Geomagnetic data assimilation with an implicit filter into a one-dimensional sparsely observed MHD system

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Geomagnetic Data Assimilation

- Satellite data for magnetic field can be used to improve models
- Secular variation of magnetic field is coupled to core velocity
- Need to answer zero-oder questions:
  To what extend *can observations of magnetic field improve model output for velocity*? *Which DA technique is most suitable (feasible)*?

Simple Model

- Velocity: $\partial_t u + u\partial_x u = b\partial_x b + \nu\partial_x^2 u + W_u(x,t)$
- Magnetic Field: $\partial_t b + u\partial_x b = b\partial_x u + \partial_x^2 b + W_b(x,t)$
- Boundary conditions:

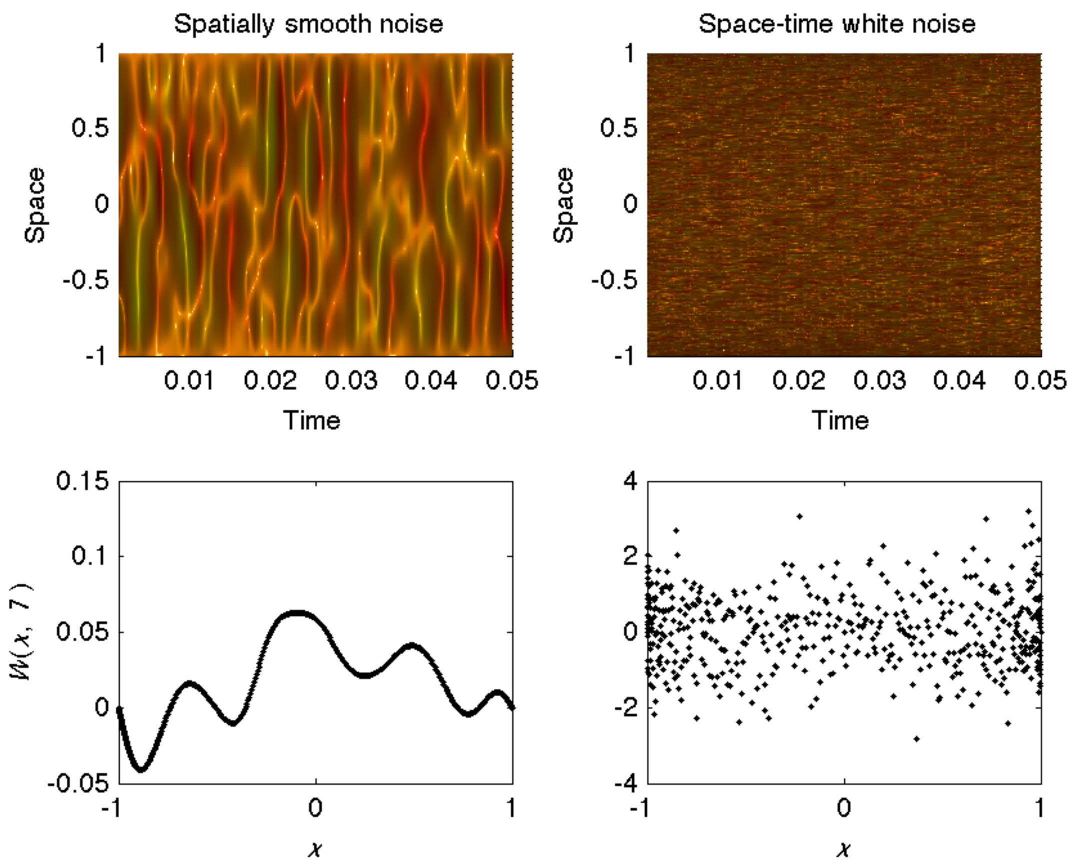
$$u(x,t) = 0, \text{ if } x = \pm 1$$

$$b(x,t) = \pm 1, \text{ if } x = \pm 1$$

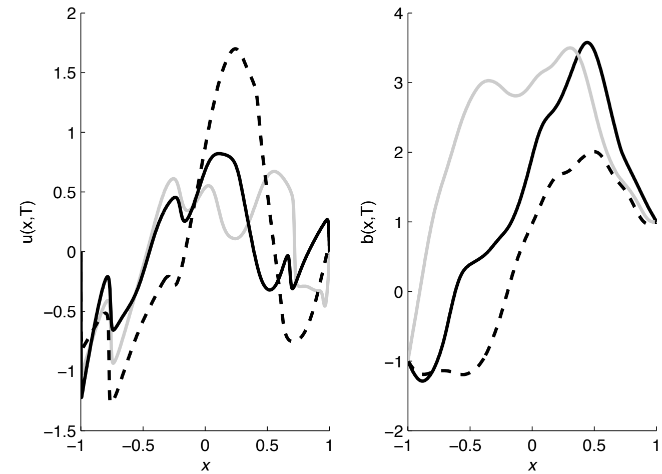
$$u(x,0), b(x,0) \text{ given}$$

Spatially Smooth- vs. Space-Time White Noise

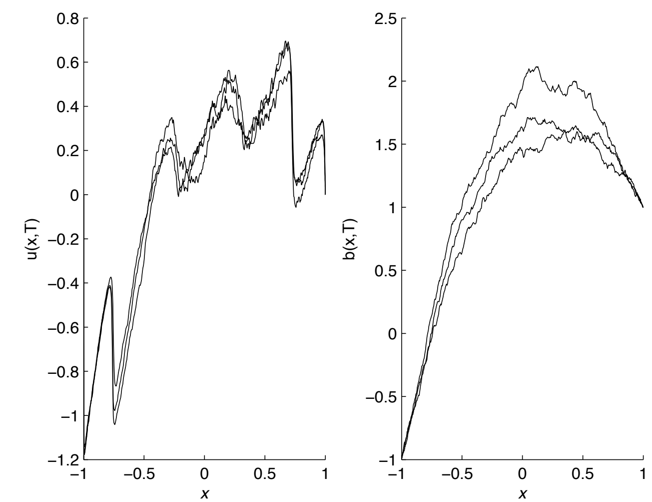
Noise Model



Solutions under spatially smooth noise



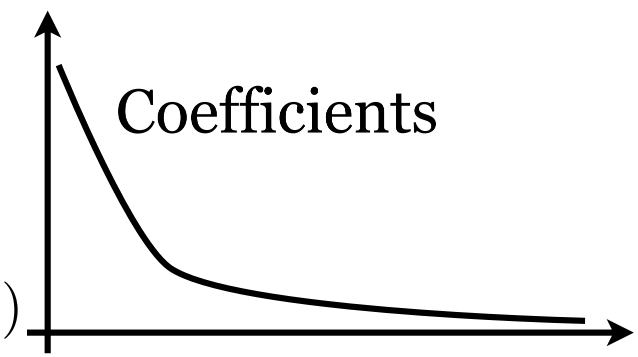
Solutions under space-time white noise



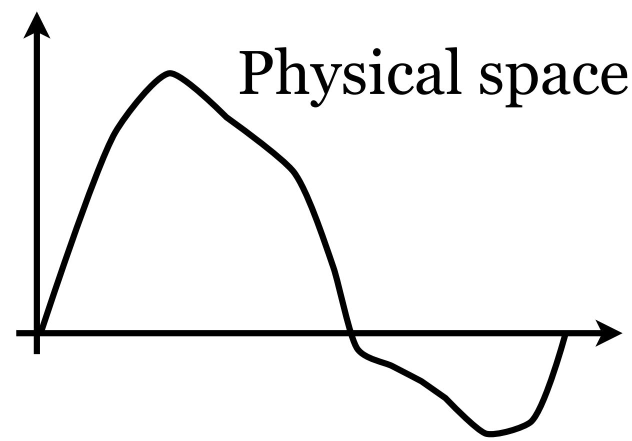
Noise Model

- Random smooth functions are easy to generate using the trigonometric series

$$W_T(x,t) = \sum_k a_j(t)\sin(k\pi x) + \sum_{k\ odd} b_k(t)\cos(k\pi/2x)$$



- Very smooth noise: pick exponentially decaying coefficients.



- Prior information on spatial distribution of the uncertainty is easy to incorporate.

Discretized MHD equations

Spatially smooth noise results in equations:

$$v_u^{n+1} = e^{\Lambda_u \delta} v_u^n + \Lambda_u^{-1}(I - e^{\Lambda_u \delta}) N(v_u^n, v_b^n) + \mathcal{N}(0, \Sigma_u)$$
$$v_b^{n+1} = e^{\Lambda_b \delta} v_b^n + \Lambda_b^{-1}(I - e^{\Lambda_b \delta}) N(v_u^n, v_b^n) + \mathcal{N}(0, \Sigma_b)$$

Another coordinate transformation highlights noisy equations:

$$\begin{aligned}
x_u^{n+1} &= G_{x_u}(x_u^n, y_u^n, x_b^n, y_b^n) + \mathcal{N}(0, \Sigma_{x_u}) \\
y_u^{n+1} &= G_{y_u}(x_u^n, y_u^n, x_b^n, y_b^n) \\
x_b^{n+1} &= G_{x_b}(x_u^n, y_u^n, x_b^n, y_b^n) + \mathcal{N}(0, \Sigma_{x_b}) \\
y_b^{n+1} &= G_{y_b}(x_u^n, y_u^n, x_b^n, y_b^n)
\end{aligned}$$

Observations are collected sparsely in physical space

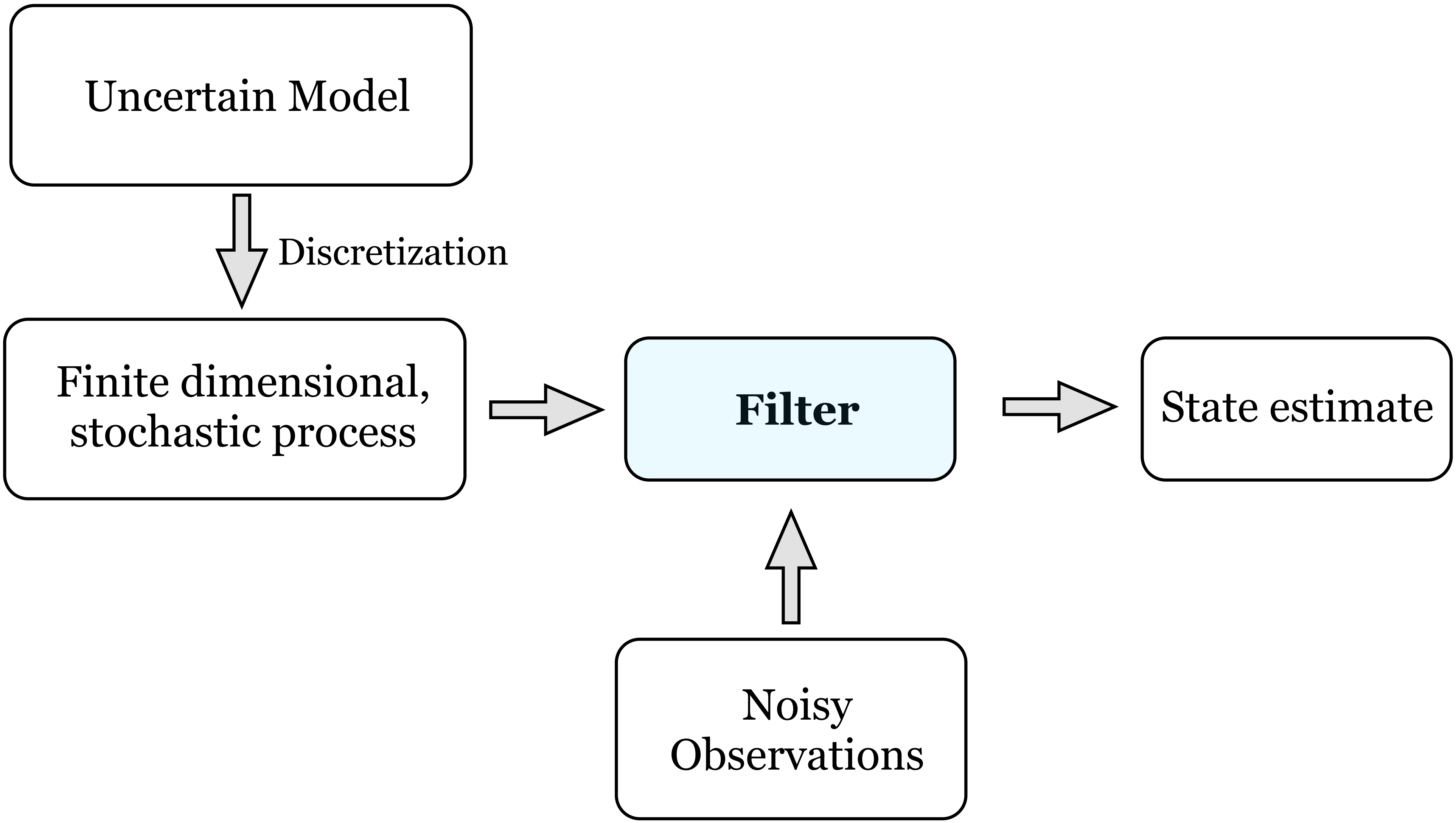
$$z^{n+1} = H b^{n+1} + \mathcal{N}(0, sI)$$

Design the filter to operate in the low dimensional, additive-noise subspace!

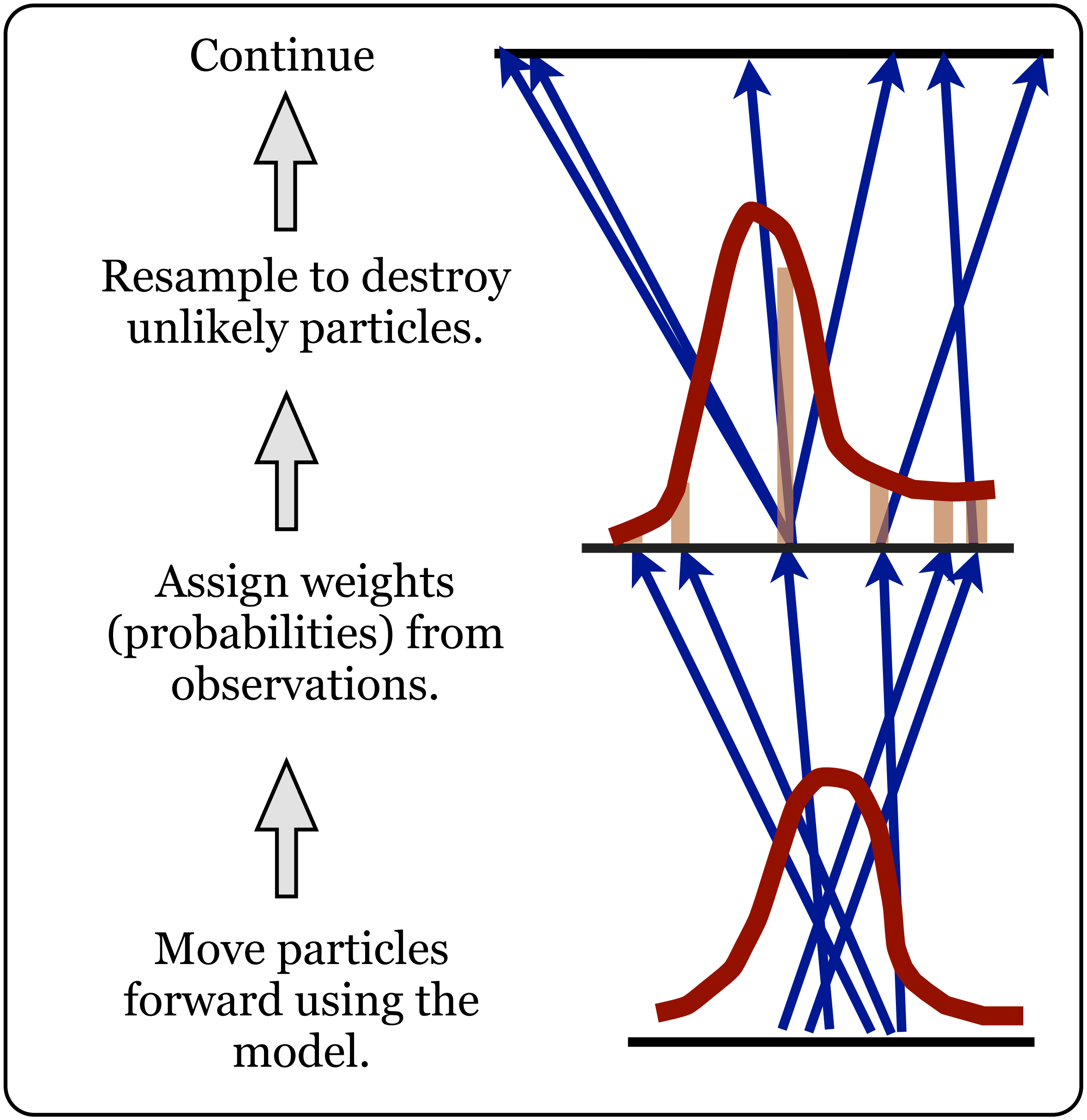
Agenda

1. Introduction
2. **Implicit Particle Filter**
3. Results

Data Assimilation: Problem Description

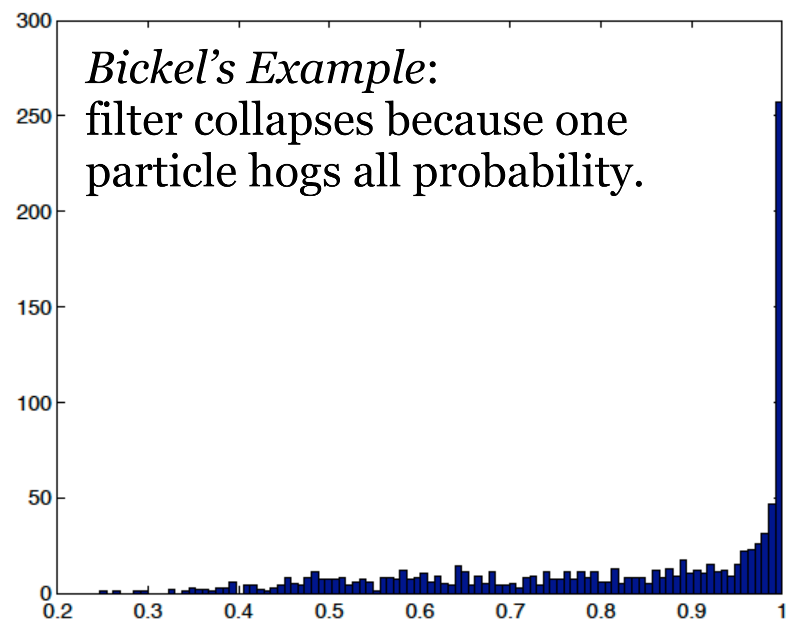


Standard Particle Filter

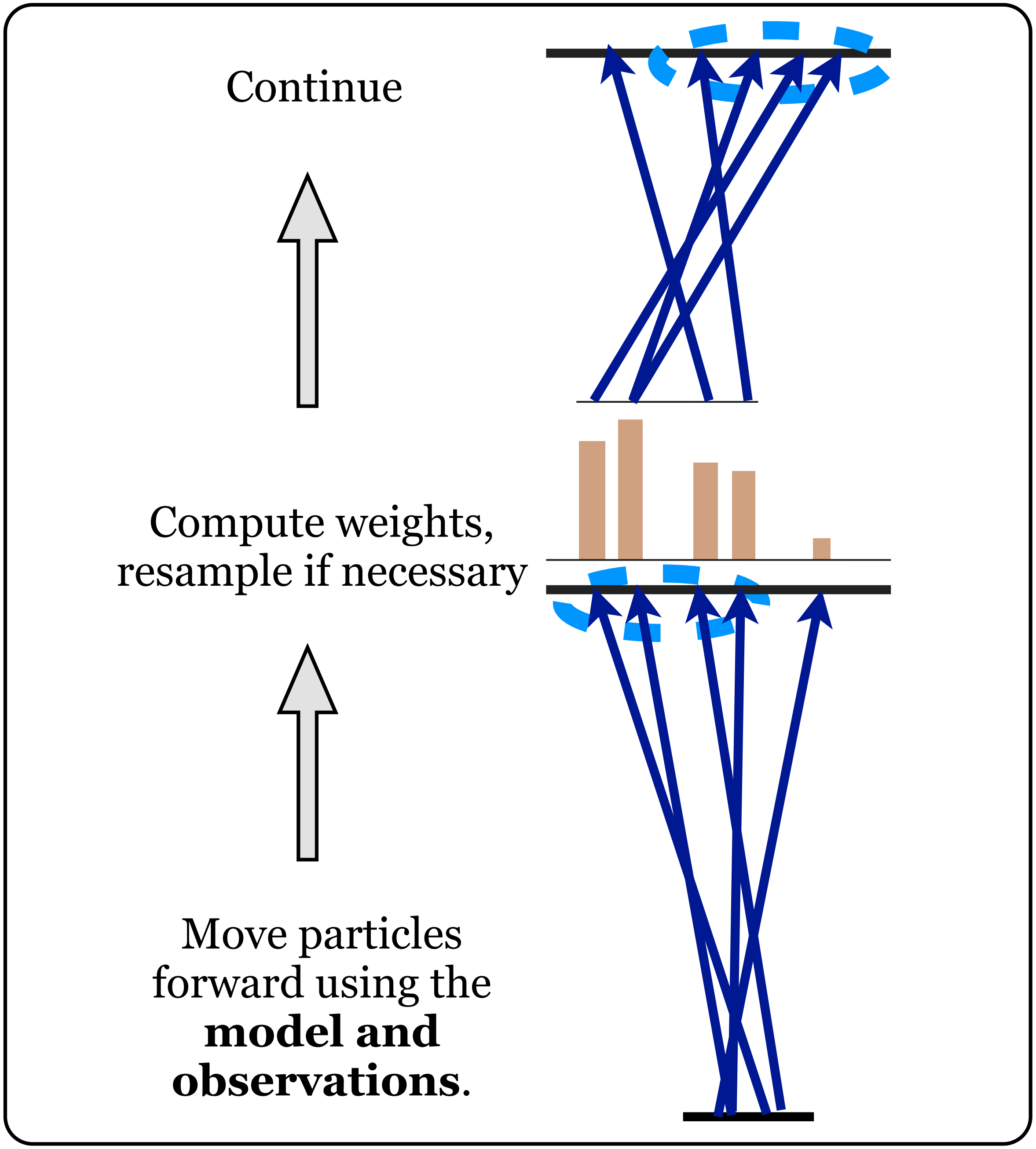


The Catch

The number of particles required can grow catastrophically with the dimensions of the problem.



Implicit Particle Filter



Idea

- Focus particles by using observations when generating the path
- Focussing effect makes number of particles manageable

Bayes' Theorem

- Conditional pdf: $p(X^{0,\ldots,n+1}|b^{1,\ldots,n+1}) \propto p(X^{0,\ldots,n}|b^{1,\ldots,n})p(b^{n+1}|X^{n+1})p(X^{n+1}|X^{n})$

Implicit Sampling

- We work particle by particle. For each one, pick a probability:

$$\xi_j \sim \mathcal{N}(0, I)$$

$$p_\xi(\xi_j) \propto \exp(-0.5\xi_j^T \xi_j)$$

- Find a sample that carries it

$$p(X_j^{n+1}|X_j^n)p(b^{n+1}|X_j^{n+1}) = \exp(-F_j(X_j^{n+1}))$$

$$F_j(X_j^{n+1}) - \phi_j = 0.5\xi^T\xi$$

- Compute importance weight

$$w_j^{n+1} = w_j^n \exp(-0.5\phi_j) \left|\frac{\partial X_j}{\partial \xi}\right|$$

Advantages

- Theoretical framework offers great freedom to construct tailor-made filters for the problem at hand.
- Observations are used to generate new positions (sharply focussed particle beam).

Implicit Particle Filter with a Random Map

Algorithm

- Sample reference density

$$\rho = \xi^T\xi,\ \xi \sim N(0, I)$$

- Minimize F

$$\phi = \min F$$

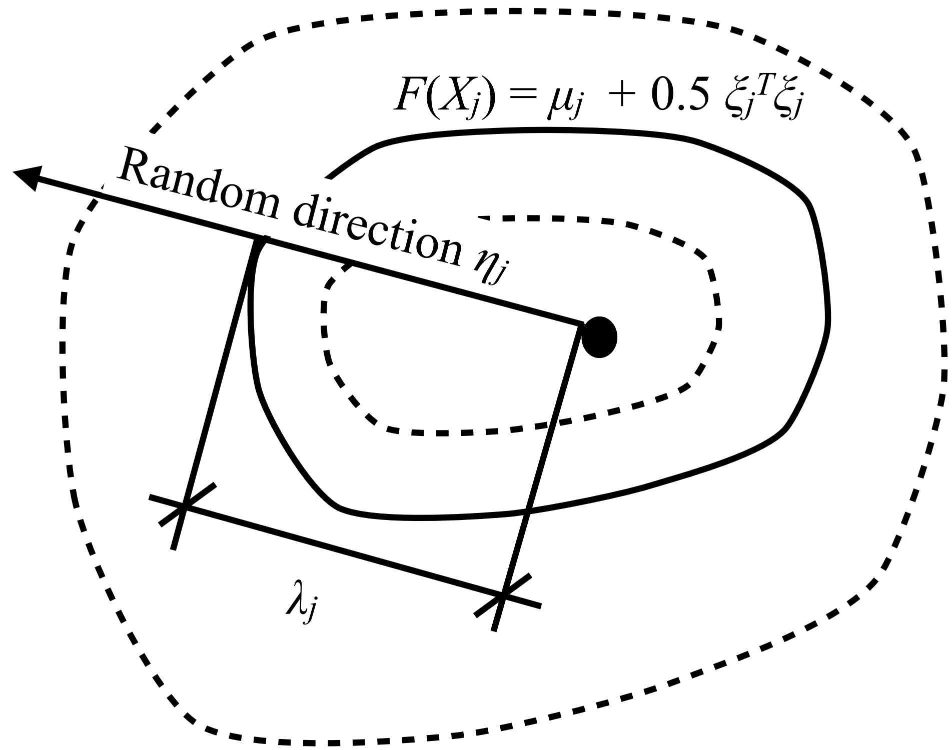
- Solve algebraic equation by random map

$$F(X) - \phi = 0.5\rho$$
$$X - \mu = \lambda L\eta$$

- Compute importance weight

$$w_j^n = w_j^{n-1} e^{-\frac{1}{2}\phi} \rho^{1-\frac{m}{2}} \left| \lambda^{m-1} \frac{\partial\lambda}{\partial\rho} \right|$$

Geometry of the random map

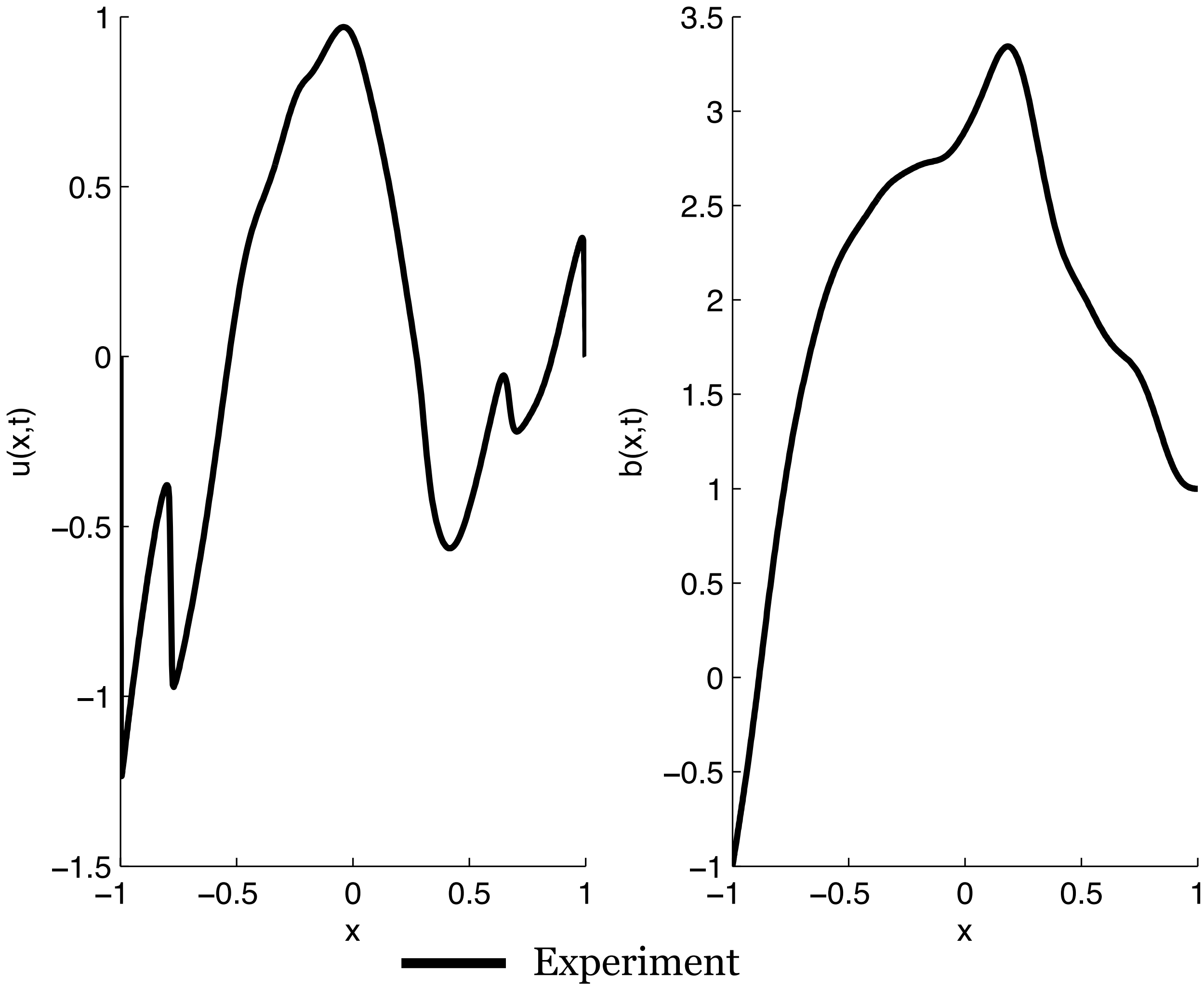


Agenda

Twin Experiments

Twin Experiment

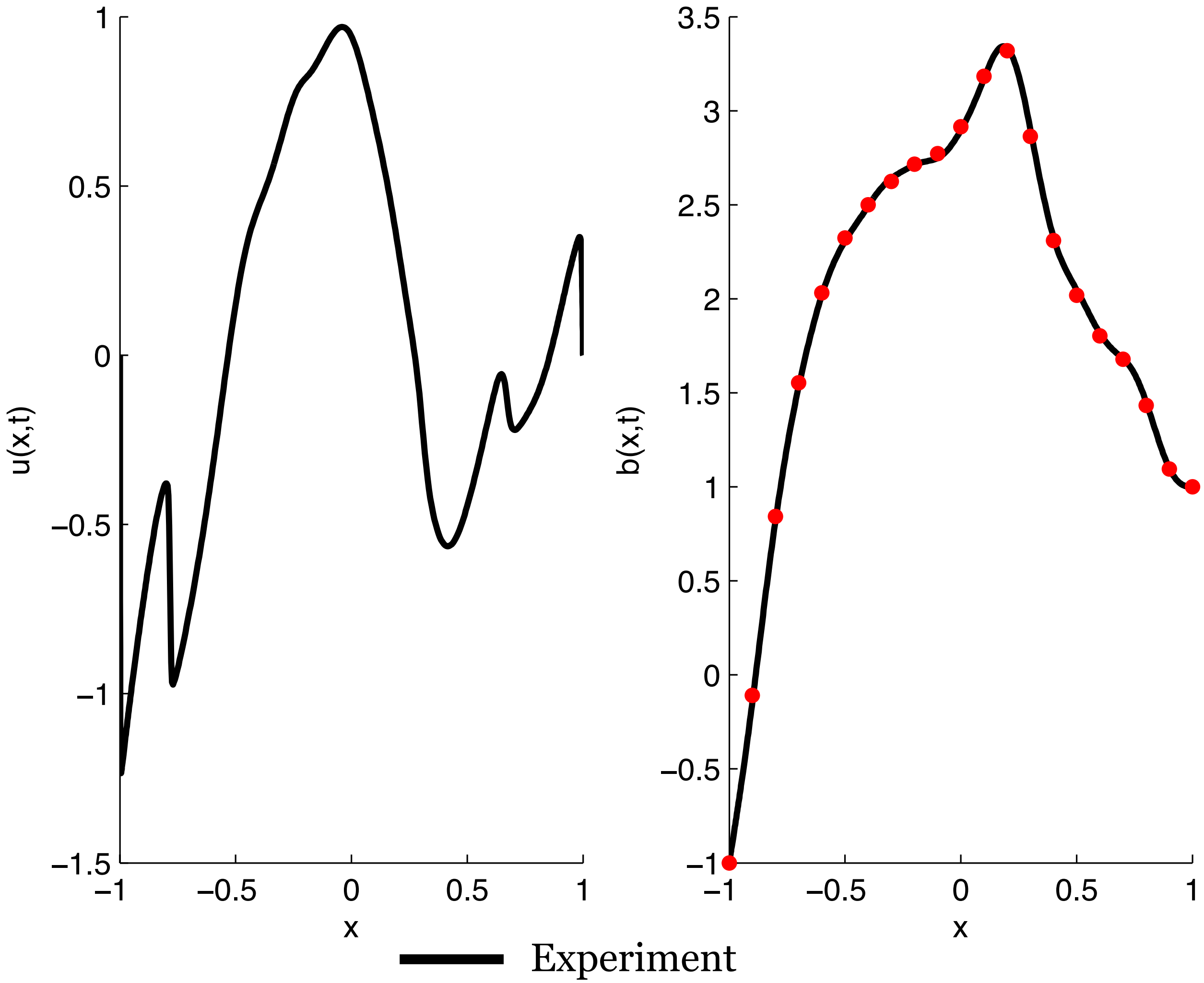
- Run the model



Twin Experiments

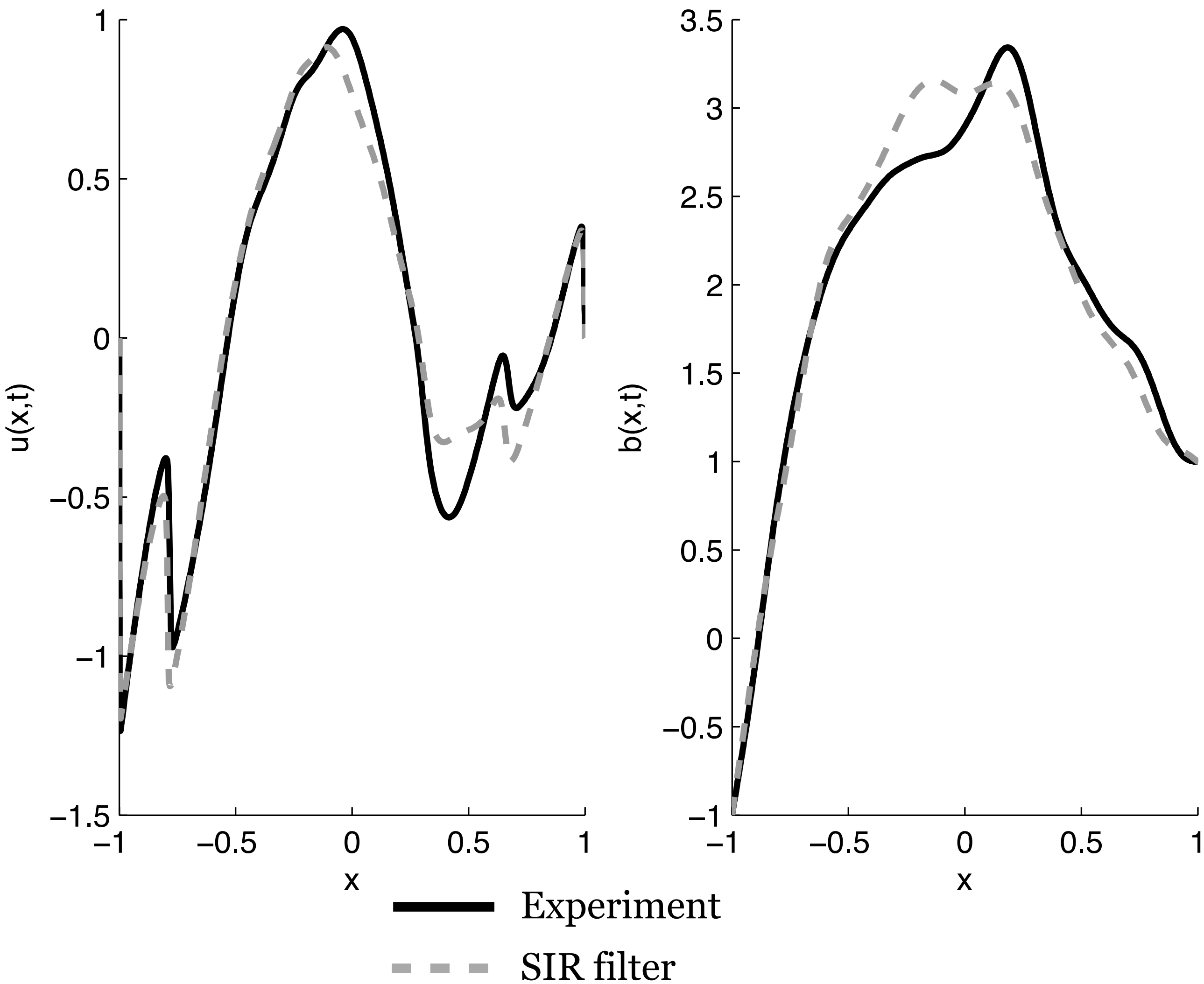
Twin Experiment

- Run the model
- Collect observations



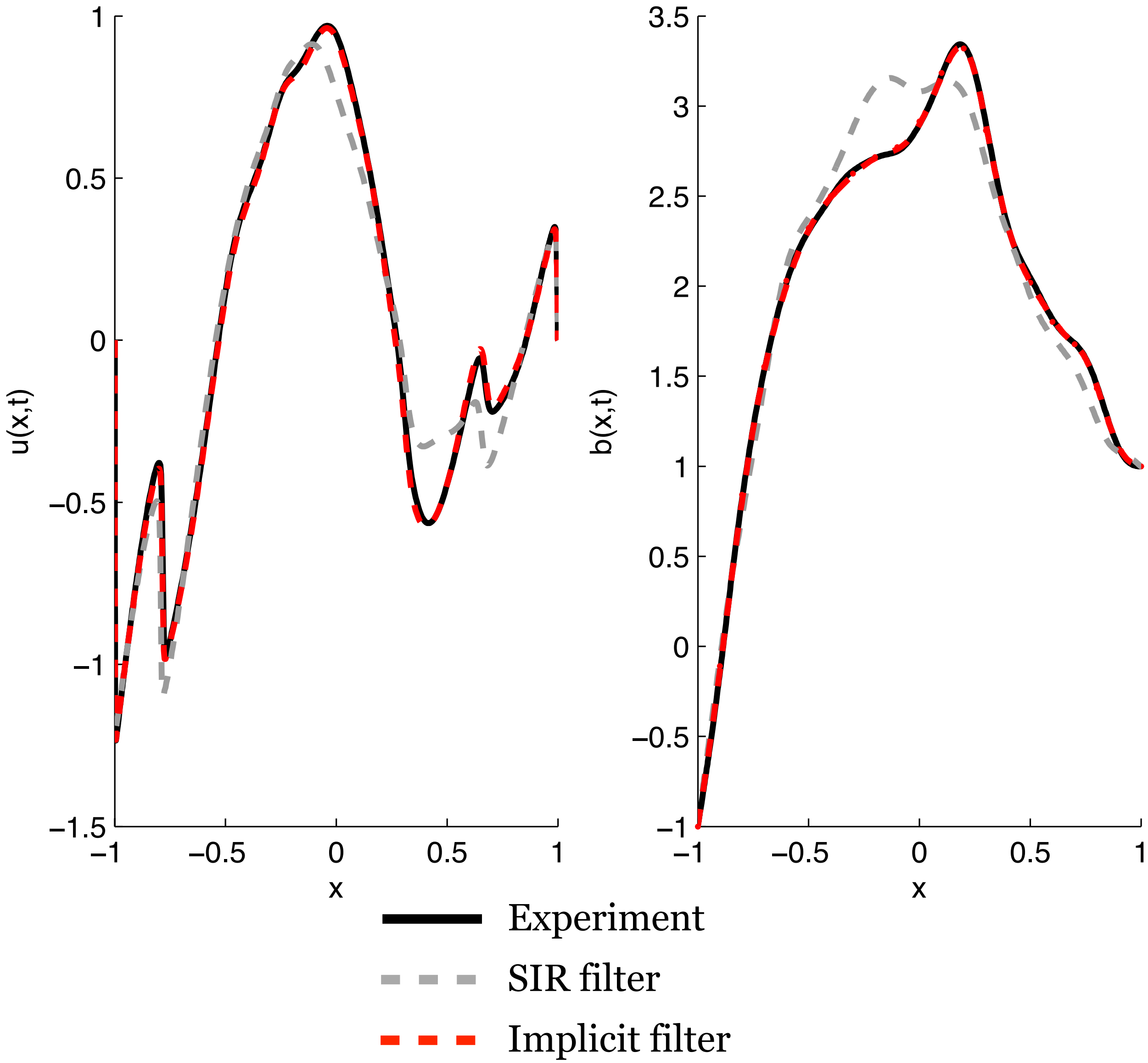
Twin Experiment

- Run the model
- Collect observations
- Run SIR filter

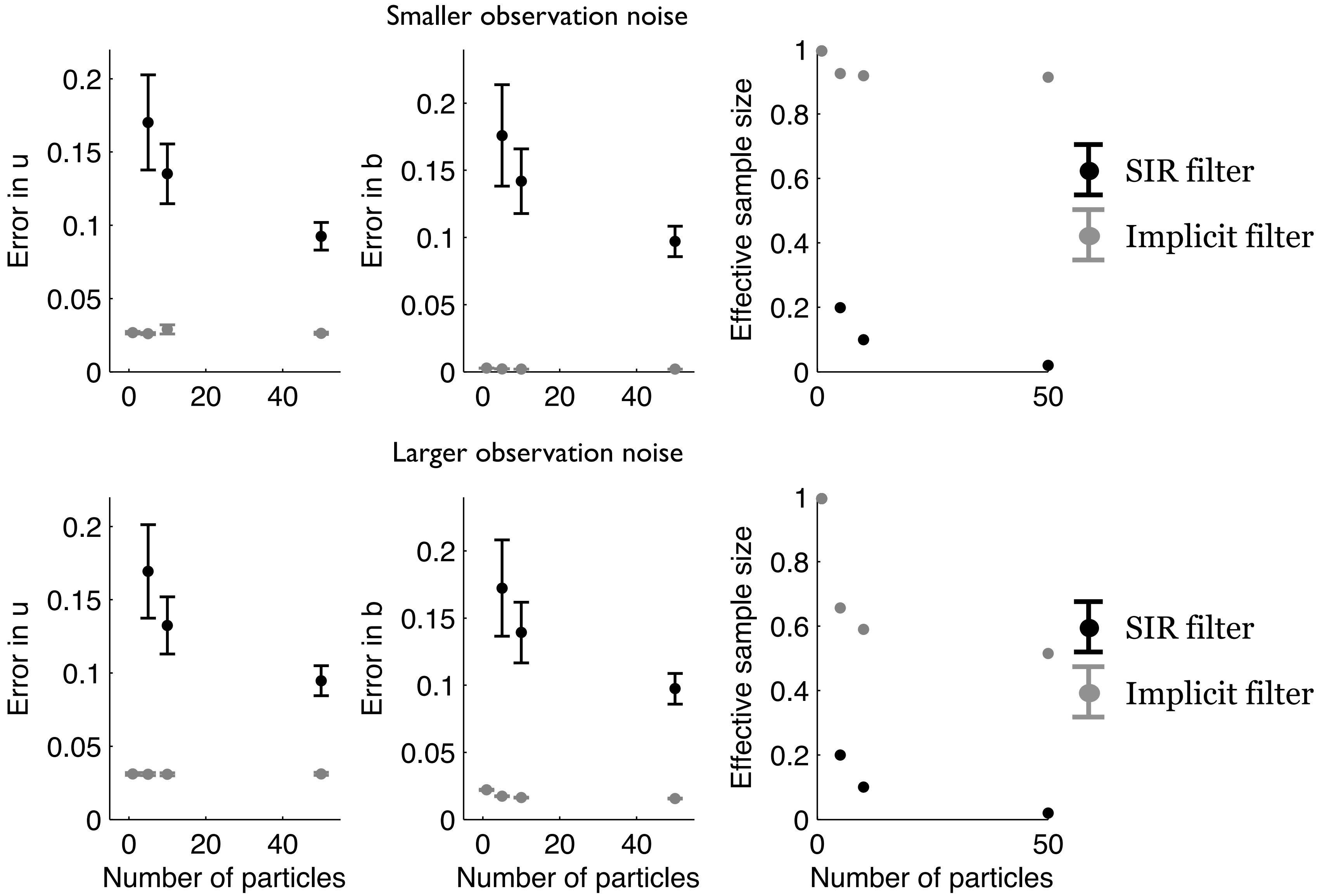


Twin Experiment

- Run the model
- Collect observations
- Run SIR filter
- Run Implicit Filter
- Compute Errors



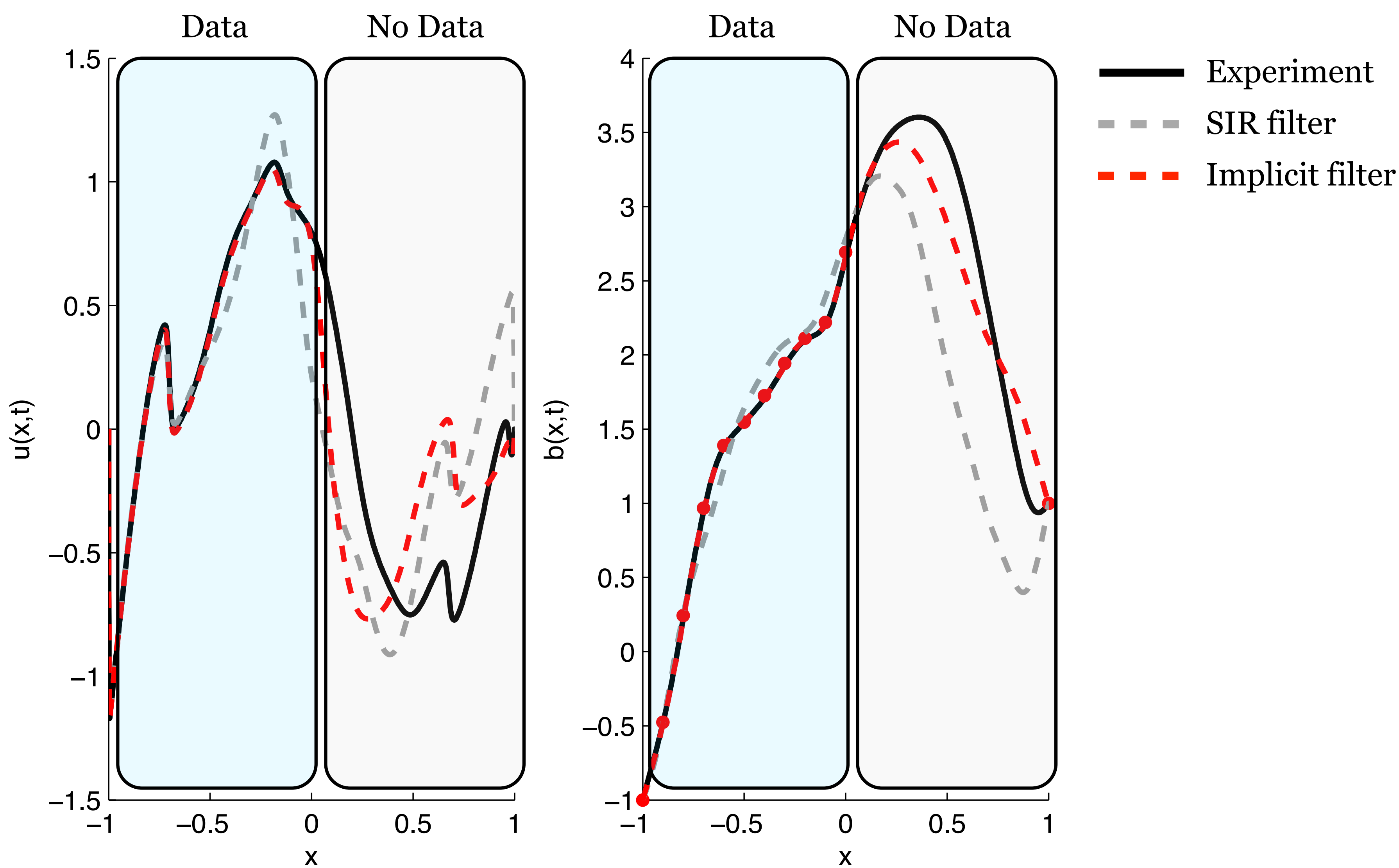
Filtering Results: Dense Data



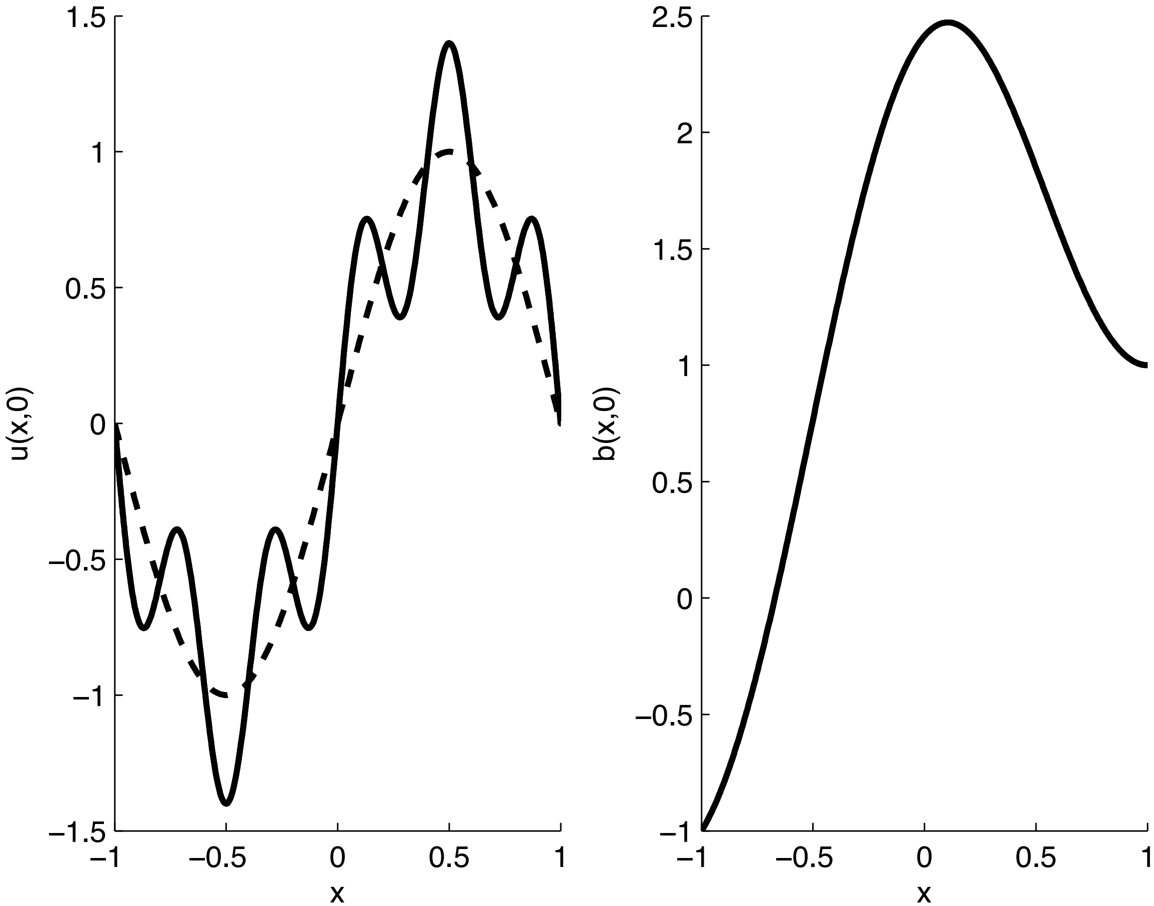
Summary

- Implicit filter requires 5-10 particles
- SIR requires significantly more particles
- SIR with 200 particles give an accuracy 4 times larger than implicit filter with 5 particles
- Implicit filter performs well as data becomes more sparse in space

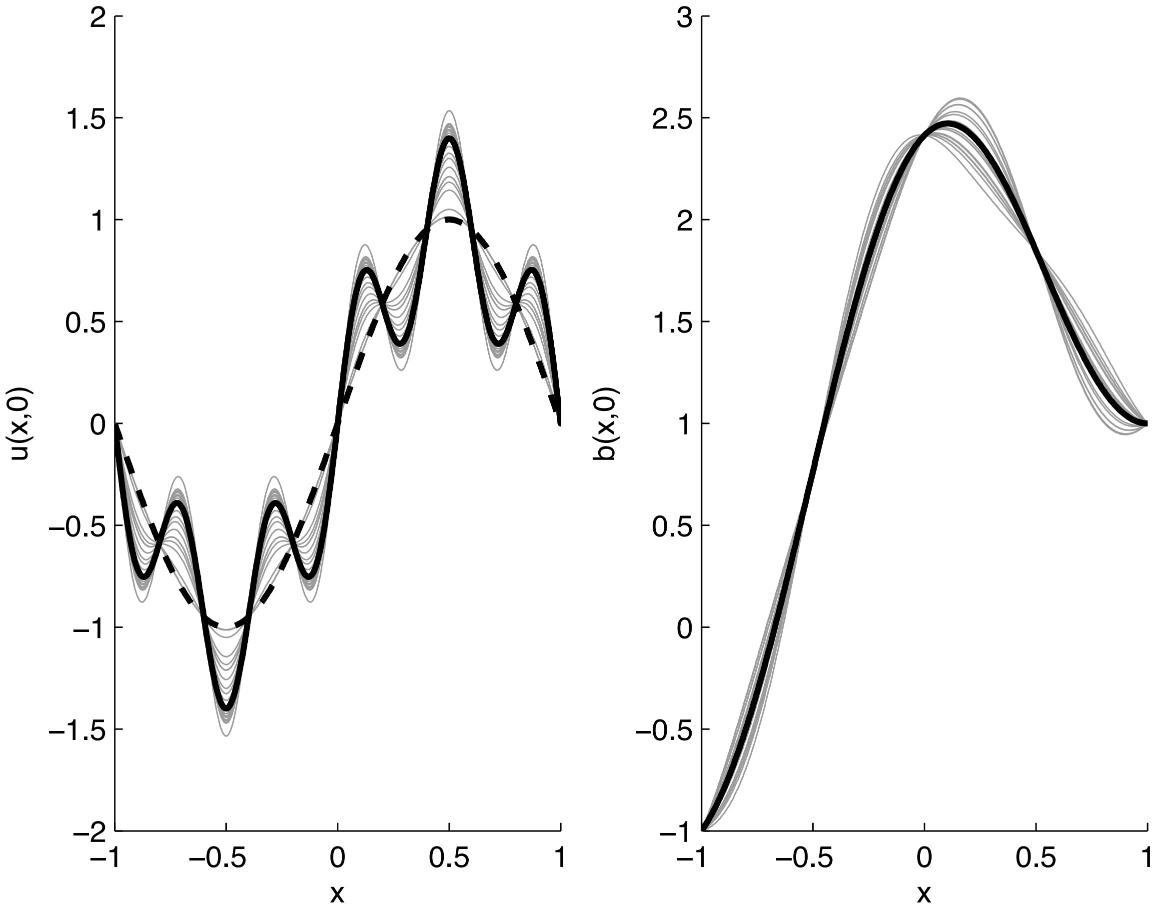
Filtering Results: Skew Data, Example



Poor Initial Guess



Initial Uncertainty



Conclusions

- Noise model allows for easy incorporation of prior information on the spatial distribution of uncertainty
- Numerical examples indicate that sequential Monte Carlo techniques are applicable to geomagnetic applications
- Implicit filter yields accurate state estimates with very few particles
- Implicit filter operates in low dimensional subspace (dimension 50 vs. dimension 1000)
- Implicit particle filter outperforms SIR filter

Thank you!

Please come talk to us at our poster

Implicit Filter: A Numerical Case Study of the Lorenz Attractor

Friday, 10:30 AM - 12:00 Noon

Hall X/Y, XY335

References:

(1) A. Fournier, C. Eymin, T. Alboussière, *A case for variational geomagnetic data assimilation: insights from a one-dimensional, nonlinear, and sparsely observed MHD system*, Nonlin. Processes Geophys. 14 (2007), pp. 163-180.

(2) M. Morzfeld, E. Atkins, X. Tu, A. J. Chorin, *A random map implementation of implicit filters*, submitted for publication.