On the “Optimal” Choice of Trial Functions for Modelling Potential Fields

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Examples of trial functions which are available on the sphere are

- Spherical harmonics (ideal frequency localization, no space localization)
- Slepian functions
- Radial basis functions (aka reproducing kernel based functions)
- Locally supported functions (very low frequency localization, high space localization)
**Spherical Harmonics**

**Pros:**
- Established system with known physical interpretation
- Efficient numerical codes available

**Cons:**
- Instabilities for irregular data grids or regional approximation
- Maximum degree is the only parameter to control the resolution
- Large data sets cause large systems of equations
- Local noise becomes global

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Local noise becomes global

Data are locally perturbed at a cap in the North and we interpolate the data by spherical harmonics (left-hand) and by radial basis functions (right-hand). The plots show the approximation errors.
Slepian Functions (due to Simons et al.)

**Pros:**
- Functions can (theoretically) be calculated for every arbitrary region
- Analysis of regional effects is easily possible

**Cons:**
- Functions are bandlimited, i.e. polynomials
- Resolution of an eigenvalue problem required (but with stable approach for simple geometries)
Radial Basis Functions

\[
K(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} k_n \ Y_{n,j}(\xi) \ Y_{n,j}(\eta) = \sum_{n=0}^{\infty} k_n \frac{2n+1}{4\pi} \ P_n(\xi \cdot \eta), \ \xi, \eta \in \Omega.
\]

Pros:

- Local noise remains local.
- Resolution can be locally controlled.
- Irregular data sets can be handled (by using splines).
- Multiresolution analysis is possible (by using wavelets).

Cons:

- Splines: large data sets require solution of large systems of equations.
- Wavelets: quadrature rule needed, but difficult for irregular data sets.

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Locally Supported Functions

\[ B_{h,k}(\xi) = \frac{(\xi \cdot \eta - h)^k}{(1-h)^k} \cdot \chi[h,1](\xi \cdot \eta), \ \xi \in \Omega \ \text{variable}, \ \eta \in \Omega \ \text{fixed centre} \]

**Pros:**
- Space-limited functions can e.g. fade out certain areas
- Numerically very easy to implement
- Smoothness can be controlled

**Cons:**
- Spectral properties are complicated (way out: up-functions by Schreiner)
- No eigenfunctions of typical geodetic/geophysical equations
We search a function $F$ such that $\mathcal{F}F = y$, where $y \in \mathbb{R}^l$ is given.

- Choose a so-called dictionary $\mathcal{D}$ of trial functions that might be useful
- Iteratively construct an expansion of the unknown signal $F$ as follows:

  - If

    $$F_n = \sum_{k=1}^{n} \alpha_k d_k$$

    with $\alpha_k \in \mathbb{R}$ and $d_k \in \mathcal{D}$ has already been constructed, then add another summand

    $$F_{n+1} = F_n + \alpha_{n+1} d_{n+1}$$

    such that the (regularized) data misfit

    $$\|y - \mathcal{F}F\|_{\mathbb{R}^l}^2 + \lambda \|F\|_{\mathcal{H}(\Omega)}^2$$

    is minimized.
We approximate the EGM 2008 potential from irregularly distributed samples (more data on the continents) with the RFMP (30 000 iterations). The dictionary contains spherical harmonics and localized trial functions (scaling function and wavelets at regular grid of centres). The approximation is finally split up into the contributions of different trial functions (for details, see Michel and Telschow 2014).

coarse to fine approximation  
added details of scale $J = 0, 1, 2$  
centres of added wavelets (dots) and coefficients (colour)
We can handle large, irregular data grids and we obtain a multiresolution analysis!
Near-surface mass anomalies from EGM2008

density approximation (from degree 3) and centres of chosen RBFs
Water mass transport in the Amazon area in 2008
References


