A SENSITIVITY ANALYSIS FOR A THERMOMECHANICAL MODEL OF THE ANTARCTIC ICE SHEET AND ICE SHELVES

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Motivations

- The outcomes of an ice sheet model depend on a number of parameters and physical quantities which are often estimated with large uncertainty, because of lack of sufficient experimental measurements in such remote environments.
- The efforts to improve the accuracy of the predictions of ice sheet models by including more physical processes and interactions with atmosphere, hydrosphere and lithosphere can be affected by the inaccuracy of the fundamental input data.
- A **sensitivity analysis** can help to understand which are the input data that most affect the different predictions of the model.
- The objective of this work is to analyze the sensitivity of an ice sheet model with respect to some of its input parameters, by computing three synthetic numerical indices: two **local sensitivity indices** and a **global sensitivity index**.

The sensitivity indices

The sensitivity of the model is investigated for some parameters, which are denoted as the elements of an array $\mathbf{X} = (X_1, X_2, \dots, X_k)^t$, whereas any model outcome is denoted with Y and is expressed as a function of the parameters: $Y = g(\mathbf{X})$. The sensitivity of the model to variations of the parameters is analyzed in a small neighborhood of their reference value $\mathbf{x}_{\mathrm{ref}}$.

Local sensitivity indices imply a linearization of the model and neglect both non-linear and joint effects of the parameters. The two following local indices have been considered:

1. Dimensionless normalized sensitivity:

$$S_i^\sigma = \frac{\sigma_i}{\sigma_Y}\frac{\partial g}{\partial X_i}(\mathbf{x}_{\mathrm{ref}}), \tag{2}$$

where σ_i and σ_Y are the standard deviations of X_i and Y respectively.

2. Prediction scaled sensitivity: relative variation of the predicted quantity Y corresponding to a unit relative variation of a model parameter X_i:

$$pss_i = \frac{dY}{y_{\mathrm{ref}}}\frac{x_{\mathrm{ref}\,i}}{dX_i}. \tag{3}$$

Global variance-based sensitivity indices, instead, take into account the complete variability of the input parameters. The following global index has been considered:

3. First-order sensitivity index:

$$S_i = \frac{\mathrm{var}_{X_i}[E_{\mathbf{X}-i}[Y|X_i]]}{\sigma_Y^2}, \tag{4}$$

where $E_{\mathbf{X}-i}[Y|X_i]$ is the expected value of the model outcome conditioned on the parameter X_i and var_{X_i} is the variance with respect to X_i.

The non-linearity of the ice sheet model makes a thorough sensitivity analysis very time consuming. Therefore, (2), (3) and (4) have been computed for a **second-order approximation of the model**, obtained by developing the model outputs in a neighborhood of the reference parameters $\mathbf{x}_{\mathrm{ref}}$.

THE MODEL

In this work an ice sheet model developed by our research group has been applied to simulate the evolution of the Antarctic ice sheet in the last 220,000 years. The model has the following basic features:

- 3D, thermomechanical
- it includes the ice shelves treatment
- it is based on the Shallow-Ice (SIA) and Shallow-Shelf (SSA) approximations

Basic equations:

- **Conservation of mass**:

$$\partial_t H = -\nabla' \cdot (H\overline{\mathbf{u}'}) + M_s - M_b. \tag{1}$$

- **Conservation of momentum**: the field of horizontal velocities $\mathbf{u}'$ in the ice sheet and ice shelves is obtained applying the SIA and the SSA to the Stokes' equations $\nabla \cdot \sigma = \rho \boldsymbol{g}$.

Ice sheet:

$$\mathbf{u}'(z) = \mathbf{u}'_b - 2(\rho g)^n |\nabla' s|^{n-1} \nabla' s \int_b^z \mathcal{A}(T^*)(s-\zeta)^n d\zeta.$$

The *vertical velocity* w is computed from the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

Ice shelves:

$$\partial_x[2\bar\eta H(2\partial_x u + \partial_y v)] + \partial_y[\bar\eta H(\partial_y u + \partial_x v)] = \rho\left(1 - \frac{\rho}{\rho_w}\right) gH\partial_x H,$$

$$\partial_y[2\bar\eta H(2\partial_y v + \partial_x u)] + \partial_x[\bar\eta H(\partial_y u + \partial_x v)] = \rho\left(1 - \frac{\rho}{\rho_w}\right) gH\partial_y H.$$

In the ice shelves, the horizontal velocity $\mathbf{u}'$ is assumed constant with depth, so that the *vertical velocity* w is negligible.

- **Conservation of energy**:

Ice sheet: $\partial_t T = \chi \partial^2_{zz} T - \mathbf{u}\cdot\nabla T + \Sigma.$ (2)

Ice shelves: $\partial_t T = \chi \partial^2_{zz} T - \mathbf{u}\cdot\nabla T.$ (3)

Boundary conditions and input data:

- The **surface temperature** $T_s(\mathbf{x}, t)$ and **surface accumulation** $M_s(\mathbf{x}, t)$ are prescribed for the Antarctic region, according to the parameterizations employed, e.g., by Huybrechts et al. (1990) and Ritz et al. (2001). These parameterizations depend on the present-day surface temperature $T_s(0)$ and accumulation $M_s(0)$, whose values are taken from Le Brocq et al. (2010), and on the difference $\delta T(t)$ of the air temperature at time t from its present-day value, derived from the Vostok ice core (Petit et al. 1999).

Ice sheet

- Basal velocity:

$$\boldsymbol{u}'_b = \begin{cases} \mathbf{0} & \text{if } T_b < T_{\mathrm{pmp}}, \\ -B_s \rho g H \nabla' s & \text{if } T_b = T_{\mathrm{pmp}}. \end{cases}$$

- Thermal condition at the base:

$$-\kappa \partial_z T|_{z=b} = \begin{cases} G & \text{if } T_b < T_{\mathrm{pmp}}, \\ G + \rho g H \mathbf{u}'_b \cdot \nabla' s - \rho L_f M_b & \text{if } T_b = T_{\mathrm{pmp}}, \end{cases}$$

where the geothermal heat flux G is assumed to be constant with time, but spatially varying as in Shapiro and Ritzwoller, 2004.

Ice shelves

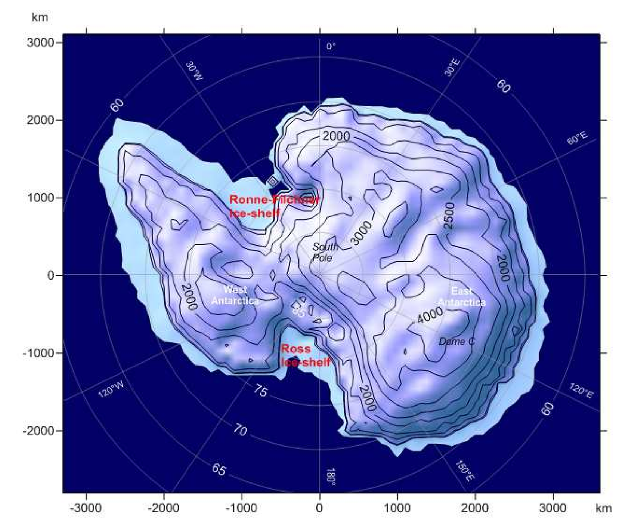
- Position of the grounding line (flotation criterion): $\rho H = \rho_w(z_{\mathrm{sl}} - b)$.
- The border of the ice shelf towards the ocean is determined by the condition $H \le 1$ m.
- Continuity of ice flux and of ice thickness at the grounding line.
- Basal temperature equal to the freezing temperature for sea water.

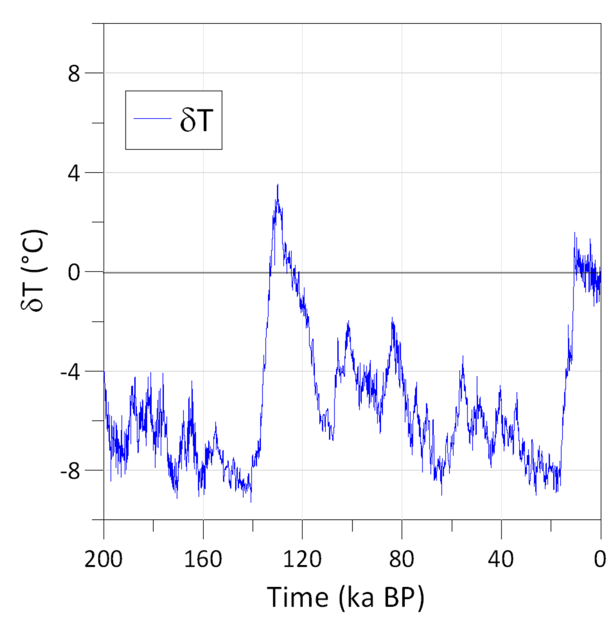
Notation

b, s	height above mean sea level of the bottom and top surfaces of the ice sheet
H	ice thickness
$\mathbf{u} = (u, v, w)$	ice velocity
$\mathbf{u}' = (u, v)$	horizontal ice velocity
$\overline{\mathbf{u}'}$	vertically averaged horizontal velocity
M_s	surface accumulation rate
M_b	basal melt rate
σ	stress tensor
ρ	ice density
$\mathbf{g}$	gravity acceleration
$\mathbf{u}'_b$	horizontal velocity at the base of the ice sheet
$\mathcal{A}$, n	flow parameters of Glen's law
T	ice temperature
T_{pmp}	temperature of pressure melting point
T_0	melting temperature of ice at atmospheric pressure
$T^* = T - T_{\mathrm{pmp}} + T_0$	homologous temperature
T_b	basal temperature
$\bar\eta$	vertically averaged ice viscosity
χ	thermal diffusivity of ice
Σ	strain heating
B_s	coefficient of basal sliding
κ	thermal conductivity of ice
G	geothermal heat flux
L_f	latent heat flux
ρ_w	sea water density
z_{sl}	sea surface elevation

Numerical model:

- finite differences
- explicit scheme for (1), Crank-Nicholson for (2) and (3)
- horizontal grid: 70×60 nodes (cell size: 100 km); 20 vertical layers.





Elevation map of Antarctica obtained with the reference simulation (left). Time evolution of the climatic forcing δT for the last 200,000 years, measured from the Vostok ice core (Petit et al., 1999) (right).

Control variables (X_i)

$\bar\eta$	ice shelves viscosity averaged over the ice thickness
bmr	basal melt rate of the ice shelves
$\tilde T$	correction for the ice shelf temperature
B_s	coefficient of basal sliding
$\langle M_s(0)\rangle$	spatial average of the present-day accumulation
$\langle G\rangle$	spatial average of the geothermal heat flux
$\langle T_s(0)\rangle$	spatial average of the present-day surface temperature

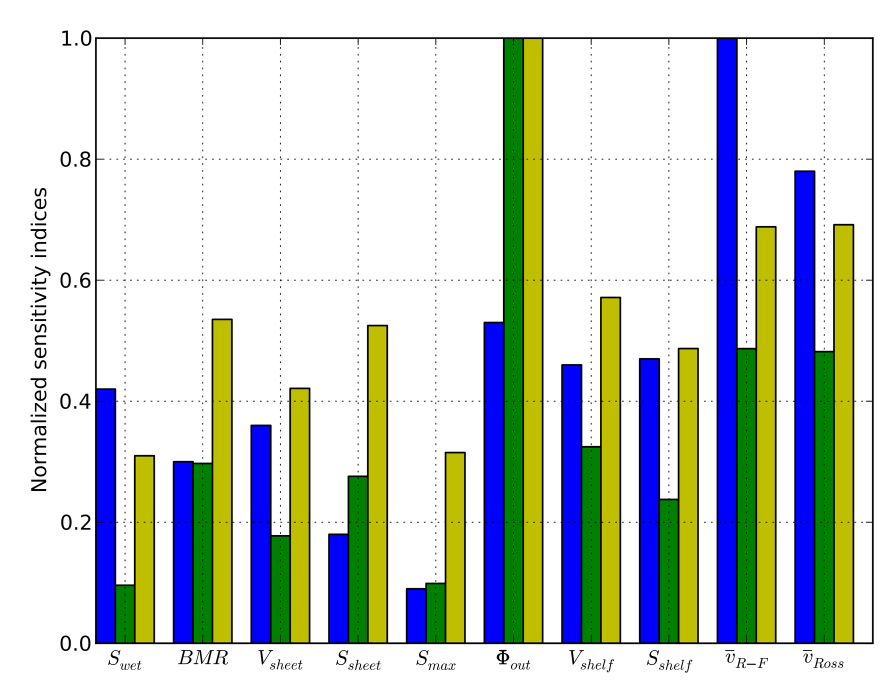
Model outcomes (Y)

S_{wet}	area of the base of the ice sheet at melting point
BMR	integral of the basal melt rate over the ice sheet
V_{sheet}	volume of the ice sheet
S_{sheet}	surface extension of the ice sheet
S_{max}	maximum surface elevation of the ice sheet
Φ_{out}	ice mass flowing through the lateral side of the ice sheet
V_{shelf}	volume of the ice shelves
S_{shelf}	surface extension of the ice shelves
$\bar v_{\mathrm{Ross}}$	averaged horizontal velocity of the Ross ice shelf
$\bar v_{\mathrm{R\text{-}F}}$	averaged horizontal velocity of the Ronne-Filchner ice shelf

Results of the sensitivity analysis:

- Most of the model outcomes are mainly sensitive to $\boldsymbol{T_s(0)}$ **and** $\boldsymbol{M_s(0)}$, which, in principle, can be measured more easily (e.g., with remote sensing techniques) than the other input parameters considered.
- Low sensitivity to B_s and $\bar\eta$;
- $\langle G\rangle$ has a rather important effect on S_{wet}, S_{max} and BMR;
- bmr and $\tilde T$ affect the model outcomes that are directly related to the ice shelves (V_{shelf}, S_{shelf}, $\bar v_{\mathrm{Ross}}$ and $\bar v_{\mathrm{R\text{-}F}}$);
- The local indices S_i^σ and pss_i also bring information about the sign of the perturbation of the model outcomes corresponding to a positive perturbation of the input parameters;
- **The information given by S_i^σ and S_i are similar**, as the corresponding hystograms have the same trend. In fact these indices are both based on the variance of the model output σ_Y, so that they both take into account the variation of the output corresponding to the variation of all the input parameters;
- The normalized S_i^σ is systematically greater than the normalized S_i. Therefore, $\boldsymbol{S_i}$ **tends to enhance the relevance of those parameters for which the sensitivity is greatest**

⇒ The approximation of the non-linear model with a second-order expansion proved to be sufficient to show some differences between the local (S_i^σ) and the global (S_i) variance-based indicators.



Normalized sensitivities of the different model outcomes to $\langle M_s(0)\rangle$, computed by the indices S_i^σ (light green bars), pss_i (blue bars) and S_i (green bars).

Dimensionless normalized sensitivity S_i^σ

	$\bar\eta$	bmr	$\tilde T$	B_s	$\langle M_s(0)\rangle$	$\langle G\rangle$	$\langle T_s(0)\rangle$
S_{wet}	0.14	-0.20	0.13	-0.06	0.27	0.72	0.53
BMR	0.31	0.10	0.14	-0.01	0.46	0.31	0.73
V_{sheet}	0.05	-0.14	0.08	0.01	0.36	-0.19	-0.89
S_{sheet}	0.12	-0.26	0.15	0.03	0.46	-0.01	-0.83
S_{max}	0.00	0.00	0.00	-0.01	0.27	-0.55	-0.78
Φ_{out}	0.03	-0.24	0.16	0.00	0.87	0.17	-0.34
V_{shelf}	-0.02	-0.49	0.16	-0.06	0.50	0.12	-0.68
S_{shelf}	-0.08	-0.54	0.17	-0.01	0.42	0.09	-0.69
$\bar v_{\mathrm{Ross}}$	-0.18	-0.47	0.22	-0.04	0.60	-0.01	-0.54
$\bar v_{\mathrm{R\text{-}F}}$	-0.04	-0.52	0.30	0.02	0.60	0.17	-0.48

Color scale: light blue for $|S_i^\sigma| \le 0.06$, blue for $0.06 < |S_i^\sigma| \le 0.5$, dark blue for $|S_i^\sigma| > 0.5$.

Prediction scaled sensitivity pss_i

	$\bar\eta$	bmr	$\tilde T$	B_s	$\langle M_s(0)\rangle$	$\langle G\rangle$	$\langle T_s(0)\rangle$
S_{wet}	0.58	-0.82	0.54	-0.24	1.11	2.96	8.76
BMR	0.53	0.17	0.24	-0.01	0.8	0.54	5.04
V_{sheet}	0.14	-0.35	0.22	0.03	0.95	-0.49	-9.27
S_{sheet}	0.13	-0.28	0.16	0.03	0.49	-0.01	-3.56
S_{max}	0.00	0.00	0.00	-0.01	0.23	-0.47	-2.67
Φ_{out}	0.04	-0.39	0.25	0.00	1.40	0.27	-2.23
V_{shelf}	-0.05	-1.19	0.38	-0.14	1.21	0.29	-6.69
S_{shelf}	-0.24	-1.61	0.50	-0.03	1.25	0.28	-8.23
$\bar v_{\mathrm{Ross}}$	-0.79	-2.10	0.96	-0.17	2.65	-0.03	-9.57
$\bar v_{\mathrm{R\text{-}F}}$	-0.16	-1.80	1.06	0.06	2.08	0.58	-6.63

Color scale: light blue for $|pss_i| \le 0.2$, blue for $0.2 < |pss_i| \le 1$, dark blue for $|pss_i| > 1$.

First-order sensitivity index S_i

	$\bar\eta$	bmr	$\tilde T$	B_s	$\langle M_s(0)\rangle$	$\langle G\rangle$	$\langle T_s(0)\rangle$
S_{wet}	0.02	0.04	0.02	0.00	0.07	0.53	0.32
BMR	0.10	0.01	0.02	0.00	0.22	0.10	0.55
V_{sheet}	0.00	0.24	0.03	0.00	0.25	0.01	0.47
S_{sheet}	0.02	0.07	0.02	0.00	0.21	0.00	0.68
S_{max}	0.00	0.00	0.00	0.00	0.08	0.31	0.62
Φ_{out}	0.00	0.06	0.02	0.00	0.76	0.03	0.13
V_{shelf}	0.00	0.02	0.01	0.00	0.13	0.04	0.80
S_{shelf}	0.01	0.30	0.03	0.00	0.18	0.01	0.48
$\bar v_{\mathrm{Ross}}$	0.03	0.22	0.05	0.00	0.37	0.00	0.32
$\bar v_{\mathrm{R\text{-}F}}$	0.00	0.27	0.09	0.00	0.36	0.03	0.24

Color scales: light blue for $S_i \le 0.01$, blue for $0.01 < S_i \le 0.1$, dark blue for $S_i > 0.1$.

Notice that in these tables '0.00' denote elements whose absolute value is less than 0.005.