3D FINITE-ELEMENT SIMULATION OF CONTROLLED-SOURCE ELECTROMAGNETIC PROBLEMS USING EDGE AND NODAL INTERPOLATION FUNCTIONS

Seyedmasoud Ansari and Colin Farquharson

Department of Earth Scinces, Memorial University of Newfoundland, St. John's, Canada

SUMMARY

We deal with the three dimensional finite-element solution of the CSEM problems in a conductive medium. 3D unstructured tetrahedral meshes are used to discretize the computational domain. Inorder to separate the galvanic and inductive natures of the EM field, an $\mathbf{A}$ - ϕ decomposition of the electric field is done. After obtaining two partial differential equations, the finite-element approximation of the system of equations is done using edge and nodal element basis functions.

METHODOLOGY

Assuming a time dependence of $e^{i\omega t}$, Faraday's law and Ampere's laws of induction are written as:

$$\nabla \times \mathbf{E} + i\omega\mu\mathbf{H} = 0 \qquad (1) \qquad (1)$$

$$\nabla \times \mathbf{H} - \tilde{\sigma}\mathbf{E} = \mathbf{J}^s \qquad (2)$$

μ = Magnetic Permeability

$\tilde{\sigma}$ = Electrical Conductivity

$\mathbf{J}^s$ = Source Current Density

Upon combining equations (1) and (2), a system of equations is obtained

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu(\tilde{\sigma}\mathbf{E} + \mathbf{J}^s) = 0 \qquad (3)$$

$$\nabla \cdot \mathbf{J} = -\nabla \cdot \mathbf{J}^s \qquad (4)$$

Decomposition of the electric field:

$$\mathbf{E} = -i\omega\mathbf{A} - \nabla\phi$$

$\mathbf{A}$ = Magnetic Vector Potential

ϕ = Electric Scalar Potential

Decomposed Equations in the form of Partial Differential Equations:

$$\boxed{\nabla \times \nabla \times \mathbf{A} + i\omega\mu\tilde{\sigma}\mathbf{A} + \mu\tilde{\sigma}\nabla\phi = \mu\mathbf{J}^s} \qquad (5)$$

$$\boxed{-i\omega\nabla \cdot (\tilde{\sigma}\mathbf{A}) - \nabla \cdot (\tilde{\sigma}\nabla\phi) = -\nabla \cdot \mathbf{J}^s} \qquad (6)$$

DISCRETIZATION USING UNSTRUCTURED TETRAHEDRAL ELEMENTS

" Blocks2mesh ", a code written by Peter Lelièvre (plelievre@mun.ca), is utilized to create a three dimensional grid of unstructured tetrahedrons. This code makes a block poly file that then get fed into *Tetgen*. Global numbers are assigned to each element and their corresponding nodes within the entire domain.

Because we mostly need the edge numbering scheme in our approach, a Fortran code is developed to convert the node numbering scheme into the edge one. Also, some auxiliary subroutines are written to figure out if any edges and nodes are shared by other neighbour cells. Moreover, edge-to-node, edge-to-element, element-to-edge, and other similar connectivity arrays have been coded.

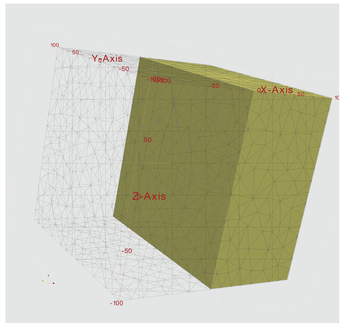

Figure 1. The finite-element mesh mesh for a homogenous Earth model depicted using Kitware Paraview 3.6.2. This mesh consists of 1233 nodes, 4432 cells and 10753 edges.

THE FINITE-ELEMENT APPROACH

Doing the finite-element solution of the system of equations requires the approximation of the vector and scalar potentials as follows:

$$\tilde{\mathbf{A}} = \sum_{j=1}^{N_{edges}} \tilde{A}_j \mathbf{N_j} \qquad (7) \qquad \mathbf{N_j} = \text{Edge-element vector interpolation function}$$

$$\tilde{\phi} = \sum_{k=1}^{N_{nodes}} \tilde{\phi}_{k_1} N_{k_1} \qquad (8) \qquad N_{k_1} = \text{Nodal-element scalar interpolation function}$$

The Galerkin Method : Seeks the solution by wighting the vector and scalar residuals of the partial differential equations

1- Vector Residual

$$R = \int_\Omega \mathbf{N_i} \cdot \mathbf{r}\, d\Omega = 0 \qquad (9) \qquad \mathbf{r} = \nabla \times \nabla \times \mathbf{A} + i\omega\mu\tilde{\sigma}\mathbf{A} + \mu\tilde{\sigma}\nabla\phi - \mu\mathbf{J}^s$$

$$\sum_{j=1}^{N_{edges}} \tilde{A}_j \int_\Omega (\nabla \times \mathbf{N_i}) \cdot (\nabla \times \mathbf{N_j})\, d\Omega + i\omega\mu \sum_{j=1}^{N_{edges}} \tilde{A}_j \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \mathbf{N_j}\, d\Omega$$

$$+\mu \sum_{k=1}^{N_{nodes}} \phi_k \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \nabla N_{k_1}\, d\Omega = \mu \sum_{j=1}^{N_{edges}} \int_\Omega \mathbf{N_i} \cdot \mathbf{J_s}\, d\Omega \qquad (10)$$

2- Scalar Residual

$$R_{scalar} = \int_\Omega N_{k_2} r\, d\Omega = 0 \qquad (11) \qquad r = i\omega\nabla \cdot (\tilde{\sigma}\mathbf{A}) + \nabla \cdot (\tilde{\sigma}\nabla\phi) - \nabla \cdot \mathbf{J}^s$$

$$i\omega\tilde{A}_j \sum_{j=1}^{N_{edges}} \int_S N_{k_2}\hat{n} \cdot (\tilde{\sigma}\mathbf{N_j})\, dS - i\omega\tilde{A}_j \sum_{j=1}^{N_{edges}} \int_\Omega \nabla N_{k_2} \cdot (\tilde{\sigma}\mathbf{N_j})\, d\Omega$$

$$+ \tilde{\phi}_{k_1} \sum_{k_1=1}^{N_{nodes}} \int_S N_{k_2}\hat{n} \cdot (\tilde{\sigma}\nabla N_{k_1})\, dS - \tilde{\phi}_{k_1} \sum_{k_1=1}^{N_{nodes}} \int_\Omega \nabla N_{k_2} \cdot (\tilde{\sigma}\nabla N_{k_1})\, d\Omega \qquad (12)$$

$$= \int_\Omega N_{k_2} \nabla \cdot \mathbf{J_s}\, d\Omega$$

NODAL ELEMENTS

Tetrahedrons are more suitable to model the curves of the complex-shaped volumes and irregular geometries, for instance, jagged topography features, faults and dipping layers. The finite-element method uses nodal and edge basis functions to determine the approximated solution for the electric field within each element.

Within each linear tetrahedral element, the unknown scalar function can be written as

$$\phi^e(x, y, z) = a^e + b^e x + c^e y + d^e z \qquad (13)$$

Coefficients a^e, b^e, c^e, and d^e can be determined by enforcing the equation (13) at four corners of the tetrahedron. By substituting these coefficients into equation (13) we obtain:

$$\phi^e(x, y, z) = \sum_{k=1}^{4} N_k^e(x, y, z)\phi_j^e \qquad (14)$$

N_k^e = Scalar basis function

$$N_k^e(x, y, z) = \frac{1}{6V^e}(a_k^e + b_k^e x + c_k^e y + d_k^e z) \qquad (15)$$

V^e = The volume of the tetrahedron

As the main property, the value of the interpolation function $N_k(x_i, y_i, z_i)$. is specified to be 1 if i = k and zero if i ≠ k .

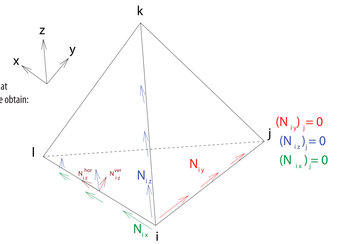

Figure 2. An illustration of the scalar basis function related to the node i . Scalar N decreases linearly in different directions away from its initial location and vanishes at the location of nodes j, l, and k. In contrast to the horizontal component of the N_{lz} , which is depicted by N_{lz}^{hor}, its vertical component, N_{lz}^{ver}, is discontinous across the edge "ll" .

EDGE-ELEMENTS

Nodal basis functions guarantee only the continuity of the tangential component of the electric field across the interface while the vertical component of the electric field is not necessarily continuous. This causes the normal current density to be no longer continous across the interface. Also, Because the electric field is allowed to vary in all three orthogonal directions, the term $\nabla \cdot (\sigma\mathbf{E})$ is not consistently zero within cells of uniform conductivity. To counteract these problems edge-element basis functions are used. Constrained to be along the edge, an edge-element combines three components of the approximated field into one collection. This guarantees the continuity of the tangential component, while allowing the normal component of the electric field to freely jump across the interelement boundaries.

The Whitney 1-form is used as the vector edge-element function. By using this form the basis function, for example, along edge 1 (see Fig. 3) is given by:

$$\mathbf{P_{12}} = L_1^e \nabla L_2^e - L_2^e \nabla L_1^e \qquad (16)$$

Where $L_k^e = N_k$ (k = 1, 2, 3, and 4)

Advantage 1:

$\mathbf{P_{12}}$ has no tangentila component along other edges exept edge 1, thus, forcing the tangential component of the field to be continuous between adjacent cells sharing this edge.

In general, the edge function is given by:

$$\mathbf{N_i^e} = l_i^e (L_{i1}^e \nabla L_{i2}^e - L_{i2}^e \nabla L_{i1}^e) \qquad (17)$$

Advantage 2:

$\nabla \cdot \mathbf{N_i} = 0$ indicates that the field is divergence-free in a source-free region

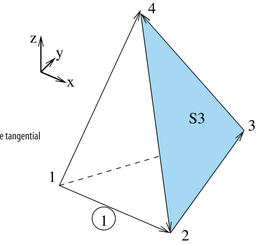

Figure 3. Six edges are depicted for a tetrahedral element. Edge 1 is picked as an example which starts from node 1 and ends at node 2. It has no tangential component on the blue surface S_3 .

CALCULATION OF RESIDULAS USING ELEMENTAL FUNCTIONS

1 - Edge - Edge Interactions :

$$\mathbf{T}_{ij} = \sum_{j=1}^{N_{edges}} \int_\Omega (\nabla \times \mathbf{N_i}) \cdot (\nabla \times \mathbf{N_j})\, d\Omega \qquad \text{i = 1, ... Nedges} \qquad (18)$$

$$\mathbf{T}_{ij} = \sum_{j=1}^{N_{edges}} \frac{4\, l_i\, l_j}{(6)^4 (V^e)^3} [\, (b_{i1}c_{i2} - c_{i1}b_{i2})(b_{j1}c_{j2} - c_{j1}b_{j2}) \qquad (19)$$

$$+(d_{i1}b_{i2} - b_{i1}d_{i2})(d_{j1}b_{j2} - b_{j1}d_{j2}) + (c_{i1}d_{i2} - d_{i1}c_{i2})(c_{j1}d_{j2} - d_{j1}c_{j2})\,]$$

To easily calculate the second edge - edge interaction integral we changed the coordinate system from cartesian to **simplex or normalized** system

$$\mathbf{U}_{ij} = \sum_{j=1}^{N_{edges}} \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \mathbf{N_j}\, d\Omega \qquad (20)$$

$$\int_\Omega \tilde{\sigma}\mathbf{N_i}\mathbf{N_j}\, d\Omega = \frac{l_i l_j}{36(V^e)^2}\tilde{\sigma}[\, F_{i2j2}G_{i1j1} - F_{i2j1}G_{i1j2} - F_{i1j2}G_{i2j1} + F_{i1j1}G_{i2j2}\,] \qquad (21)$$

where $F_{ij} = a_i a_j + b_i b_j + c_i c_j$

$$G_{ij} = \int_\Omega L_i(x, y, z) L_j(x, y, z)\, d\Omega$$

$$G_{ij} = \int_\Gamma L_i(u, v, w) L_j(u, v, w)\, |\mathbf{J}|\, du\, dv\, dw$$

$$G_{ij} = \frac{V^e}{20} \qquad i \neq j$$

$$G_{ij} = \frac{V^e}{10} \qquad i = j$$

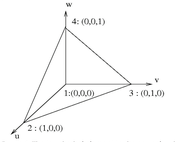

Figure 4. The tetrahedral element in the normalized coordinate system

2 - Edge - Node Interactions :

$$\mathbf{W_{ik}} = \sum_{i=1}^{N_{edges}} \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \nabla N_k\, d\Omega \qquad (22)$$

$$\int_\Omega L_i\, dxdydz = \int_\Gamma L_i(u, v, w)|\mathbf{J}|\, dudvdw = \frac{1}{24}|\mathbf{J}|$$

$$\mathbf{W_{ik}} = \sum_{i=1}^{N_{edges}} \frac{1}{144\, V^e} l_i\tilde{\sigma}\, [\, (b_{i2}b_k + c_{i2}c_k + d_{i2}d_k) - (b_{i1}b_k + c_{i1}c_k + d_{i1}d_k)\,] \qquad (23)$$

3 - Node - Node Interactions :

$$\mathbf{Z_{k2k1}} = \sum_{k_1=1}^{N_{nodes}} \int_\Omega \nabla N_{k_2} \cdot \tilde{\sigma}\nabla N_{k_1}\, d\Omega \qquad (24)$$

$$\mathbf{Z_{k2k1}} = \tilde{\sigma}(\frac{1}{6V^e})[b_{k2}b_{k1} + c_{k2}c_{k1} + d_{k2}d_{k1}] \qquad (25)$$

ELECTROMAGNETIC SOURCE

A line source of electric current is chosen to be as a delta function of finite length :

$$\mathbf{J} = \delta(y')\, \delta(z')\, Box(x')j_s\, \hat{\mathbf{x}}' \qquad (26)$$

where $\hat{j}_s$ is a scalar arbitrary function and $\mathrm{Box}(\mathrm{x}')$ is the boxcar function; it indicates that the source current is zero over the entire domain except for a specific interval, l, on which it is given a constant value.

Source - Edge interaction :

$$\mathbf{S} = \int_\Omega \mathbf{N_i} \cdot \mathbf{J}\, d\Omega \qquad (27)$$

The secondary system, $(\mathrm{x}', \mathrm{y}', \mathrm{z}')$ is chosen in a way that the source line has the simplest equation. Because the tetrahedral elements are set up in the cartesian coordinate system a transformation has to be done in order to easily calculate the source term integrals.

$$\mathbf{S} = \mu \sum_{i=1}^{N_{edges}} (\frac{1}{6V^e})^2 (a_{i1}b_{i2} - a_{i2}b_{i1}) j_s l \qquad (28)$$

Source - Node interaction :

$$\mathbf{SS} = \int_\Omega N_{k_2} \nabla \cdot \mathbf{J_s}\, d\Omega \qquad (29)$$

$$\mathbf{SS} = \int_\Omega N_{k_2} \nabla \cdot \mathbf{J_s}\, d\Omega = \sum_{k=1}^{N_{nodes}} b_k\, j_s (\frac{1}{6V^e})\, l \qquad (30)$$

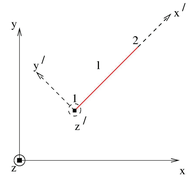

Figure 5. A line source of current shown in two cartesian and target coordinate system

NUMERICAL SOLUTION TO THE SYSTEM OF EQUATIONS

Plugging equations (19), (21), (23), (25), (28), and (30) into both equations (10) and (12) gives us a system of equations in the form of :

$$L.u = F \qquad (31)$$

Separating the potentials into their real and imaginary parts, we end up with the following equations.

$$\begin{pmatrix} \mathbf{T_{ij}} & -\omega\mu\mathbf{U_{ij}} & \mu\mathbf{W_{ik_1}} & 0 \\ \omega\mu\mathbf{U_{ij}} & \mathbf{T_{ij}} & 0 & \mu\mathbf{W_{ik_1}} \\ 0 & \omega\mathbf{W_{k_2j}} & -\mathbf{Z_{k_2k_1}} & 0 \\ -\omega\mathbf{W_{k_2j}} & 0 & 0 & -\mathbf{Z_{k_2k_1}} \end{pmatrix} \begin{pmatrix} \mathbf{A_j^r} \\ \mathbf{A_j^i} \\ \phi_k^r \\ \phi_k^i \end{pmatrix} = \begin{pmatrix} \mathbf{S} \\ 0 \\ \mathbf{SS} \\ 0 \end{pmatrix} \qquad (32)$$

Boundary Conditions:

Boundary conditions are applied on the truncation boundary of the domain. Subject to the Dirichlet boundary condition, if the source is considered far enough from the truncation boundary, the following equations will be satisfied.

$$\hat{\mathbf{n}} \times \mathbf{A}|_{\partial\Omega} = \mathbf{0} \qquad \phi|_{\partial\Omega} = 0$$

To code up the system of equations a **Fortran** code is written. The sparse structure of the coefficient matrix and the non zero elements are shown by the figure 6.

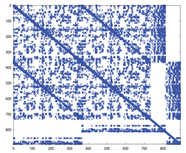

Figure 6. The sparse structure of the coefficient matrix. It is designed for a mesh of 216 cells, 80 nodes and 366 edges. This matrix consists of four main blocks . The dominant block, (1:732 , 1:732), is related to the edge–edge interactions (top left block). Also, the dimension of the matrix is 2(# of Edges + # of Nodes) = 892

The nonzero values of the sparse coefficient matrix on the left is stored in the CSR format for the sake of lower memory usage. The non-symmetric system of equations will be solved using **BICGSTAB** solver with an **LU** preconditioner.

CONCLUSIONS AND FUTURE PLANS

The code is still under development. We hope to test it on simple Earth models. The performance of the discussed approach will be compared with a modification in which the Lorentz gauge condition is used. Also, the relative contributions to the electric field from the inductive and galvanic terms will be investigated in different situations.

REFERENCES

Haber, E., Ascher, U. M., Aruliah, D. M., & Oldenberg, D., 2000. Fast Simulation of 3D Electromagnetic Problems Using Potentials. Journal of Computational Physics 163, 150-171.

Jin, J.-M., 2002. The finite element method in electromagnetics, 2nd ed., John Wiley & Sons, New York.

Pelosi, G., Coccioli, R., & Selleri, S., 2009. Quick finite-elements for electromagnetic waves, 2nd ed., Artech House, MA.

Si, H., 2007. TetGen: A quality tetrahedral mesh generator and 3D Delauny triangulator, Weierstrass Institute for Applied Analysis and Stochastics, available from http://tetgen.berlios.de