Gradient descent assimilation for the point-vortex model

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Gradient descent assimilation for the point-vortex model

Background

- **The aim of data assimilation is to estimate the current state of a system (or model parameters)**
  - **Partial, noisy observations are incorporated into imperfect models**
  - **Trajectories are generated, consistent with measured data and model dynamics**

- **Much of the data for atmospheric and ocean science are Lagrangian**
  - **A sequence of position measurements in a flow field**

- **Assimilating Lagrangian data into models can be complicated**
  - **Models usually compute fields of flow in Eulerian coordinates**
  - **Lagrangian data is not in terms of model variables**
  - **Transforming Lagrangian data to Eulerian variables poses many problems**
  - **Position data from Langrangian tracers assimilated directly into model**

- **Nonlinear effects in relatively simple flow fields**
  - **Linear DA methods are known to fail (e.g. around saddle points)**
  - **Gradient descent assimilation provides a nonlinear approach**

Gradient descent assimilation for the point-vortex model

The point-vortex model: I

- **A toy model example employed in many areas of physics**
  - **Used here as a test bed for gradient descent assimilation**

- **Equations describe the flow of N point-vortices**
  - **Dynamics of vortices described by superposition of local fluid velocity in 2D**
  - **Flow states exhibit regular (2-vortices) or chaotic (> 3-vortices) motion**

$$\frac{dz_m}{dt} = \frac{i}{2\pi} \sum_{l=1, m\neq l}^{N_v} \frac{\Gamma_l}{z_m^* - z_l^*}$$

where

N_v = **Number of vortices**

Γ_l = **Circulation strength**

$$z_m(t) = x_m(t) + i y_m(t)$$

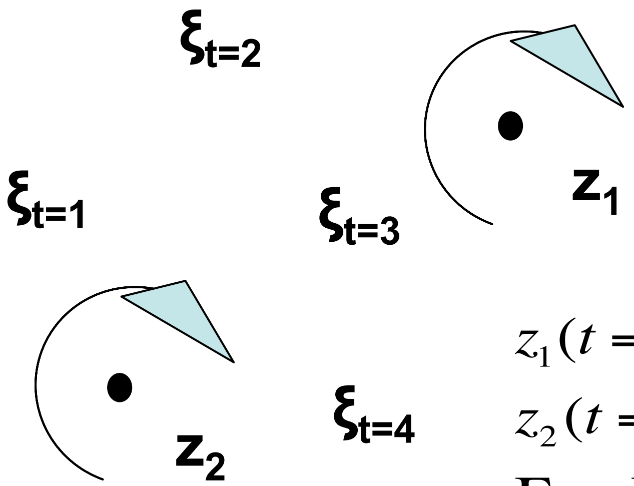
  - **Passive tracers are advected according to:**

$$\frac{d\xi_n}{dt} = \frac{i}{2\pi} \sum_{n,l=1}^{N_v} \frac{\Gamma_l}{\xi_n^* - z_l^*}$$

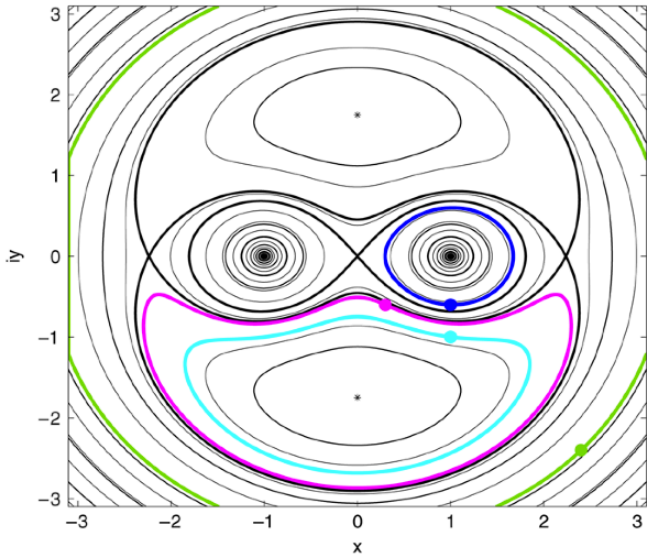
Gradient descent assimilation for the point-vortex model

The point-vortex model: II

- 2 point-vortices, one passive tracer
  - Regular flow – rotation frequency inversely dependent on vortex separation
  - Several types of flow exhibited (different tracer initial conditions explore this)
  - Nonlinear effects at saddle points can cause problems for data assimilation
  - Rapid dispersion of tracer trajectories around saddle points is also useful



$$z_1(t=0) = -1$$
$$z_2(t=0) = 1$$
$$\Gamma_1 = \Gamma_2 = 2\pi$$



E. T. Spiller et al., Physica 237 (2008)

- We want to assimilate noisy tracer positions into model for flow

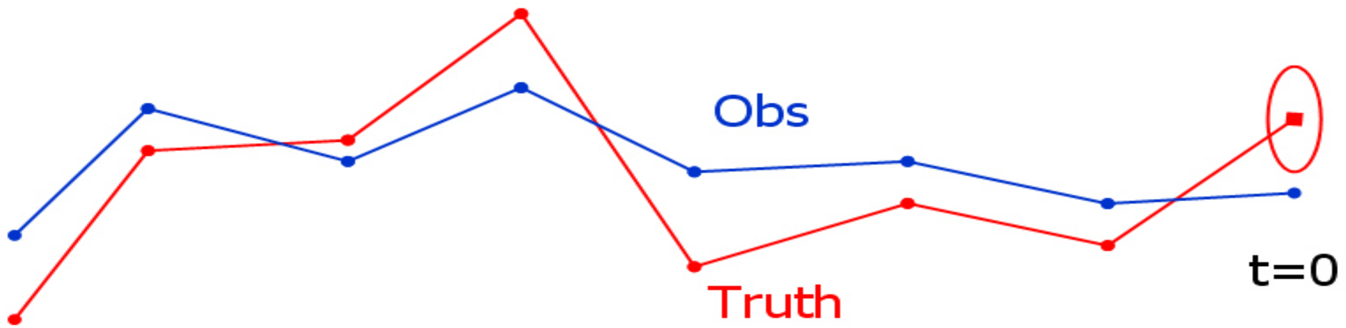
Gradient descent assimilation for the point-vortex model

Gradient descent assimilation

- **A fully nonlinear approach to DA**
  - **Relies on Minimising a cost function in an extended state space**
- **Let $u_t \in \mathbb{R}^m$ be the trajectory of the model at *t=1…n***

$$u_{t+1} = F(u_t)$$

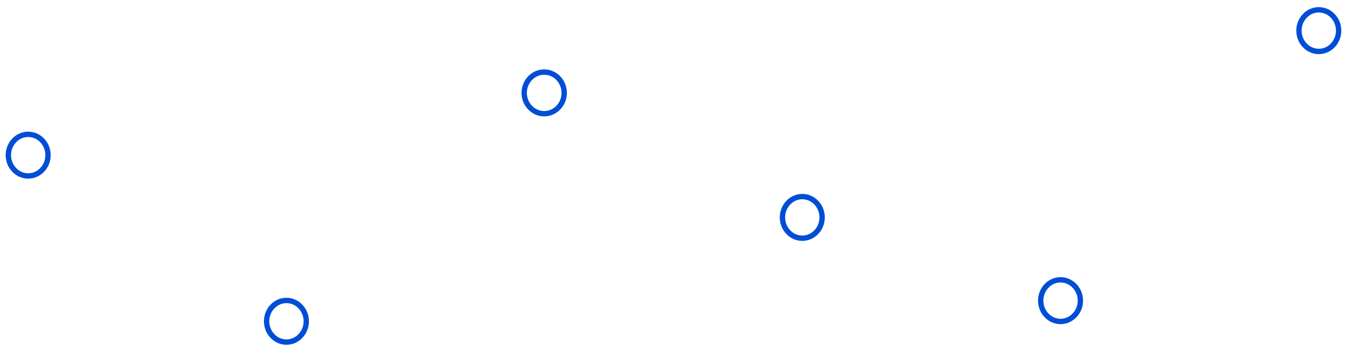
  - **Model *F* given by 2D equations for 2-vortex, 1-tracer system (*m* = 6)**
- **We have a sequence of *n* noisy observations, s_t**
- **<u>Goal</u>: Generate a trajectory consistent with model and observations**



Gradient descent assimilation for the point-vortex model

Gradient descent assimilation

- **Start with a pseudo-orbit defined by noisy observations**

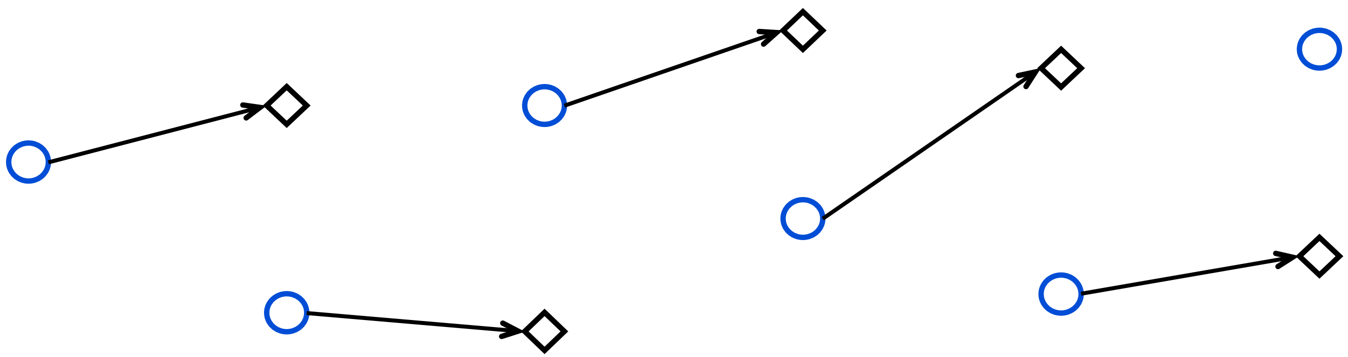


Kevin Judd, Leonard Smith, Antje and Weisheimer, Physica D 190 (2004)

Gradient descent assimilation for the point-vortex model

Gradient descent assimilation

- **Start with a pseudo-orbit defined by noisy observations**
- **Create 1-step ahead forecasts from each observation**



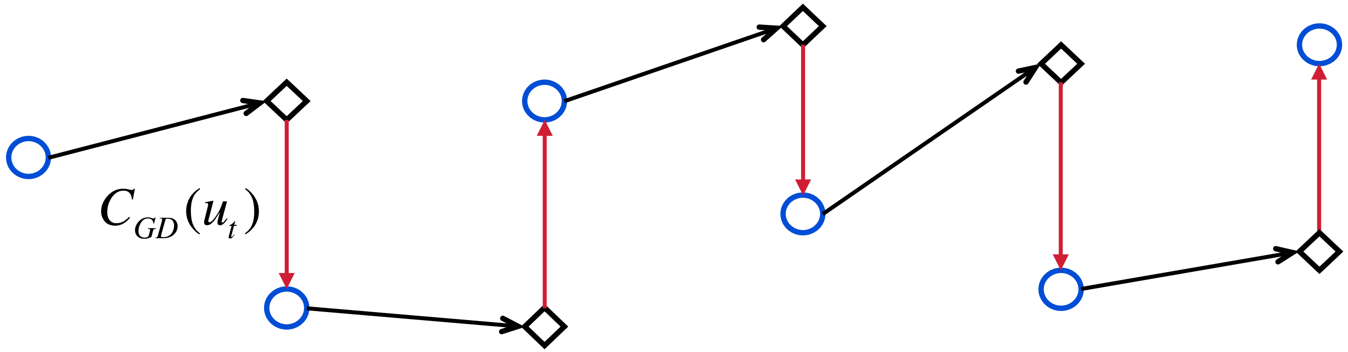
Kevin Judd, Leonard Smith, Antje and Weisheimer, Physica D 190 (2004)

Gradient descent assimilation for the point-vortex model

Gradient descent assimilation

- **Start with a pseudo-orbit defined by noisy observations**
- **Create 1-step ahead forecasts from each observation**
- **Define the mismatch (error cost function) and minimise**

$$C_{GD}(U) = \sum_{t=1}^{n} \left| F(u_t) - u_{t+1} \right|^2$$



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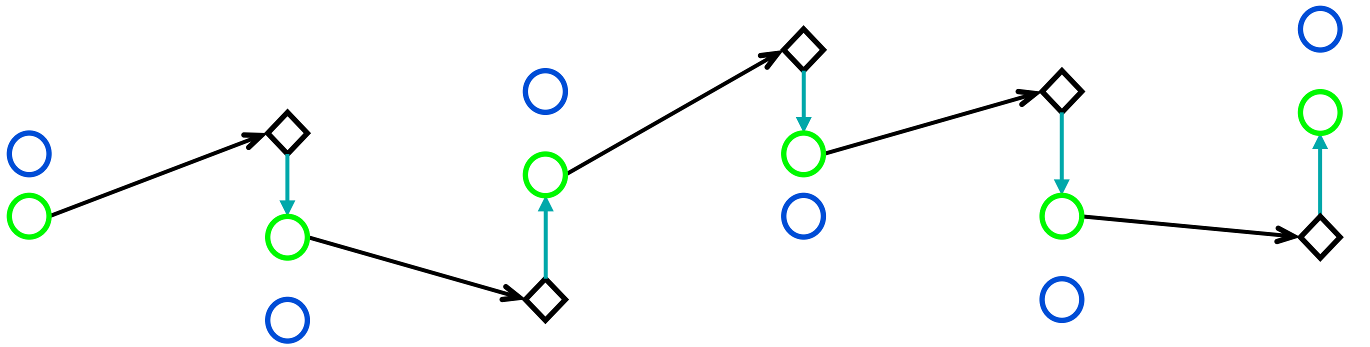

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Gradient descent assimilation for the point-vortex model

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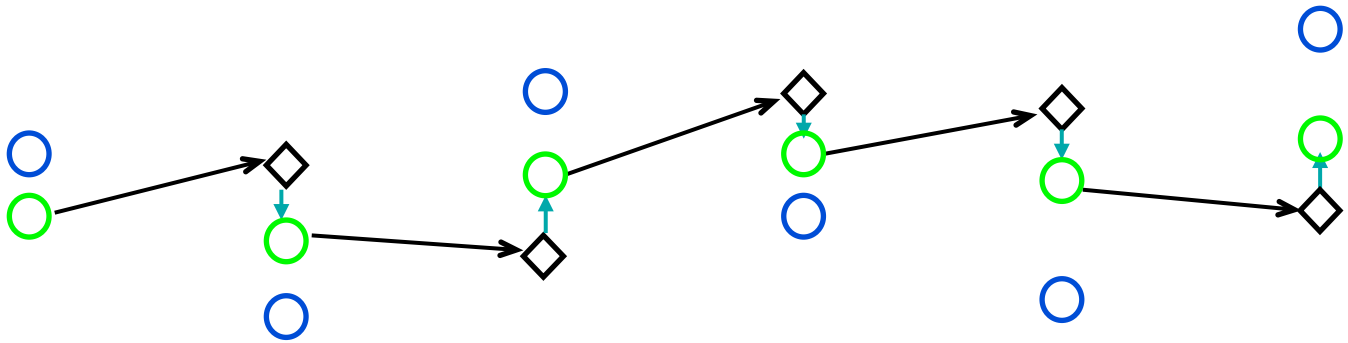

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Gradient descent assimilation for the point-vortex model

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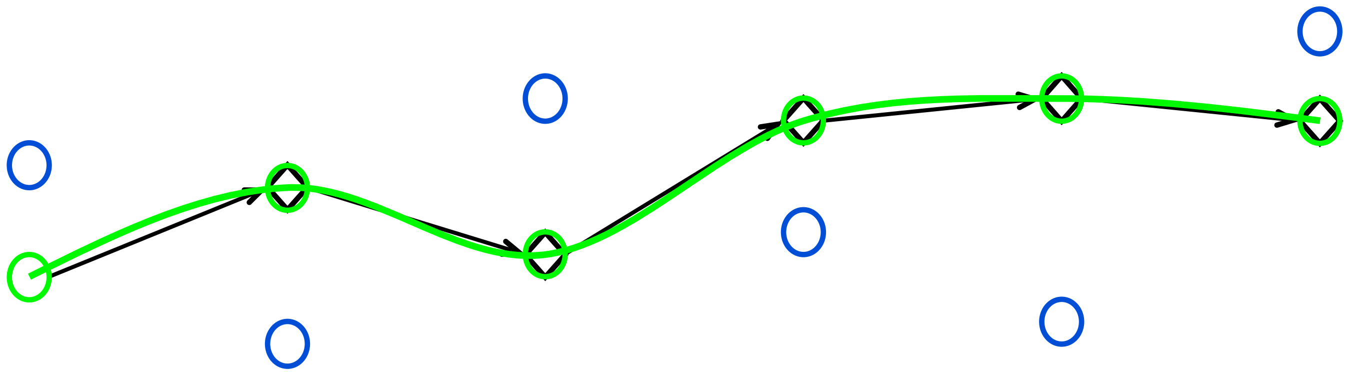
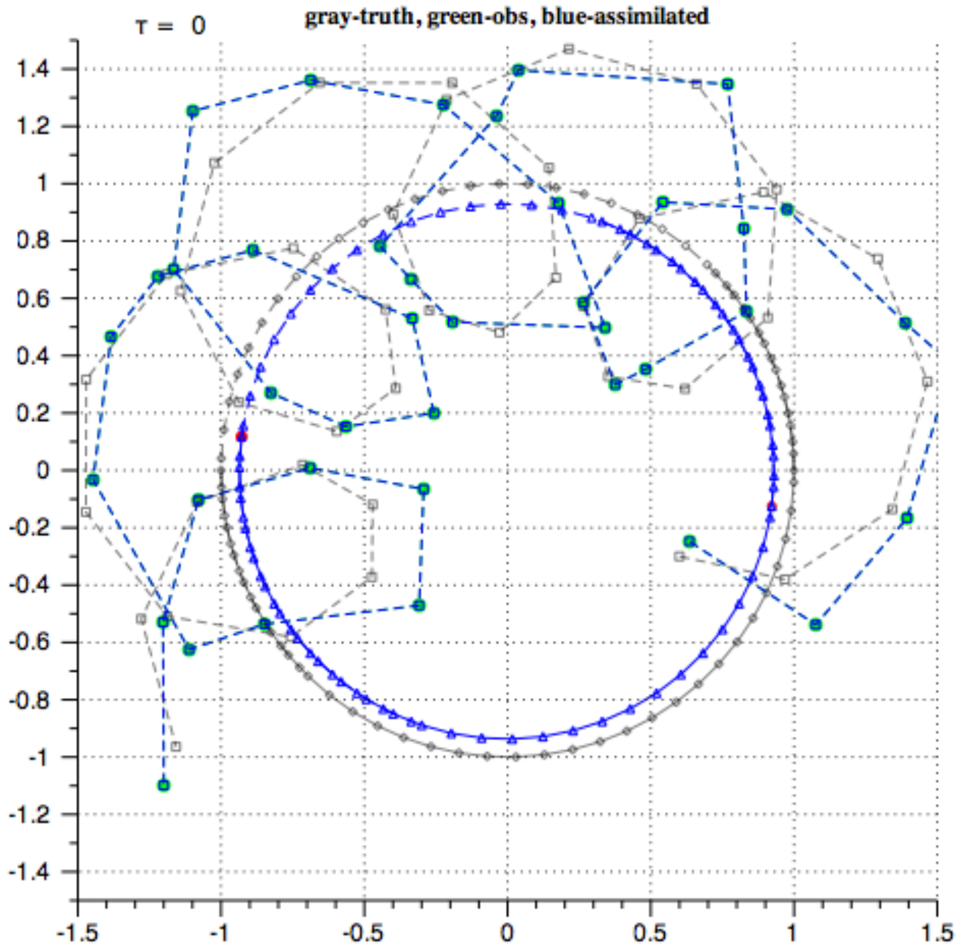
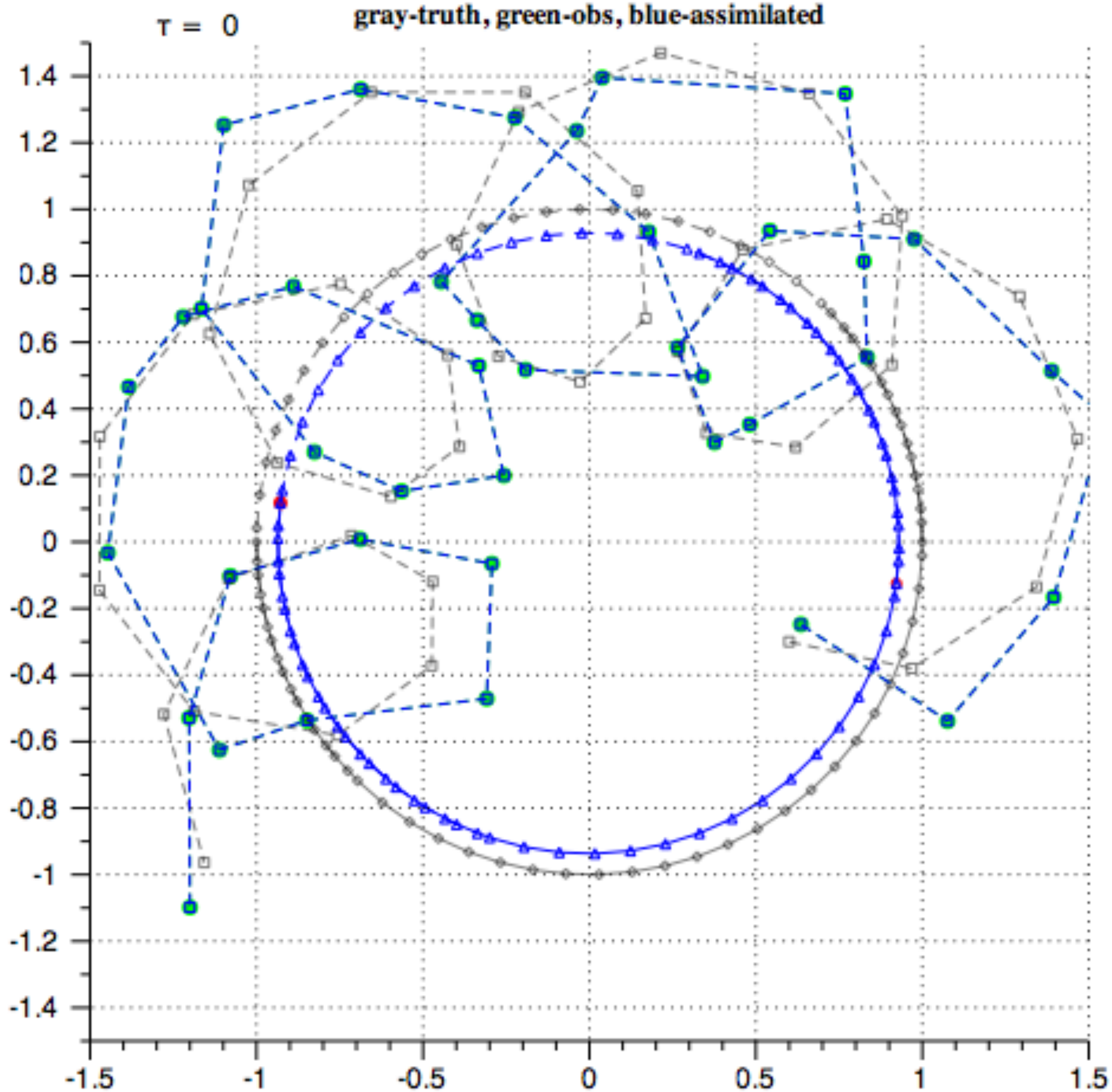


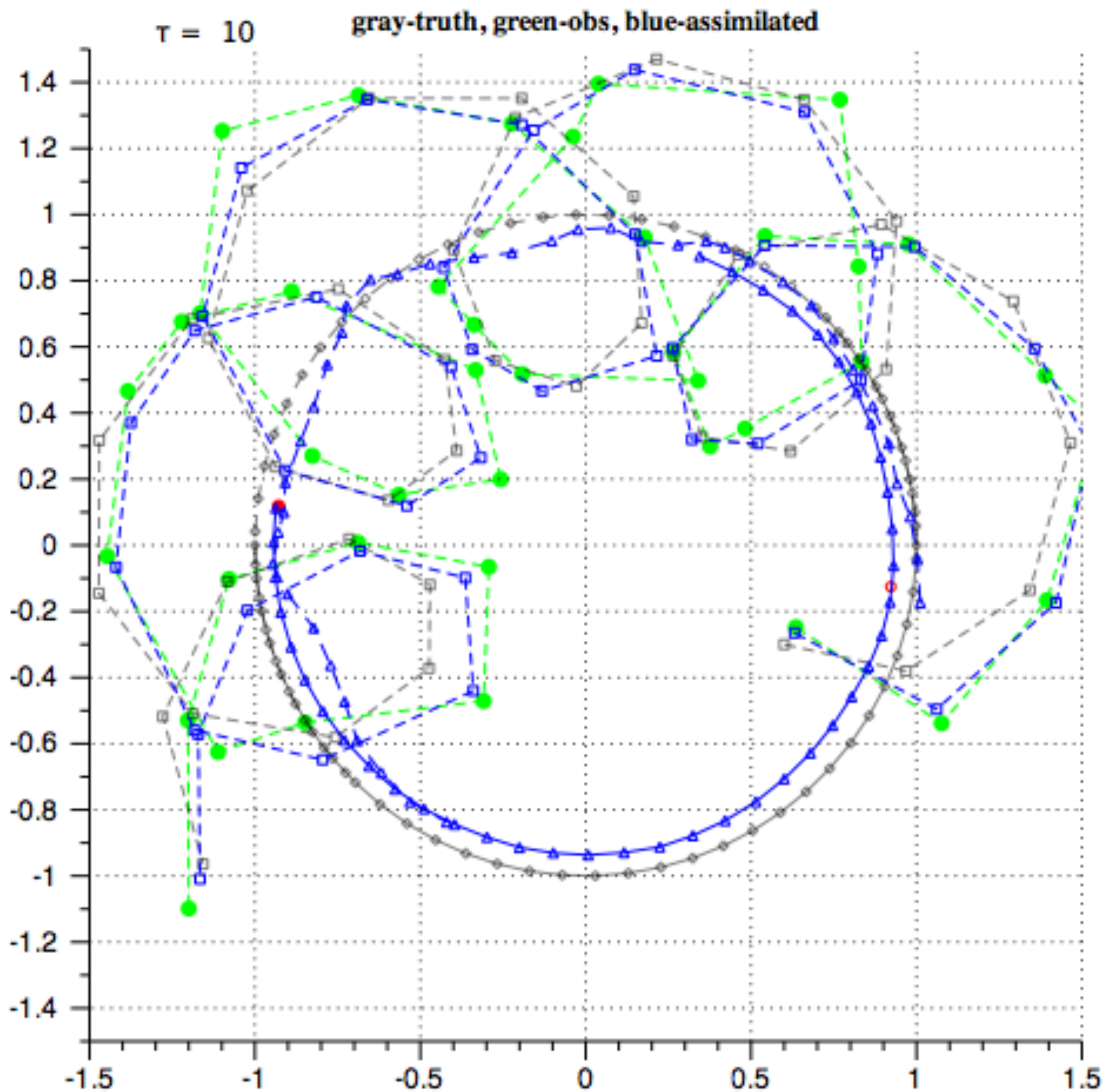
- **Converges to a trajectory 'near' the 'true' trajectory**

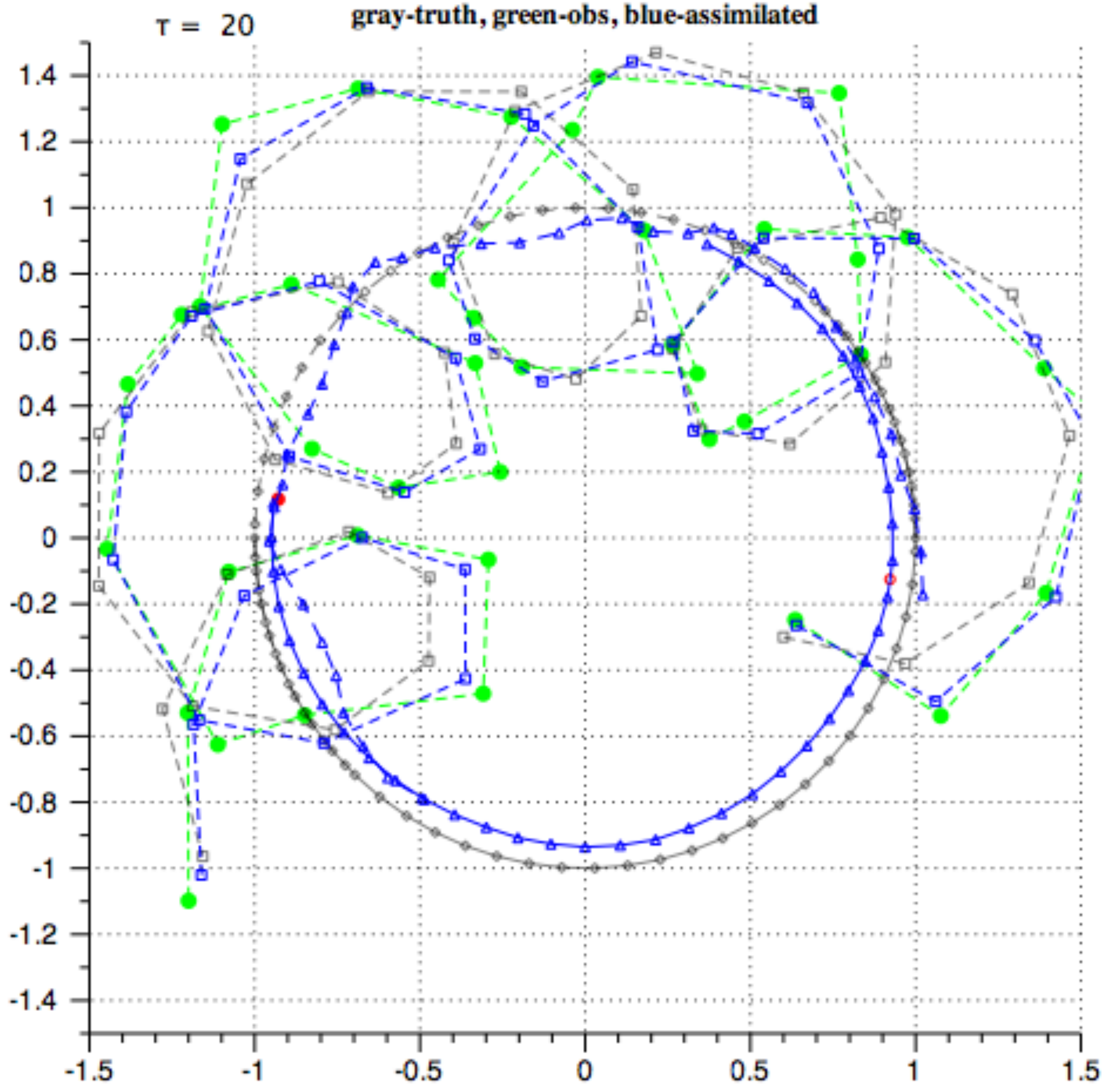
Gradient descent assimilation for the point-vortex model

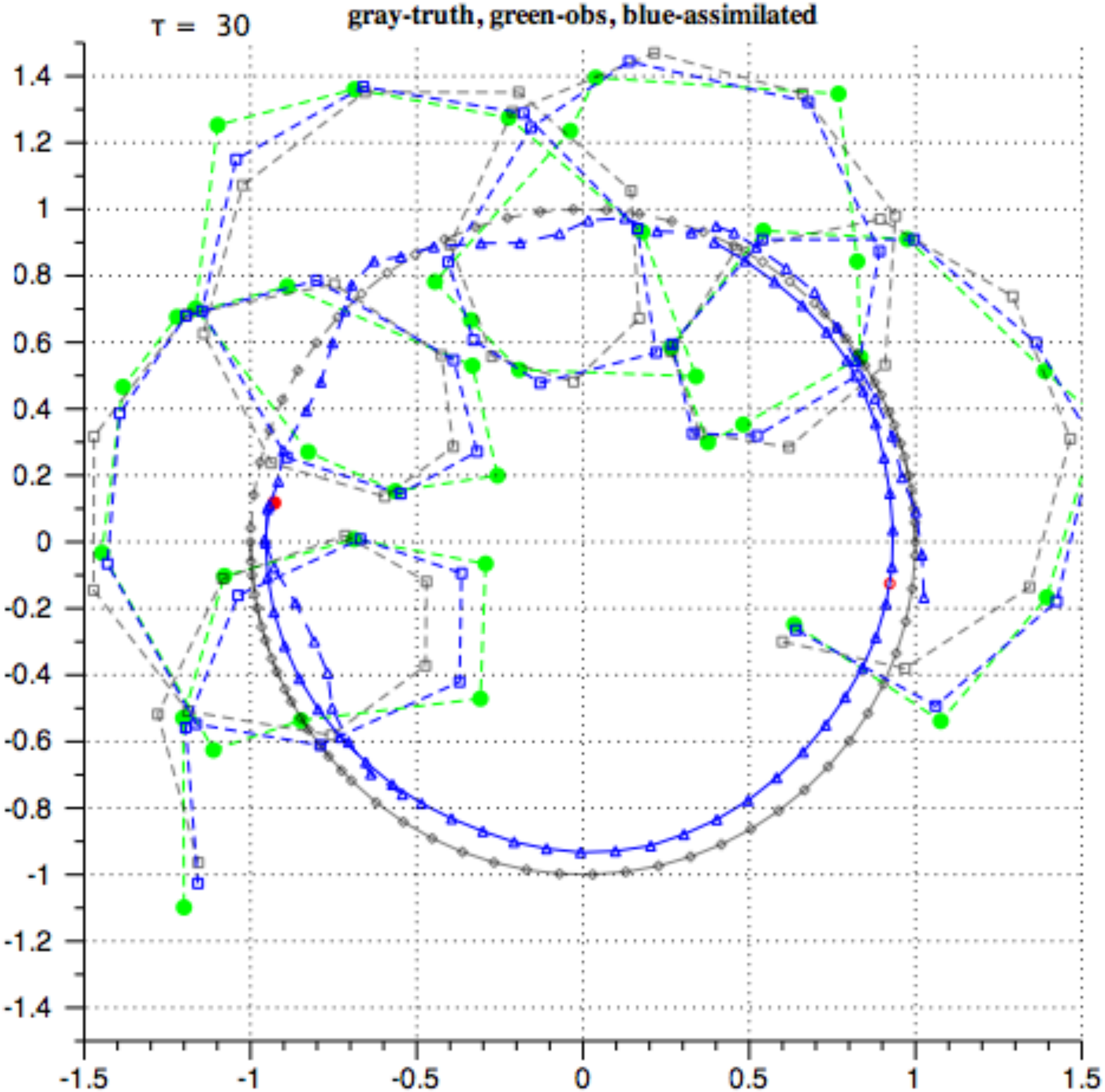
2-vortex model: Full observations

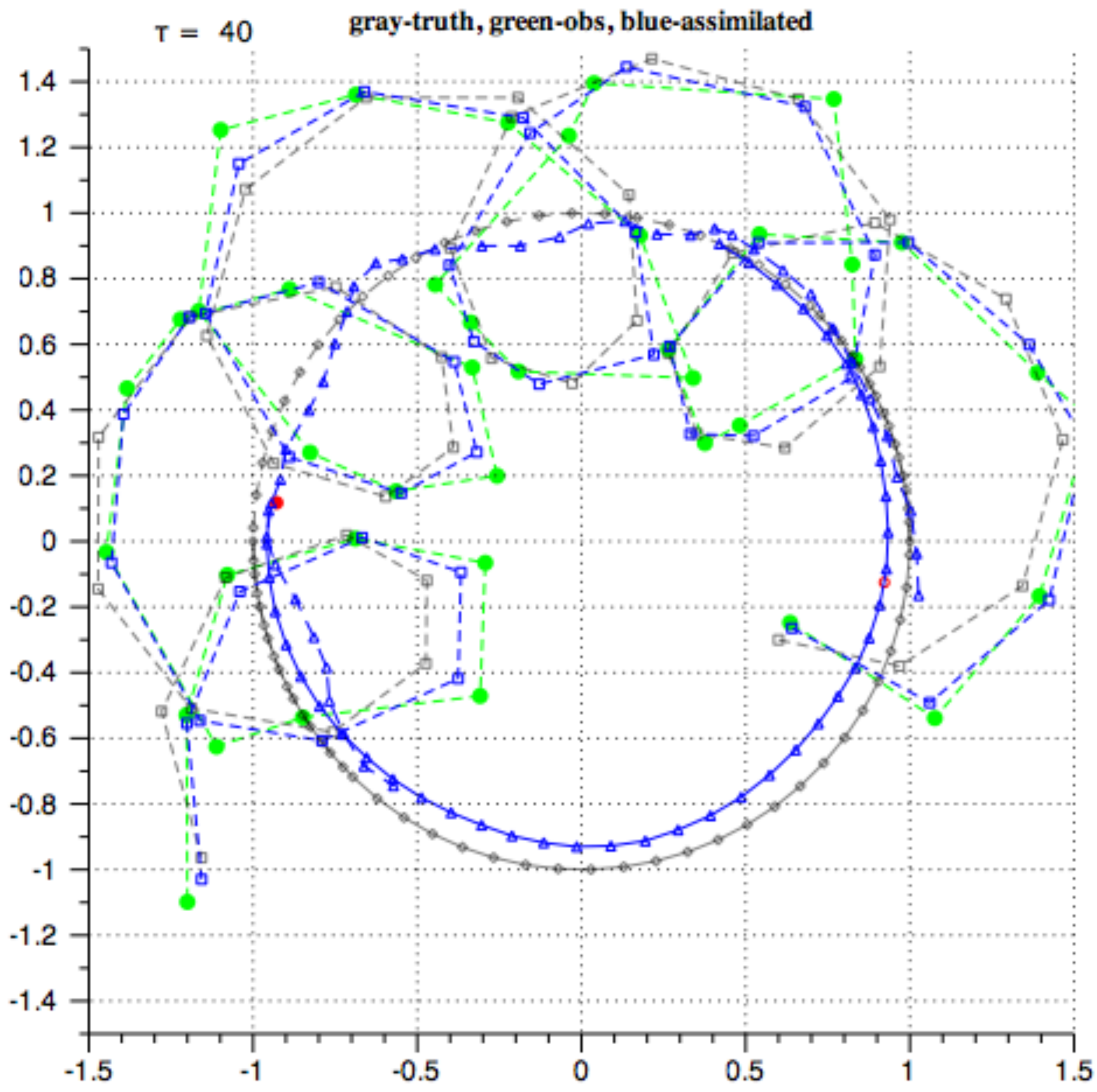


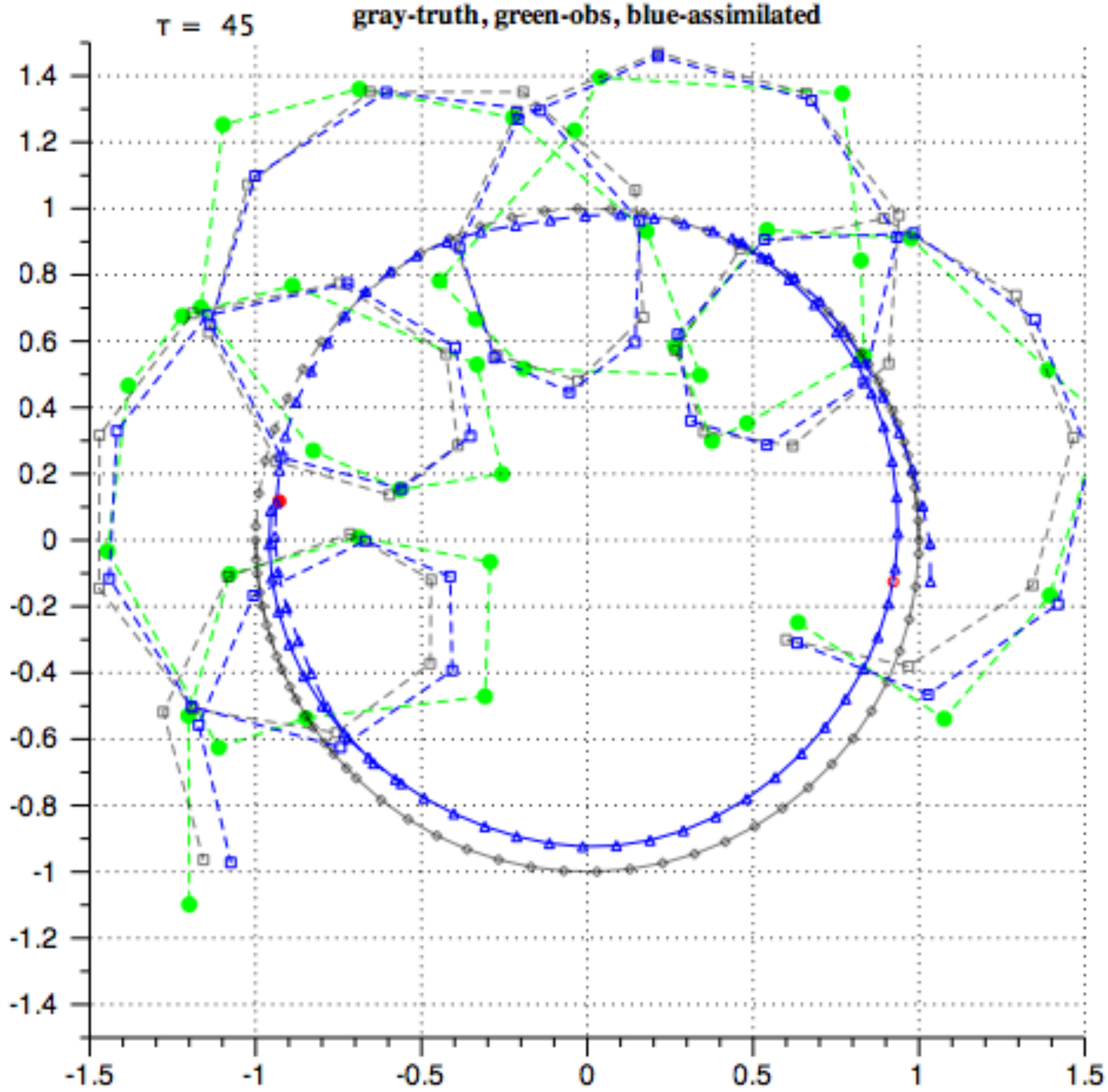

τ = 0
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

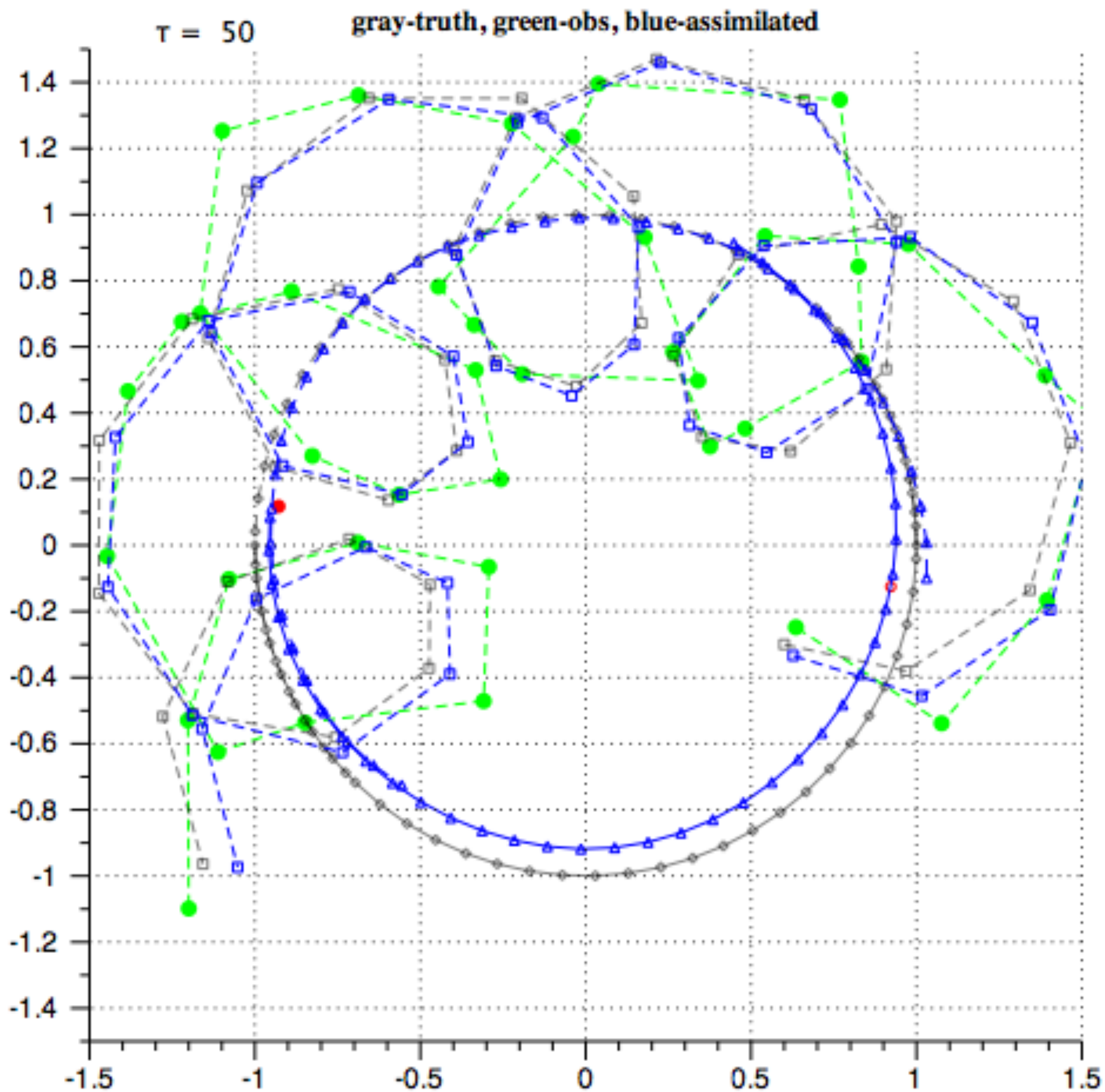

τ = 10
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

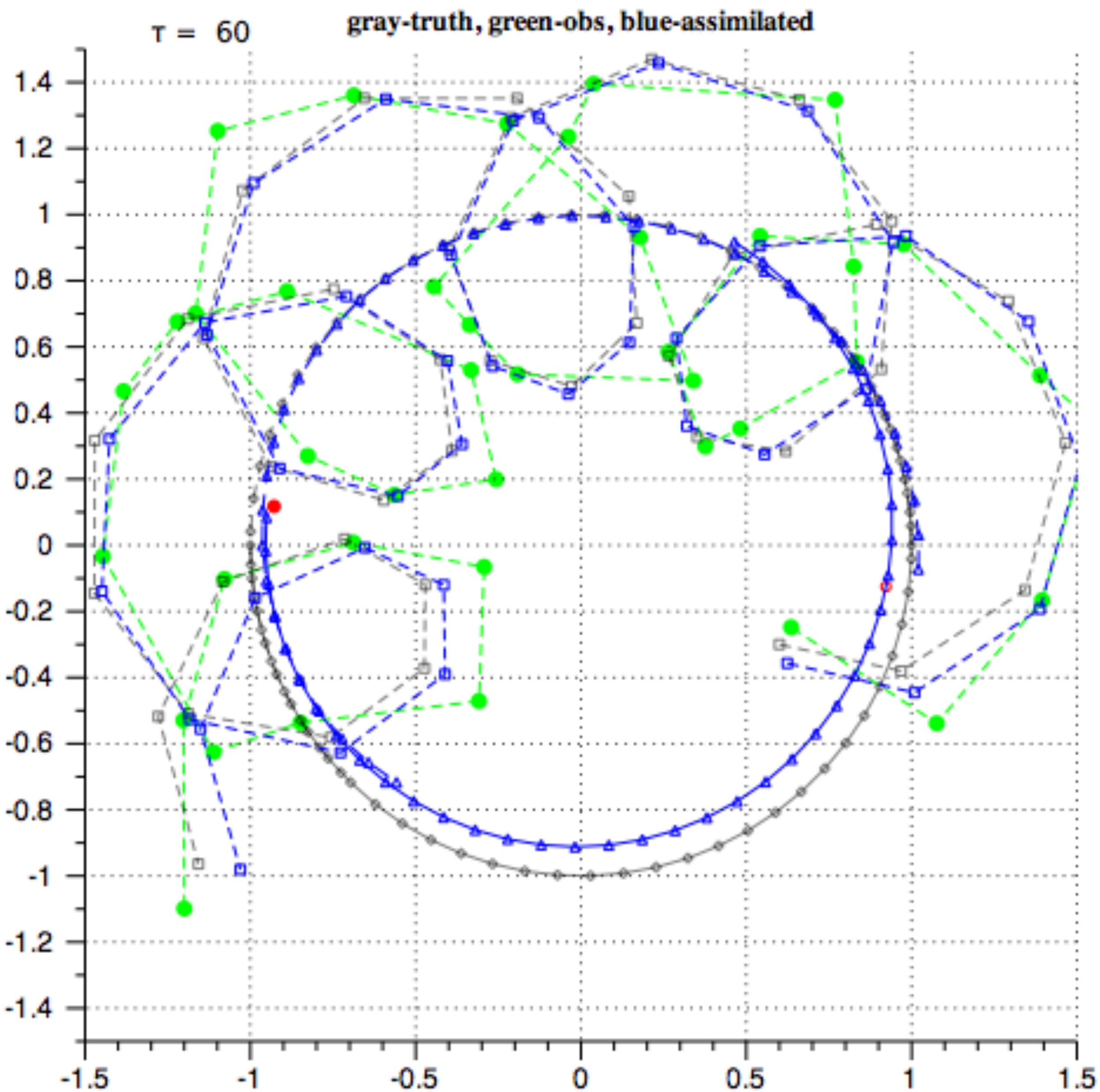

τ = 20
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

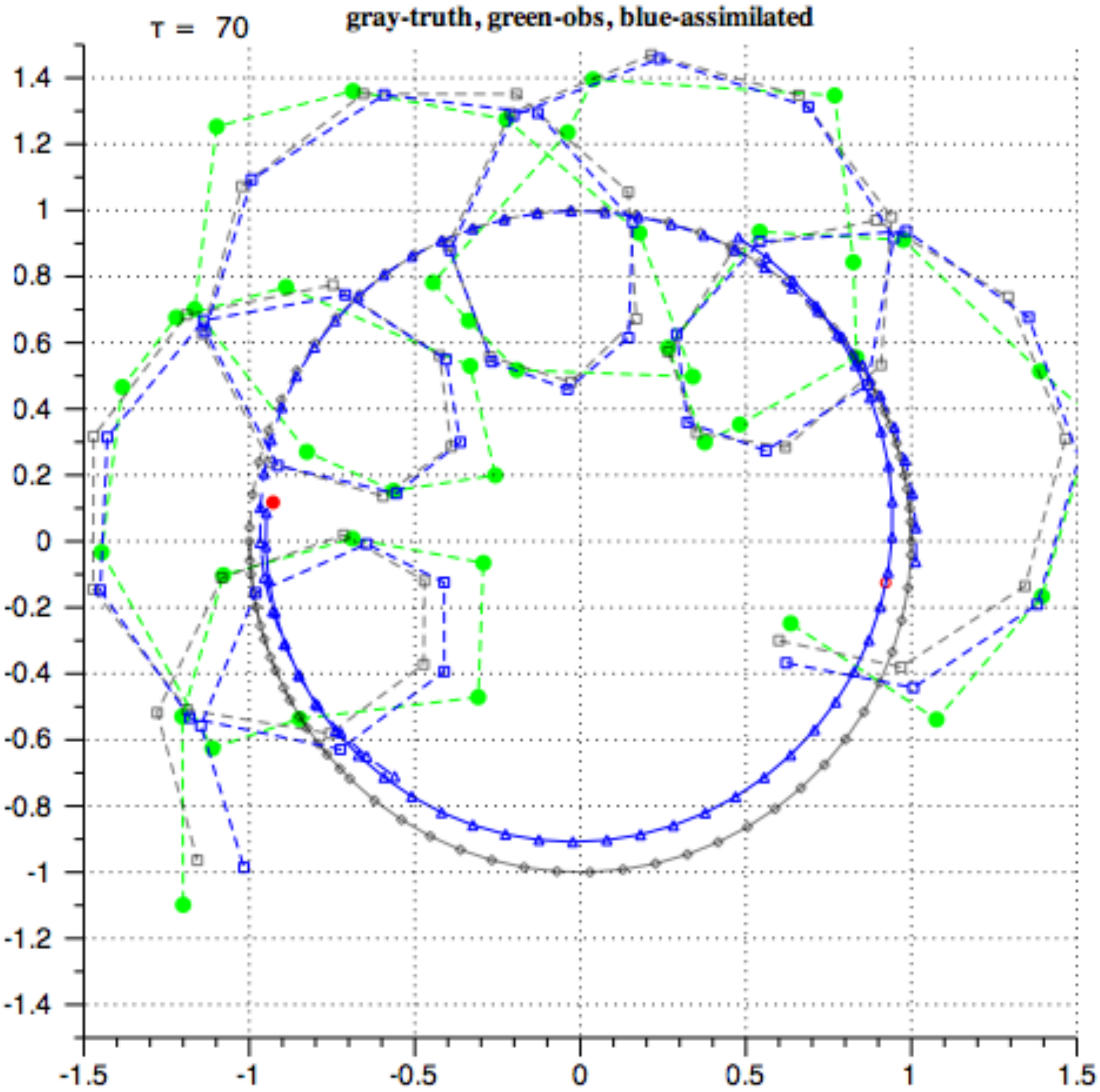

τ = 30
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

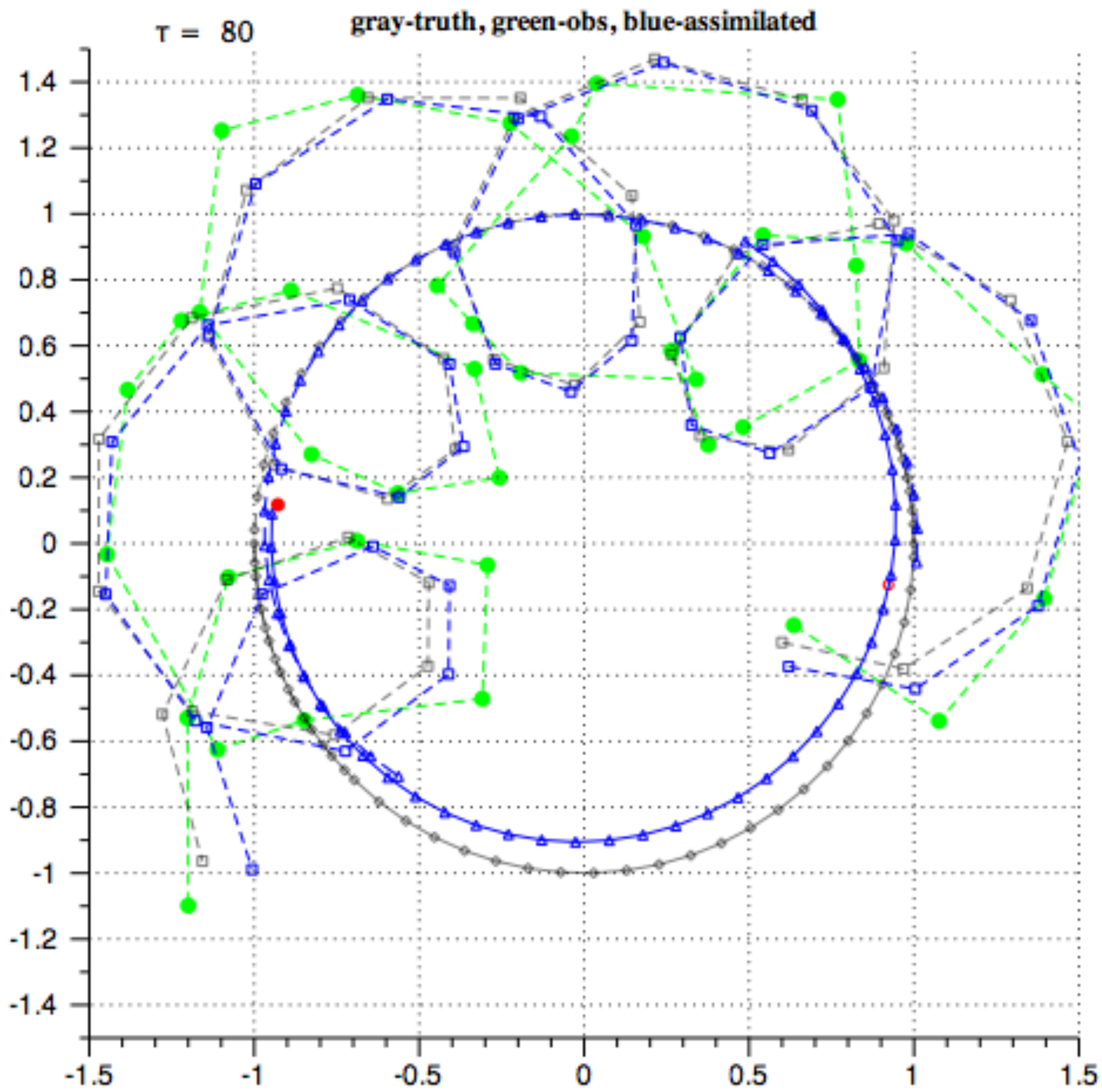

τ = 40
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5


τ = 45
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5


τ = 50
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5


τ = 60
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5


τ = 70
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

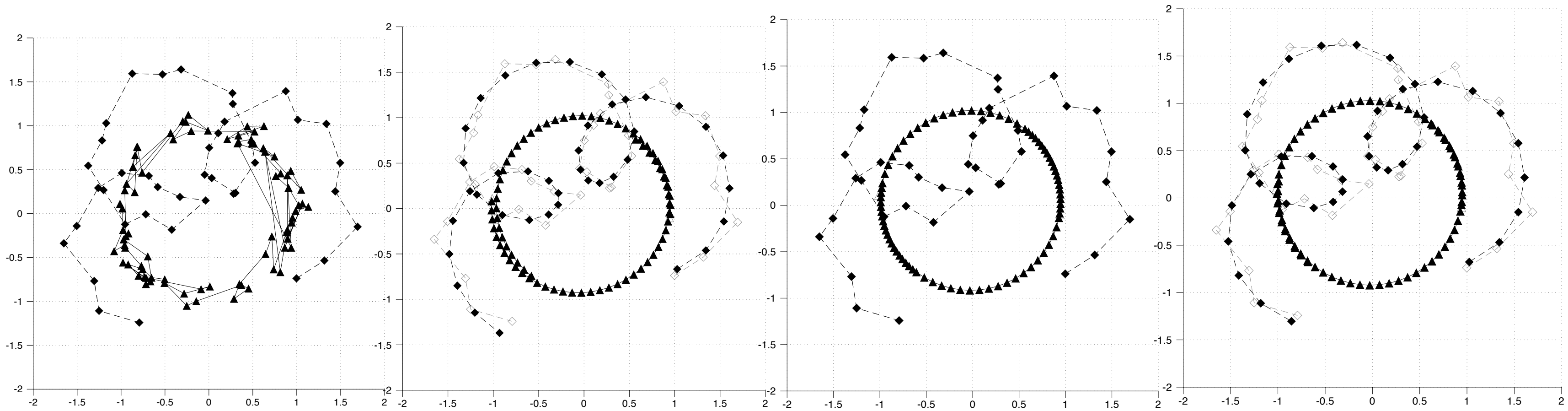

τ = 80
gray-truth, green-obs, blue-assimilated
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.5
-1
-0.5
0
0.5
1
1.5

Gradient descent assimilation for the point-vortex model

2-vortex model: Partial observations

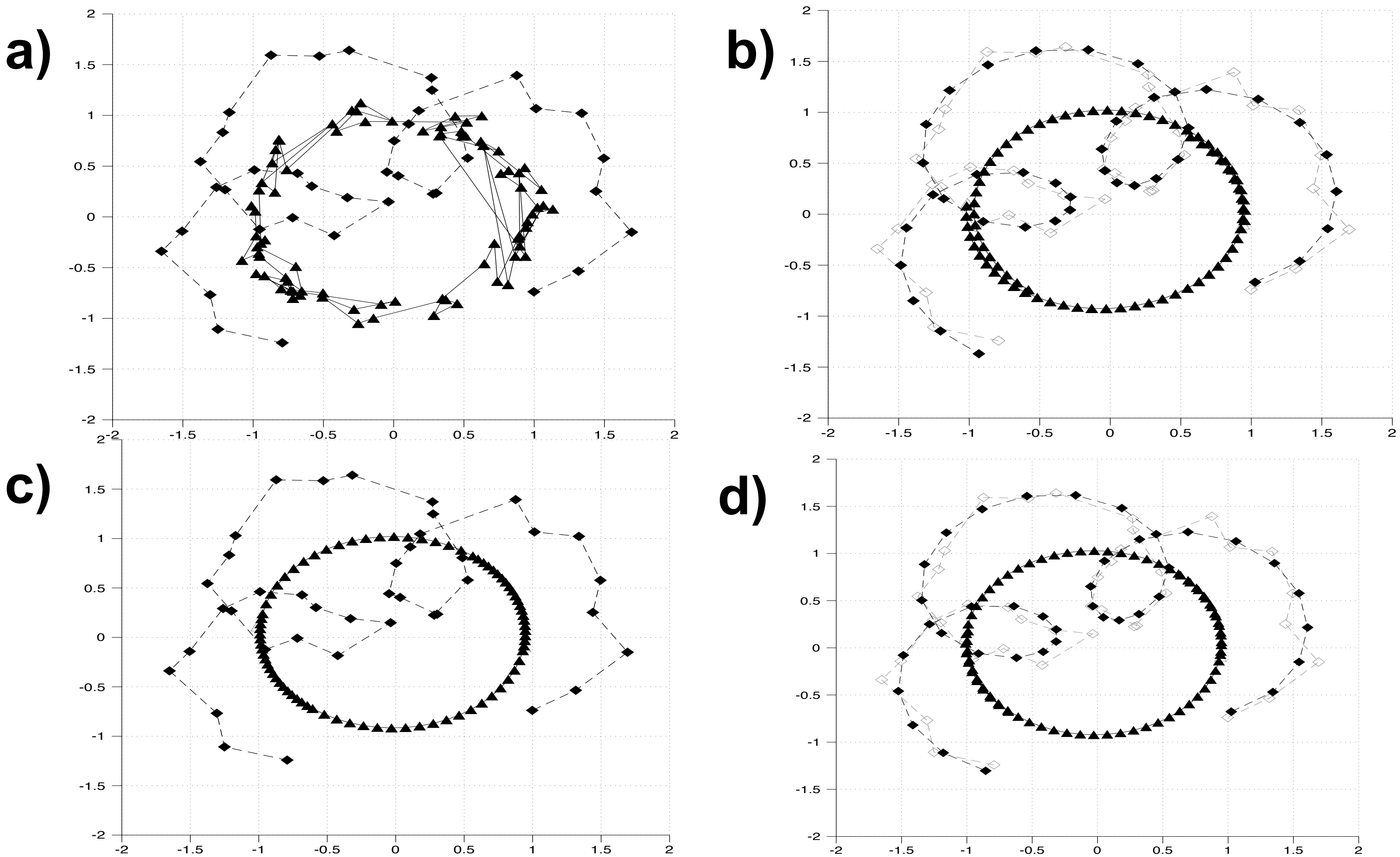
- **Tracer positions Initial locations of vortices are observed**

- **Assimilation performed in a two-stage process**
  - **Estimate the unobserved vortex positions**
  - **Start from vortex initial positions**
  - **Form noisy trajectory of vortices**
  - **Apply gradient descent**
  - **Reset tracer positions to observed and repeat with improved vortex positions**



Gradient descent assimilation for the point-vortex model

2-vortex model: Partial observations around the saddle points



Gradient descent assimilation for the point-vortex model

Summary

- **Gradient descent assimilation has been applied to the 2-vortex model**
- **Tracer and vortex trajectories are successfully generated for:**
  - **Full observations**
  - **Partial observations**
  - **Many Initial conditions including around saddle points**
- **Work is ongoing for further comparisons with other DA methods**

E. B. Suckling and L. A. Smith, Gradient descent assimilation for the point-vortex model, ***in preparation.***

Thank You!

Contact Me

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