A Stochastic Fractional Dynamics Model of Rainfall Statistics

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Motivation

- Seek a unified model of space-time statistics of rain at various averaging scales (L,T)
- Experimental need to compare radar and gauge measurements during ground validation
- Need to capture inter-dependence of space/time averaging scales and fall-off rates of spatiotemporal correlations
- Estimate the sampling error of radar and gauge measurements

[Kundu & Travis, 2013; submitted to JGR-Atmospheres]

Basic Idea

- Describe rain in terms of a random field $R(\mathbf{x},t)$ – the instantaneous point rain rate (a mathematical abstraction) obeying a stochastic dynamical equation.
- Space-time stationary, homogeneous, isotropic statistics
- Derive statistical properties (2nd moment statistics) of spatial averages at an instant $r_A(t)$ (radar estimates) and time averages at a point $r_T(\mathbf{x})$ (gauge estimates) from a common parameterized framework.
- Model parameters tuned to radar statistics should describe gauge statistics *without any further adjustment*.

Description of the Model

- Linear stochastic differential equation of fractional order β for the Fourier amplitudes $a(\mathbf{k},t)$ (notation: ${}_{-\infty}D_t^{\beta} \sim (d/dt)^{\beta}$) :

$$
\begin{aligned}
{}_{-\infty}D_t^{\beta} a(\mathbf{k},t) &= -\tau_k^{-\beta} a(\mathbf{k},t) + f(\mathbf{k},t) \\
\tau_k &= \tau_0 (1 + k^2 L_0^2)^{-\alpha/2} \\
\langle f(\mathbf{k},t) f^*(\mathbf{k}',t') \rangle &= (2\pi)^{3/2} F_0 \delta(\mathbf{k}-\mathbf{k}') \delta(\tau)
\end{aligned}
$$

$f(\mathbf{k},t)$ = white noise random force of amplitude F_0.

- Model parameters: strength parameter F_0, characteristic length L_0, characteristic time τ_0, spectral indices β and α
- Relaxation time of the Fourier mode **k**: $\tau_k \sim k^{-\alpha}$ $(k \to \infty)$ (short wavelength), $\tau_k \sim \tau_0$ $(k \to 0)$ (long wavelength)
- $\beta = 1$ case: Langevin Equation ('Brownian Motion').

Definition of Fractional Order Derivative

- The fractional order time derivative ${}_{-\infty}D_t^{\beta}$ is defined as the $a \rightarrow -\infty$ limit of an integral kernel

$$
{}_aD_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\beta)} \displaystyle\int_a^t \dfrac{du f(u)}{(t-u)^{1+\beta}} & ; \operatorname{Re} \beta < 0 \\ \left(\dfrac{d}{dt}\right)^n \dfrac{1}{\Gamma(n-\beta)} \displaystyle\int_a^t du\, f(u)(t-u)^{n-\beta-1} & ; n-1 < \operatorname{Re} \beta < n,\ n > 0 \end{cases}
$$

called the Riemann-Liouville derivative operator.

- The limit called the Liouville-Weyl operator has the important property

$$
{}_{-\infty}D_t^{\beta} f(t) \Leftrightarrow (-i\omega)^{\beta} F(\omega)
$$

under Fourier transform.

Description of the Model (cont.)

- Power spectrum of the model

$$S(k,\omega) = F_0\left[|\omega|^{2\beta} + 2\cos(\beta\pi/2)|\omega|^{\beta}\tau_k^{-\beta} + \tau_k^{-2\beta}\right]^{-1}$$

-- Fourier transform of space-time covariance $c(\rho,\tau)$

- The spatial covariance at zero lag has the Matérn form

$$c(\rho,0) = \gamma_0(\rho/2L_0)^{\nu}K_{\nu}(\rho/L_0),$$
$$\alpha(2\beta-1) = 2(1+\nu).$$

- Two distinct cases: (i) $\nu > 0$: point variance $c(0,0)$ is finite; (ii) $\nu < 0$: $c(0,0)$ is divergent, $c(\rho, 0) \sim \rho^{-2|\nu|}$.
- Radar data strongly indicates $\nu < 0$.

(β = 1 case: Bell and Kundu J. Climate 1996, Kundu and Bell WRR 2003)

Space-time Statistics of Radar Data

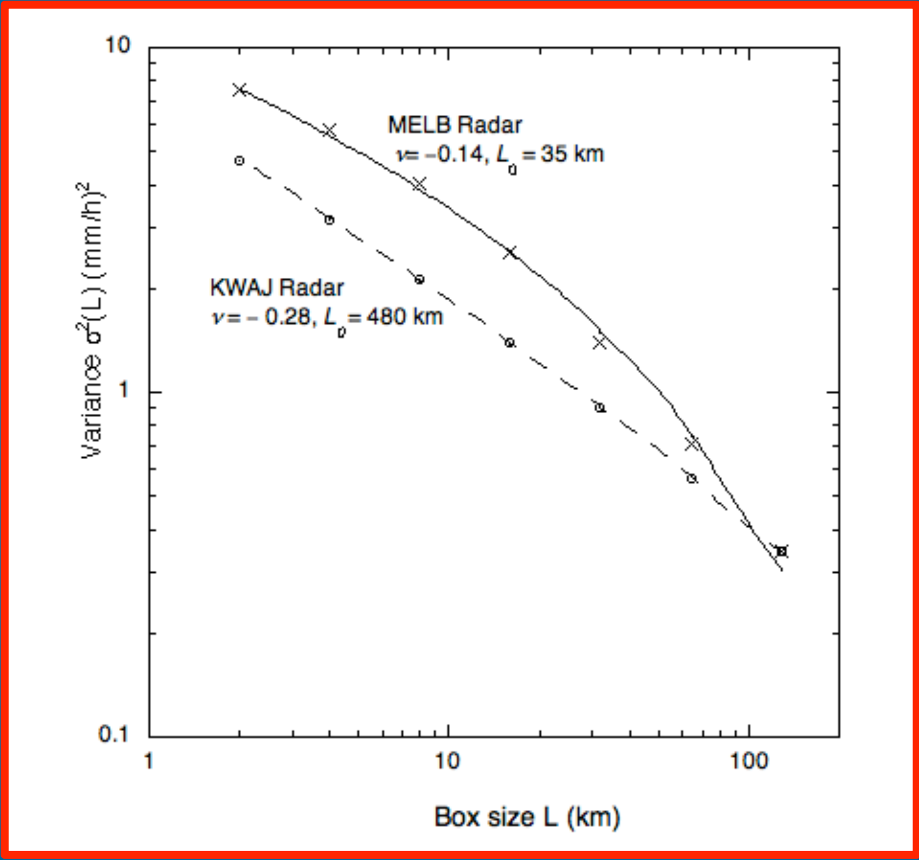
- Lagged covariance of rain rate area-averaged over two $L \times L$ squares A and A' spatially separated by distance **s** and time τ :

$$\Gamma_{AA'}(\mathbf{s},\tau) = \left(1/A^2\right)\int_A d^2\mathbf{x} \int_{A'} d^2\mathbf{x}'\, c(\mathbf{s}+\mathbf{x}'-\mathbf{x},\tau)$$

- Variance for a box A: $\sigma_A{}^2 = \Gamma_{AA}(0,0) \approx A + BL^{-2|\nu|}$ as $L \rightarrow 0$.
- Spatial correlation at zero lag: $\Phi_{AA'}(\mathbf{s},0) = \Gamma_{AA'}(\mathbf{s},0)/\sigma_A{}^2$
- Lagged autocorrelation for a box A: $\Phi_{AA}(0,\tau) = \Gamma_{AA}(0,\tau)/\sigma_A{}^2$

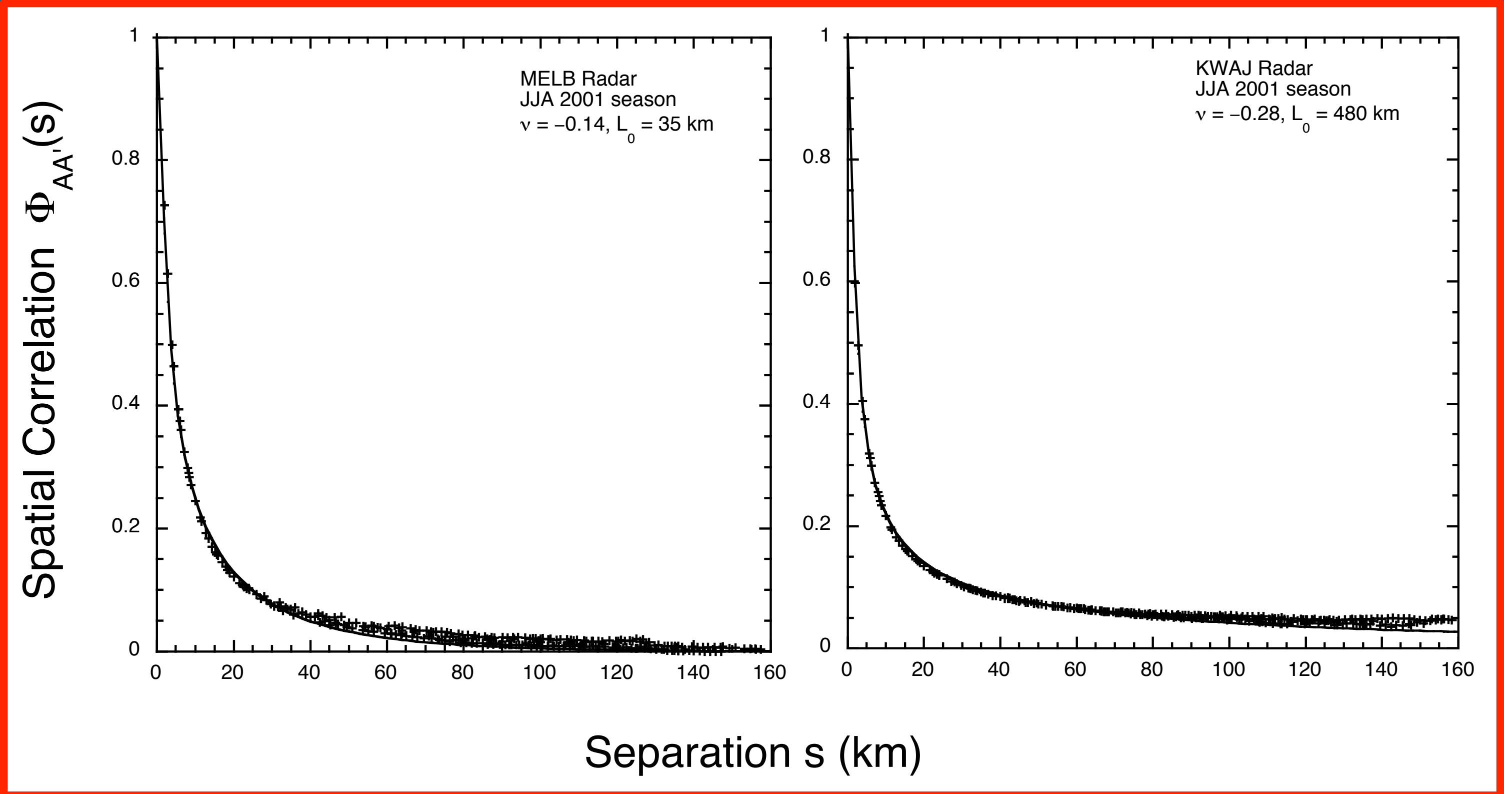
Model Fit to Radar Data

- TRMM GV Data (2A53) :
  MELB radar (Melbourne FL)
  KWAJ radar (Kwajalein Atoll, Rep. Marshall Islands, Pacific Ocean)
- Radar data gridded into 151 × 151 array of 2 × 2 km pixels.
- Results from JJA 2001 season
- Model parameters γ_0 (F_0), ν, L_0 and τ_0 fit from radar data



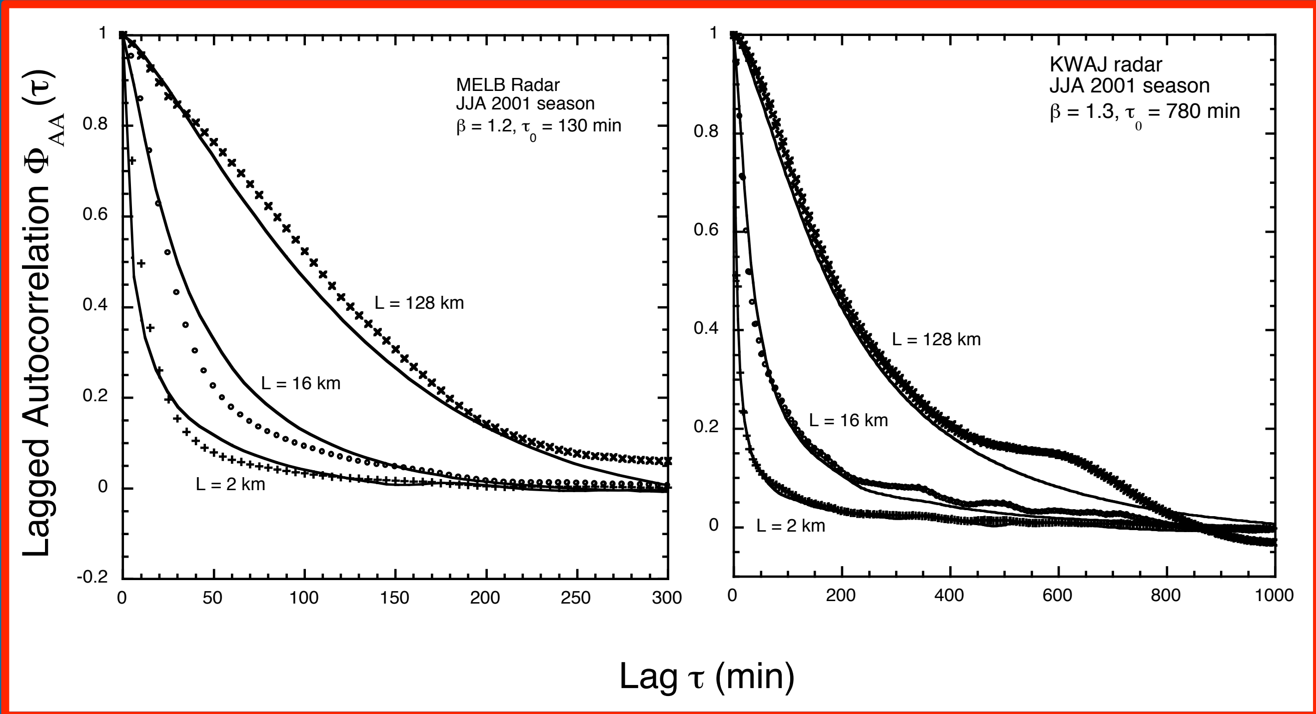
Variance of Radar Averages

Model Fit to Radar Data (cont.)



Spatial Correlation of 2 km Radar Pixels

Model Fit to Radar Data (cont.)

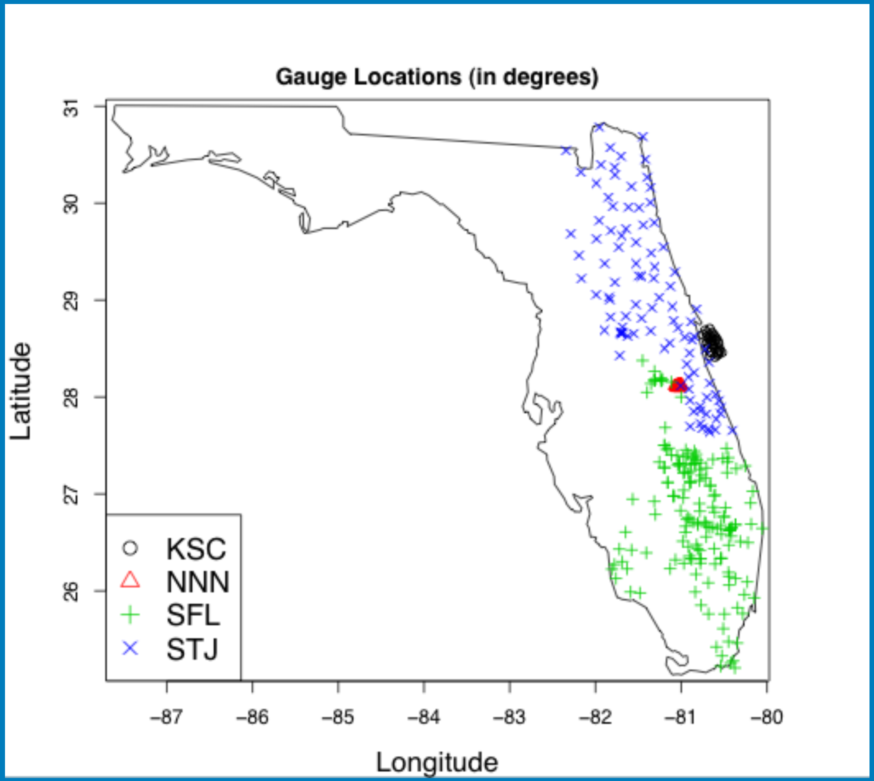


Lagged Autocorrelation for MELB and KWAJ Radars

- Model Parameters: JJA 2001 season
- MELB: $\gamma_0 = 0.994\ (\text{mm/h})^2$, $\nu = -0.14$, $L_0 = 35$ km, $\beta = 1.2$, $\tau_0 = 130$ min
- KWAJ: $\gamma_0 = 0.056\ (\text{mm/h})^2$, $\nu = -0.28$, $L_0 = 480$ km, $\beta = 1.3$, $\tau_0 = 780$ min

Comparing with Observed Gauge Statistics

- TRMM GV Data
- Radar (2A53) & Gauge (2A56)
- Nov. 1997 – present
- Radar FOV 300 km diameter
- 1-min averaged rain rates
- 300+ Tipping Bucket gauges
- Eastern Florida



- Statistics computed for 3 month season JJA 2001
- Radar statistics computed for the central 128 km box
- 1 min. data aggregated to yield gauge statistics
- Some artifacts from cubic spline fitting of TB data for $T < 10$ min

Definition of the Gauge Statistics

- Spatial Covariance of a gauge pair separated by distance ρ averaged over a time window T

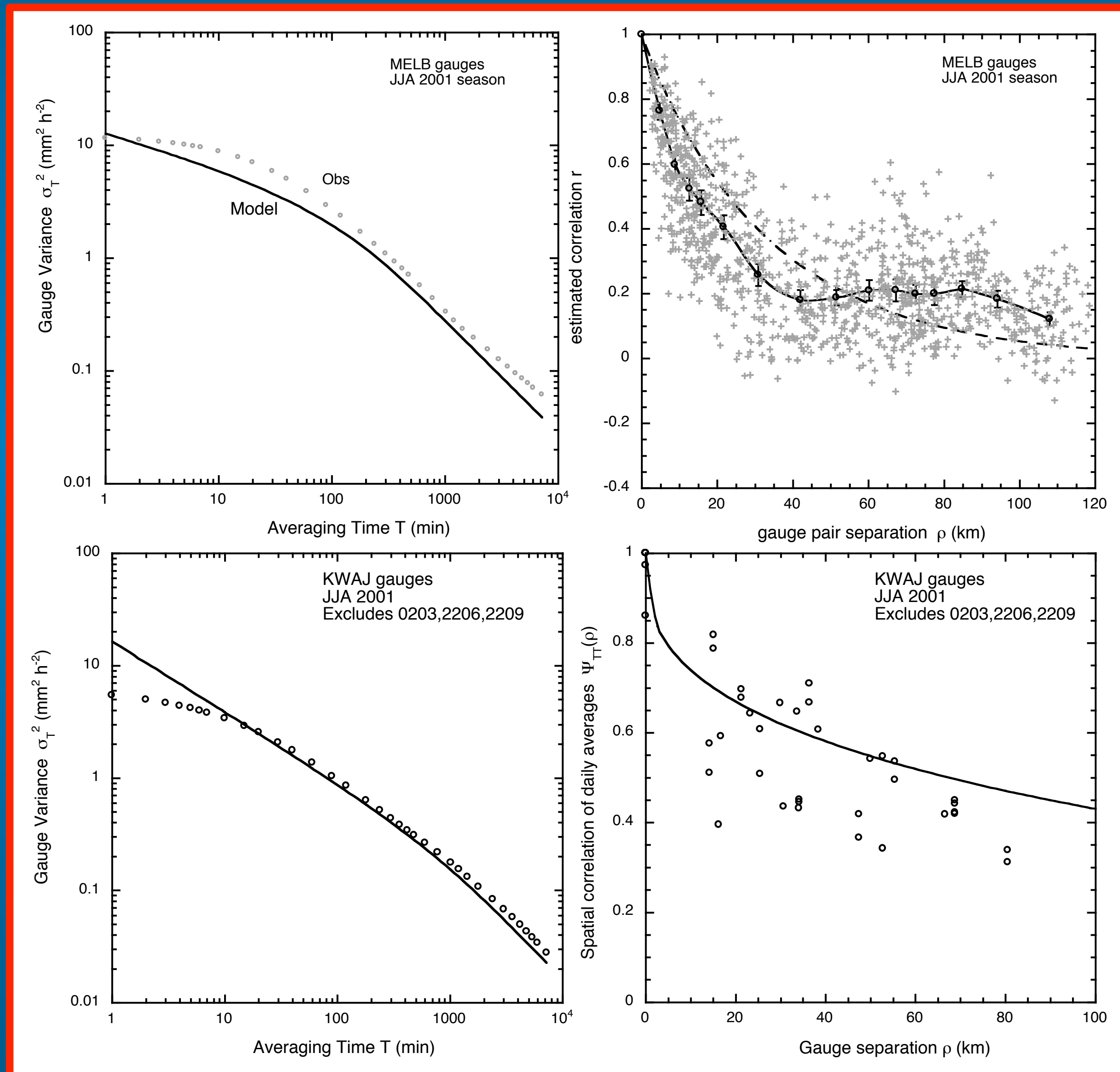
$$\Gamma_{TT}(\rho) = (1/T)^2 \int_0^T dt \int_0^T dt' \, c(\rho, t - t')$$

- Variance of the time average: $\sigma_T^2 = \Gamma_{TT}(0) \approx$ const. $T^{-2|\nu|/\alpha}$ as $T \to 0$.
- The spatial correlation of a gauge pair: $\Psi_{TT}(\rho) = \Gamma_{TT}(\rho)/\sigma_T^2$

Results: Fits to Gauge Statistics

TRMM GV Data
2A56 MELB

JJA 2001



TRMM GV Data
2A56 KWAJ

JJA 2001

A Short Distance Cut-off

Asymptotic behavior of radar and gauge variances in the $\nu < 0$ case:

$$\sigma_A^2 \approx A + BL^{-2|\nu|} \;,\; \sigma_T^2 \approx \text{ const. } T^{-2|\nu|/\alpha}$$

- Power-law behavior apparent from the model plots on a log-log scale.
- Gauge data show a tendency for gauge variance σ_T^2 to approach a constant value σ_0^2 as $T \rightarrow 0$ contrary to radar data.
- A possible solution to this dilemma:
  Introduce a short distance (“ultraviolet”) wave number cut-off

$$\tau_k = \begin{cases} \tau_0 (1 + k^2 L_0^2)^{-\alpha/2} & ; k < 1/\Lambda \\ 0 & ; k > 1/\Lambda \end{cases}$$

This renders the small scale limit σ_0^2 finite:

$$\sigma_0^2 = \frac{1}{2} \gamma_0 |\Gamma(\nu)| \cdot \left[\left(1 + L_0^2 / \Lambda^2\right)^{|\nu|} - 1 \right]$$

- Consistency with radar data requires Λ to be small compared to the radar resolution (2 km). MELB data yields the estimate $\Lambda \approx$ 0.19 km and KWAJ data yields $\Lambda \approx$ 0.36 km

Conclusions

- We have described a spectral model of rainfall in terms of a stochastic differential equation of fractional order.
- The model gives a unified description of the second moment statistics of both radar and rain gauge observations. When the parameters are determined from radar data, they also fit the gauge statistics without any further adjustment.
- The new feature of the model is the use of a fractional order time derivative, which signifies the presence of memory.
- We plan to apply the model to radar-gauge statistical inter-comparison studies in the context of GPM ground validation.

Thank You!