Exploring linear and nonlinear solutions to the tidal hydraulic equations

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Introduction

Analytical solutions of tidal hydraulic equations in convergent estuaries are investigated through linear and quasi-nonlinear models. Building on the work by Savenije et al. (2008), some of the assumptions made previously are addressed and neutralised, leading to a modified quasi-nonlinear model to reproduce the dynamics of tidal wave propagation along the estuary axis. Different versions of the analytical solutions are compared with numerical results for a wide range of parameters, which provide insight into the strengths and weaknesses of the modified quasi-nonlinear model.

Objective

- To develop a modified quasi-nonlinear model.
- To investigate the difference between linear and nonlinear models

Method

1. We introduce a modified coefficient that accounts for the non-zero tidal amplitude to depth ratio and consider the difference between the damping of tidal amplitude and velocity, leading to a new set of equations.
2. Energy consideration in tidal damping
3. The modified solutions are compared with solutions obtained by other linearised approaches (as reported by Toffolon and Savenije, 2010) and to numerical solutions of the complete St. Venant equations.

Results

① Revised set of equations

Two assumptions are relaxed:

- ❑ Tidal amplitude to depth ratio is very small.
- ❑ The damping of tidal amplitude and velocity is equal.

Equations	Sav. 2008	Sav. modified
Phase lag equation	$\tan(\varepsilon)=\frac{\lambda}{\gamma-\delta}$	$\tan\varepsilon=\frac{\lambda}{\gamma-\delta_U}$
Scaling equation	$\mu=\frac{\sin(\varepsilon)}{\lambda}=\frac{\cos(\varepsilon)}{\gamma-\delta}$	$\mu=\frac{\sin\varepsilon}{\lambda}\kappa=\frac{\cos\varepsilon}{\gamma-\delta_U}\kappa$
Damping equation	$\delta=\frac{\mu^2}{\mu^2+1}\left(\gamma-\chi\mu^2\lambda^2\right)$	$\delta_\eta\frac{\kappa}{\mu^2}+\delta_U=\gamma-\frac{\chi\mu^2\lambda^2}{\kappa}$
Celerity equation	$\lambda^2=1-\delta\frac{\cos(\varepsilon)}{\mu}=1-\delta(\gamma-\delta)$	$\lambda^2=1-\frac{\delta_\eta}{\mu}\cos(\varepsilon)=1-\frac{\delta_\eta}{\kappa}(\gamma-\delta_U)$

Table 1. Comparison between original and modified set of equations

Notation:

ε = phase lag
γ = estuary shape number
χ = friction number
μ = velocity number
λ = celerity number
δ = damping number
δ_U = velocity damping number
δ_η = amplitude damping number
κ = a modified coefficient which $\kappa=\ln[(1+\zeta)/(1-\zeta)]/(2\zeta)$
ζ = tidal amplitude to depth ratio

② Influence of the non-zero tidal amplitude to depth ratio

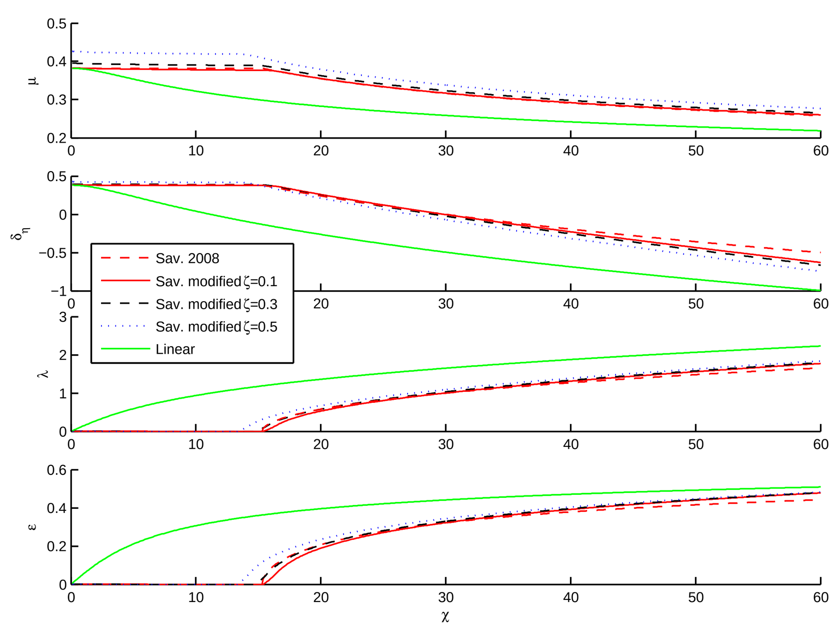

Fig. 1. Main dimensionless parameters obtained with various analytical solutions as functions of friction number χ for constant γ=3.

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The modified quasi-nonlinear model can account for the influence of tidal amplitude to depth ratio on tidal dynamics while the original quasi-nonlinear model and the linear model obtain the same results with different tidal amplitude to depth ratio.

Results

③ Relation between tidal amplitude and velocity damping

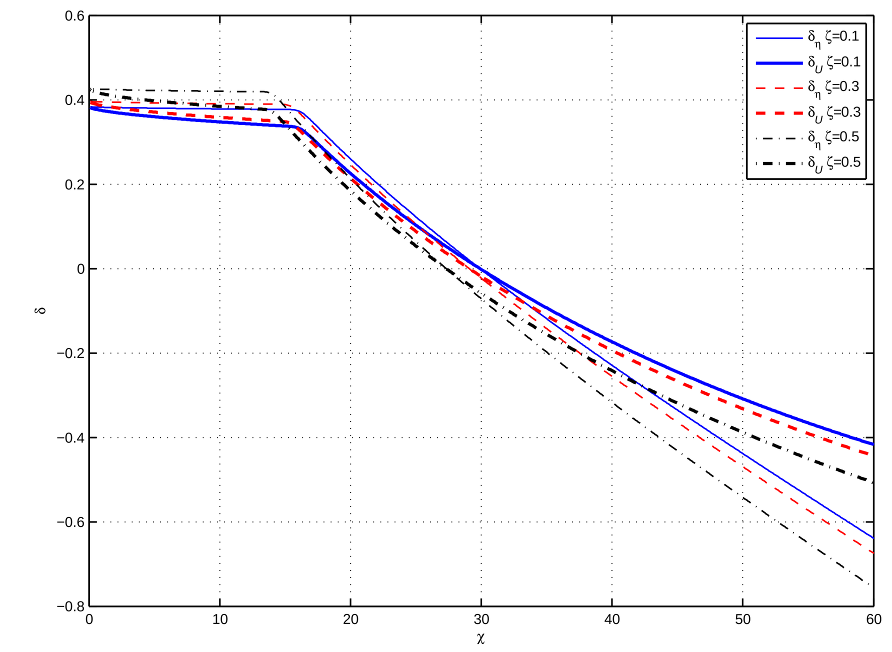

Fig. 2. Relation between the damping number of velocity δ_U and tidal amplitude δ_η with different tidal amplitude to depth for constant γ=3

When the second assumption is relaxed, i.e. considering the difference between the damping of tidal amplitude and velocity, we obtain a more accurate solution than the solution obtained before.

④ Comparison between linear and quasi-nonlinear solutions

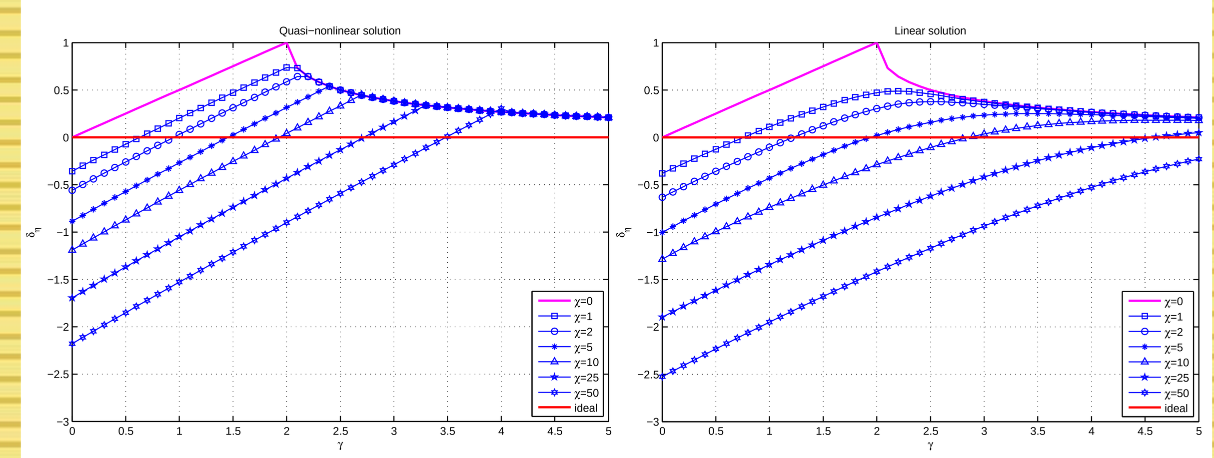

Fig. 3. Relation between the damping number of tidal amplitude and the estuary shape number obtained with quasi-nonlinear and linear models

From an energy perspective, we prove that the linear model exploits a linear damping equation while the quasi-nonlinear model makes use of a quadratic damping equation.

- Linear damping equation: $\delta=\frac{\gamma}{2}-\frac{4}{3\pi}\frac{\chi\mu}{\lambda}$
- Quadratic damping equation: $\delta=\frac{\gamma}{2}-\frac{1}{2}\chi\mu^2$

Results

⑤ Comparison of analytical solutions against numerical solutions

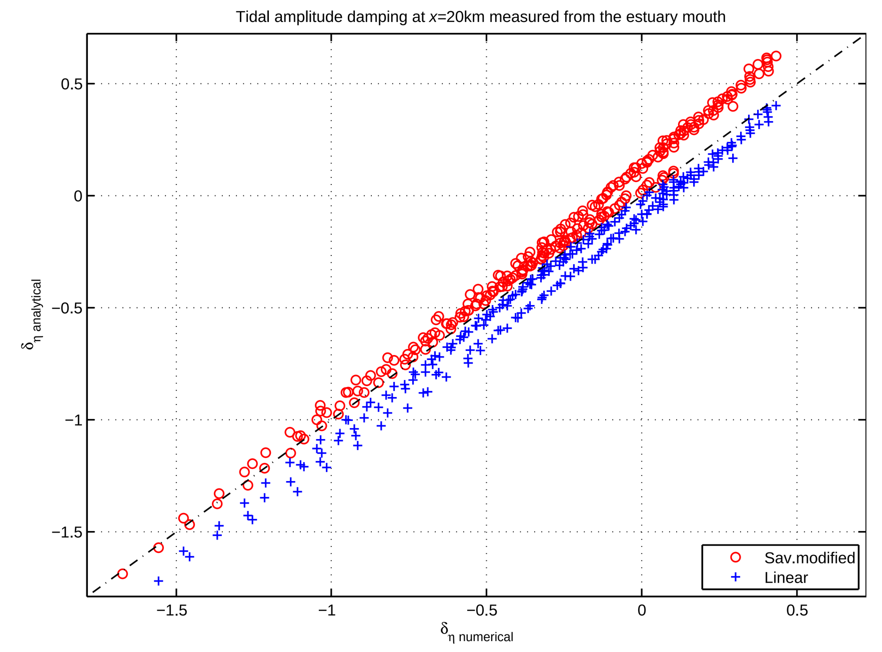

Fig. 4. Comparison of tidal damping computed with various analytical models and numerical solutions

Numerical simulations show that none of the analytical models is fully correct for a finite amplitude wave, but each of these approaches provide a difference perspective on the real solution. The average of the two comes close to the numerical result.

Conclusions

1. A modified quasi-nonlinear model can be obtained by relaxing some of the effects of simplification made earlier.
2. The major difference between quasi-nonlinear and linear model is the damping equation.
3. Numerical simulations indicate that the two models approach the numerical results from a different side.

References

Savenije, H. H. G., M. Toffolon, J. Haas, and E. J. M. Veling (2008), Analytical description of tidal dynamics in convergent estuaries, *J Geophys Res-Oceans, 113,* C10025.

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