Instability of Planetary Flows based on Riemannian Geometry

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BASICS

Hydrodynamics in 2D is a geodesics on a Riemannian manifold with kinetic energy metric (Arnold, 1966)
Instability: negative sectional curvature (Jacobi Eqn)
Torus: unstable (Arnold)
Sphere ($L \leq 2$): Dowker and Mo-Zheng (1990), Apps (2008), a.o.

SUMMARY

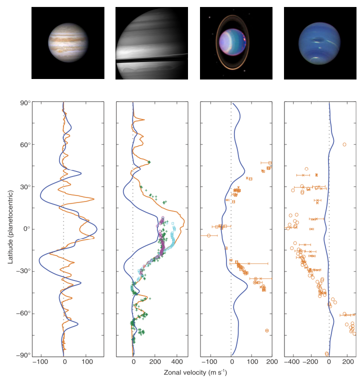
Numerical analysis for higher total wavenumbers $L > 2$ (based on: Arakelyan and Savvidy, 1989)
Complete perturbation space $Y_{\ell,m}$

- Superrotation stabilizes flow (Y_{10}, normalization)
- Stablity of odd L flows for perturbations with $m = \ell$
- Scaling: Sectional curvature for Y_{LM}

$$K(\ell,m) \sim K(m/\ell)$$
$$L\, K_{L'}(Lx/L') \approx L' K_L(x)$$

MOTIVATION

Zonal flows on giant planets*: observation and GCM simulation
* NASA



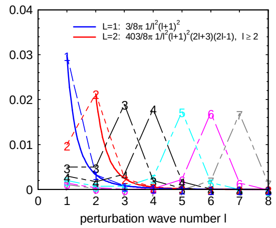
Upper atmosphere zonal velocity [m/s]
obs orange, simulation dark blue
Liu and Schneider (2010)

MAIN RESULTS

Instability is dermined by the sectional curvature K
K describes growth of perturbations perpendicular to the geodesics along a basic flow (stationary here)
K is given by the Riemann curvature tensor: $K(X,Y) = \langle R(Y,X)X,Y \rangle$
Basic flow Y_{LM}, perturbations $Y_{\ell,m}$

Sectional curvature for Y_{L0}

$m = 1$



$$K(L=1, M=0) = \frac{3}{8\pi}\frac{1}{\ell^2(\ell+1)^2}$$

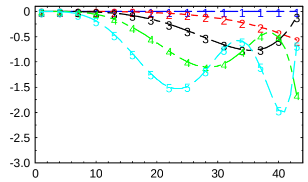
$$K(L=2, M=0) = \frac{403}{8\pi}\frac{1}{\ell^2(\ell+1)^2(2\ell+3)(2\ell-1)}, \quad \ell \geq 2$$

Dowker and Mo-Zheng (1990)

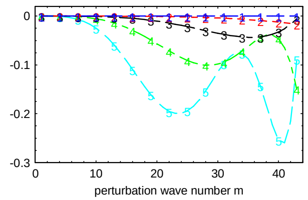
Sectional curvature with superrotation

constant $\ell = 42$, versus m

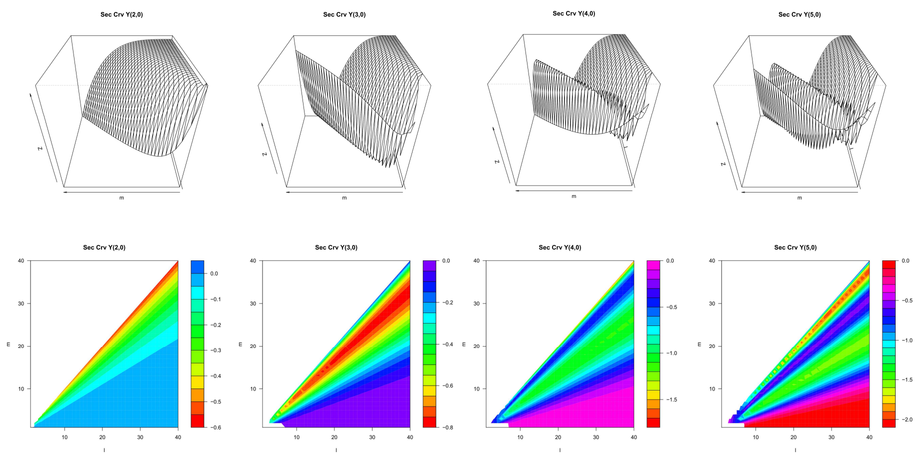
Zonal flow
Y_{L0}



Zonal flow and Superrotation
$Y_{10} + 0.1\, Y_{L0}$

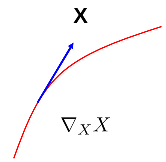


Sectional curvature for Y_{L0}: L = 2, 3, 4, 5 vs ℓ,m



GEOMETRY

Geodesic flow (Arnold, 1966)

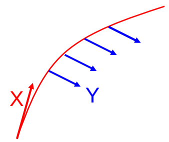


Parallel transport of X along X

$$\frac{D}{\partial t}X = \frac{\partial X}{\partial t} + \nabla_X X + \nabla p = 0$$

$$X = \frac{\partial \eta}{\partial t}$$

Deviation along geodesics: Jacobi equation



$$\left(\frac{D}{\partial t}\right)^2 Y + R(Y,X)X = 0$$

MATHEMATICAL BACKGROUND

Manifold: Sphere (S^2), basis $Y_{\ell m}$ stream functions
Metric: kinetic energy $\langle X, X \rangle$, Riemannian Manifold
Lie algebra, structure constants
Parallel Transport: $\nabla_Y X$ Levi-Civita connection Γ
Geodesics: $\partial_t + \nabla_X X$ Eulerian flow
Geodesic deviation equation (Jacobi)
Riemann tensor R_{XYZW}

Sectional curvature $K(X,Y)$ determines instability

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