\alpha|Q_0|^2d\Omega dt+\frac{1}{2}\int_{t_{j-1}}^{t_j}\int_D\alpha|U_c|^2dDdt+$$

$$+\frac{1}{2}\int_{t_{j-1}}^{t_j}\int_\Omega|T-T_{obs}^{(j)}|^2d\Omega dt+\frac{1}{2}\int_{t_{j-1}}^{t_j}\int_D|T-T_{obs}^{(j)}|^2dDdt.$$

Here $\alpha\geq 0$, is a "regularization" or "penalty" function, that may be constant.

Data assimilation problem: *find the solution* ϕ *of the Problem* I *and functions* $\{Q_0^{(j)}\},U_c^{(j)}$, *such that, the cost functional is minimal on the set of the solutions.*

This problem can be numerically solved using " adjoint equation and optimal control approaches" (Parmuzin E.I., EGU-2011).

After solving the First step and obtaining ϕ – the solution of Problem I with $T\approx T_{obs}$, the "fictitious control" U_c and vertical turbulent heat fluxes Q_0,q (– the Second step) it is possible to calculate the new turbulent heat exchange coefficient $\tilde{\nu}_T$.

5. The problem of finding the turbulent heat exchange coefficient

Assume that $T_{obs} \in W_\infty^1(D)$. Let us introduce the domains D_i, $i=1,2,3$.

$$D_1 = \{(\lambda,\theta,z): \left|\frac{\partial T_{obs}}{\partial z}\right| \neq 0\} \qquad D_2 = \{(\lambda,\theta,z): \frac{\partial T_{obs}}{\partial z} > 0\}$$

$$D_3 = \{(\lambda,\theta,z): \varphi(\lambda,\theta,z,t) \geq 0\}$$

where the function φ is introduced according to the "semiempirical turbulence theory":

$$\varphi(\lambda,\theta,z,t) = \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \frac{\alpha_T g \gamma_T}{\rho_0}\frac{\partial T}{\partial z} - \frac{\alpha_S g \gamma_S}{\rho_0}\frac{\partial S}{\partial z}.$$

Here $\alpha_T,\dots,\gamma_S$ are given constants. Here we suppose that D_1, D_2, D_3 do not depend of t on each subinterval $[t_{j-1}, t_j]$ for any j.

Assume the following "parametrization":

$$q = \tilde{\nu}_T \frac{\partial T}{\partial z} \quad \text{in } D_1 \text{ or } D_2$$

Since $T = T_{obs}$, it is possible to calculate the "turbulent heat exchange coefficients" using the regularization techniques by the following formula:

$$\tilde{\nu}_T \cong \tilde{\nu}_{T,\alpha} = \frac{\nu_T\left(\frac{\partial T}{\partial z}\right)^2 - \frac{\partial T}{\partial z}\left(\frac{1}{r^2}\int_z^H r^2 U_C dz\right)}{\alpha_2 + \left(\frac{\partial T}{\partial z}\right)^2}$$

In D_3 the coefficient $\tilde{\nu}_T$ is calculated by:

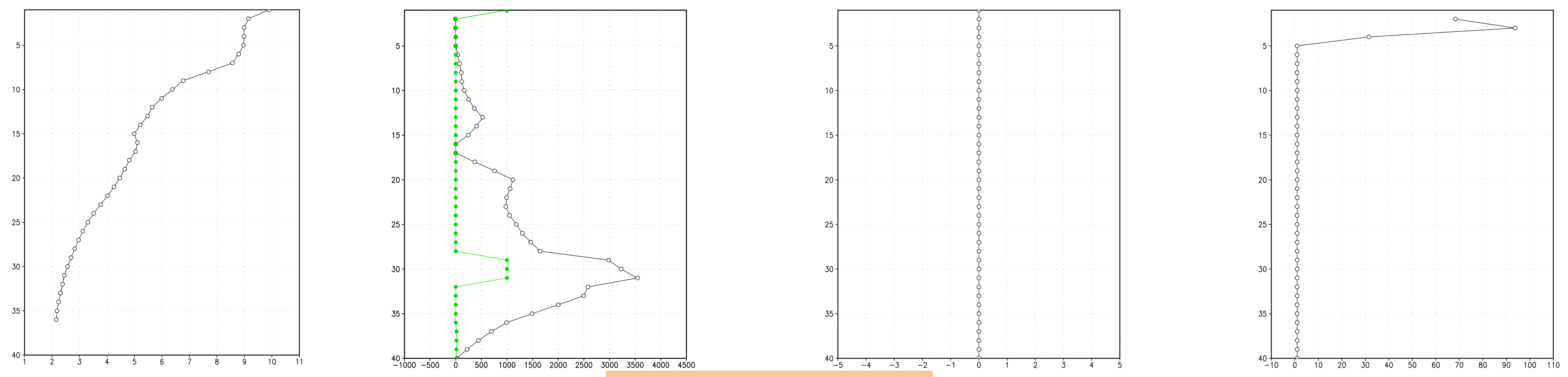
$$\tilde{\nu}_T = \alpha_T \begin{cases} (c_0 h)^2\sqrt{\varphi} + \nu_T & \text{in } D_3 \\ \nu_T & \text{in } D \backslash D_3 \end{cases}$$

where c_0, h are given positive functions.

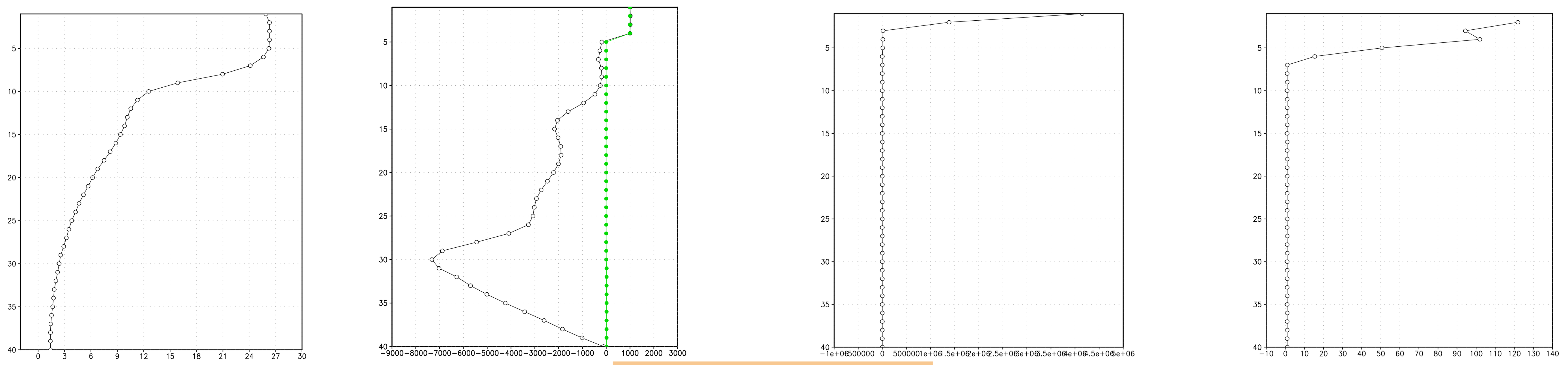
6. Numerical experiments

(Agoshkov V.I., Parmuzin E.I., Zakharova N.B.)

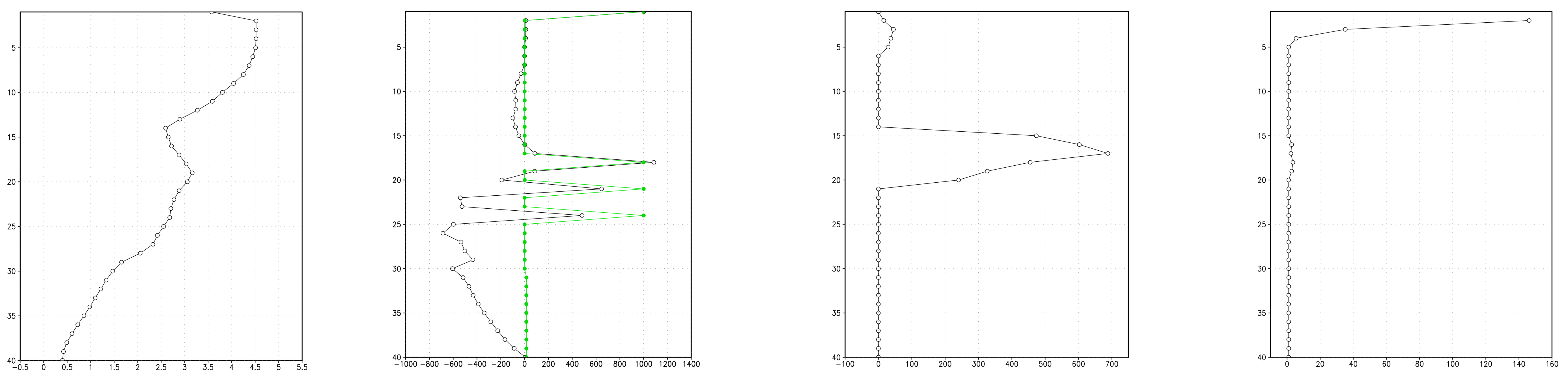
The object of simulation is the World Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 360x337x40 (latitude × longitude × depth); the first mesh point is the point with coordinates 22.5 E and 33.5 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour. The data T_{obs} was obtained from ARGO buoys system and from the satellite observations. These data were used for the construction of the function T_{obs} at all time steps at each grid points (Zakharova N.B., 2010). Other data required for the model for January 2008 was calculated using the database of NCEP (National Centers for Environmental Prediction). The data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the World Ocean developed in INM RAS. The time period taken in experiments is 10 days (January, 2008). We take also (in CGS units system): $\rho_0 = 1.0285$, $\gamma_T = 7.35\cdot 10^{-5}$ $\gamma_T = 0.8\cdot 10^{-3}$, $\alpha_T = 0.1$, $\alpha_S = 0.73$, $\nu_T = 1.$, $c_0 = 0.05$, $h = (1000 \div 6000)$



Point A (179.5° W, 49.75 ° S)



Point B (138.5° E, 6.25 ° N)



Point C (119.5° E, 60.25 ° S)

The temperature profile, thermal diffusivity coefficient defined in D_1 (2nd figure in each row) (green dots- model values, **black dots** – calculated values), thermal diffusivity coefficient defined in D_2 (3rd figure in each row), thermal diffusivity coefficient defined in D_3 (4nd figure in each row) in different points.

References

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