THE OPEN BOUNDARY EQUATION

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ANALYTICAL

Analytical approximation

We pose an approximate solution

$$\zeta(x,t) = \eta(x)\Psi(\varphi(x,t)+\epsilon(x)), \quad (11)$$
$$u(x,t) = \upsilon(x)\Omega(\varphi(x,t)), \quad (12)$$

- Ω and Ψ are any periodic function,
- η and υ are amplitudes of water level and velocity,
- φ is the variable part of the argument which is the same for ζ and u,
- ϵ is the phase lag.

$$\zeta_t = \eta\Psi_\varphi\varphi_t, \quad \zeta_x = \eta\Psi_\varphi(\varphi_x+\epsilon_x)+\eta_x\Psi,$$
$$u_t = \upsilon\Omega_\varphi\varphi_t, \quad u_x = \upsilon\Omega_\varphi\varphi_x + \upsilon_x\Omega. \quad (13)$$

$$\zeta_t u_x = \eta\Psi_\varphi\varphi_t\upsilon\Omega_\varphi\varphi_x + \eta\Psi_\varphi\varphi_t\upsilon_x\Omega, \quad (14)$$
$$u_t\zeta_x = \upsilon\Omega_\varphi\varphi_t\eta\Psi_\varphi(\varphi_x+\epsilon_x) + \upsilon\Omega_\varphi\varphi_t\eta_x\Psi. \quad (15)$$

For illustration, equation (10) applies if

1. $\epsilon_x = 0$,
   which is valid for a wave with a constant phase lag between water level and velocity along x.
2. $\frac{1}{\eta}\eta_x\frac{1}{\Omega}\Omega_\varphi = \frac{1}{\upsilon}\upsilon_x\frac{1}{\Psi}\Psi_\varphi$,
   which, if there is proportionality, represents a requirement on the degree of damping of the velocity and water level amplitudes,
   if: $\Omega = k\Psi$, then: $\frac{1}{\eta}\eta_x = \frac{1}{\upsilon}\upsilon_x$.

Derivation in a 'perfect' channel

In a prismatic, frictionless channel with a horizontal bed, equations (1), (2), (4) and (5) reduce to

$$h_t + uh_x + hu_x = 0, \quad (16)$$
$$u_t + uu_x + gh_x = 0, \quad (17)$$
$$R_{1,t} + (u+\sqrt{gh})R_{1,x} = 0, \quad (18)$$
$$R_{2,t} + (u-\sqrt{gh})R_{2,x} = 0, \quad (19)$$

We assume

$$x = \{0,\infty\}, \quad R_2(x,0) = C. \quad (20)$$

For sub-critical flow, any $R_2(x_2,t_2)$ will have traveled from some initial condition $R_2(x_1,0)$ within the domain, of which the value did not change according to equation (19), so

$$R_2(x,t) = C. \quad (21)$$

Differentiation of equation (6b) in space or time yields

$$\frac{1}{g}{u_x}^2 = \frac{1}{h}{h_x}^2, \quad (22)$$
$$\frac{1}{g}{u_t}^2 = \frac{1}{h}{h_t}^2. \quad (23)$$

By combining equation (22) with (23) we obtain

$$h_t u_x = u_t h_x. \quad (24)$$

For a horizontal bed $\zeta_x = h_x$, we have found equation (10).

GOVERNING EQUATIONS

'De Saint-Venant' vs 'Riemann'

In estuaries, the 'Saint-Venant' equations (1871) describe the relation between water level ζ and velocity u,

$$h_t + uh_x + hu_x - \beta uh = 0, \quad (1)$$
$$u_t + uu_x + g\zeta_x + W = 0, \quad (2)$$

with

$$h = \zeta - Z, \quad \beta = -\frac{B_x}{B}. \quad (3)$$

The observer can also choose to move with the Riemann invariants (1860), so that (1) and (2) become

$$R_{1,t} + (u+\sqrt{gh})R_{1,x} = \beta u\sqrt{gh} - gZ_x - W, \quad (4)$$
$$R_{2,t} + (u-\sqrt{gh})R_{2,x} = -\beta u\sqrt{gh} - gZ_x - W, \quad (5)$$

with

$$R_1 = u + 2\sqrt{gh}, \quad R_2 = u - 2\sqrt{gh}. \quad (6)$$

Geometry and friction

$$B = B_\infty + (B_0 - B_\infty)\exp\left(-\frac{x}{b}\right), \quad (7)$$
$$Z = -\overline{h}_0\exp\left(-\frac{x}{d}\right), \quad (8)$$
$$W = g\frac{u|u|}{K^2 h^{4/3}}. \quad (9)$$

- $B(x)$ is the width,
- $Z(x)$ is the bed level and
- $W(x,t)$ is the frictional term (Strickler).

NUMERICAL

Range

Symbol	Unit	Range	Description
$\overline{h}_0$	m	$2-40$	Avg depth at mouth
α	-	$0.01-0.6$	Amplitude/depth
γ_b	-	$0-5$ [a]	Width convergence
γ_d	-	$0-0.5$ [b]	Depth convergence
K	$m^{1/3}s^{-1}$	$20-80$	Friction constant

[a,b] Approximately 15% of the cases the value 0 is chosen.

Dimensionless parameters

$$\alpha = \frac{\eta_0}{\overline{h}_0}, \quad \gamma_b = \frac{c_0}{b\,\omega}, \quad \gamma_d = \frac{c_0}{d\,\omega}, \quad (25)$$

- η_0 is the tidal amplitude at the mouth (M2),
- $\omega = 2\pi T^{-1}$ is the tidal frequency and
- $c_0 = \sqrt{g\,\overline{h}_0}$ is the linearized tidal wave celerity.

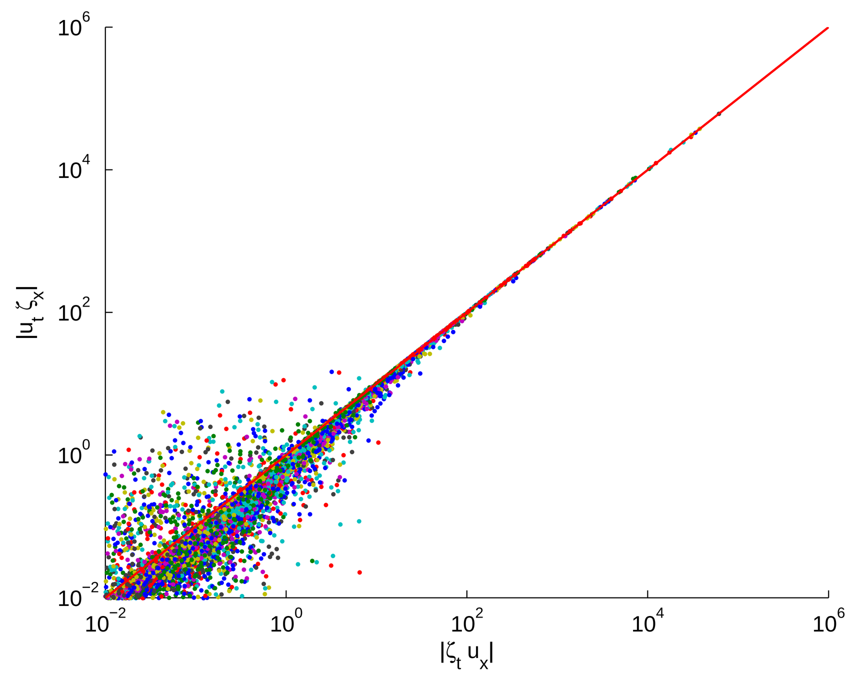

Figure 2: Picked from 100 simulations where equations (1) and (2) are approximated, each dot represents an evaluation of equation (10) at a time in the tidal cycle on a different location $x = \{0.3L, 0.7L\}$ within the estuary domain.

NEW OPEN BOUNDARY EQUATION

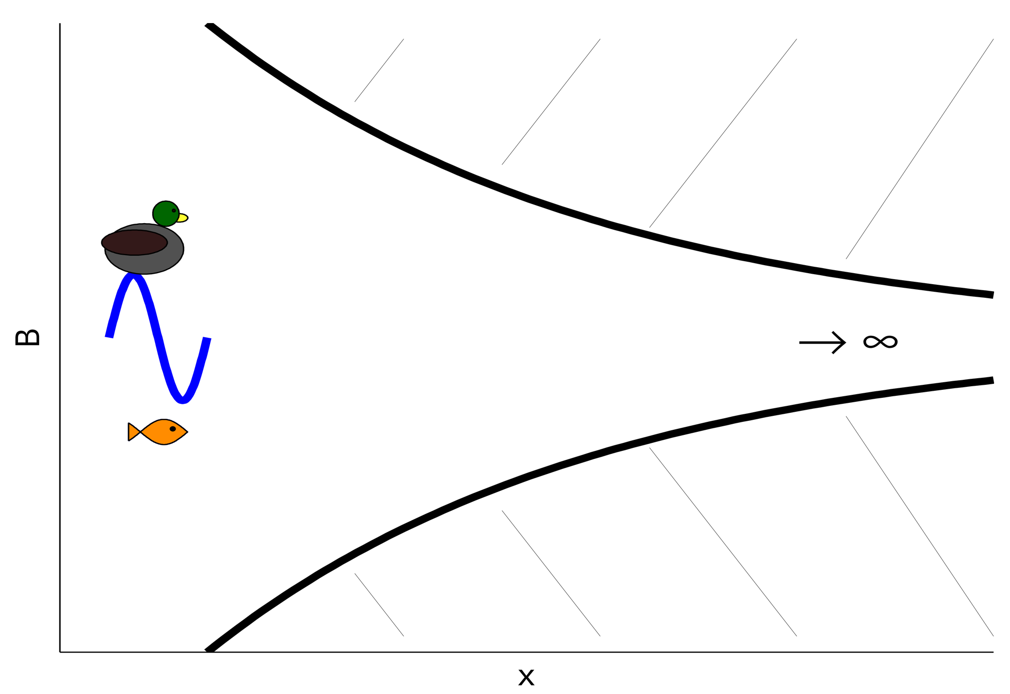

Figure 1: Tidal forcing in a converging channel of infinite length.

On the seaward boundary a tidal forcing can be imposed. But what to do on the landward side?

There is no landward boundary condition, since the domain is infinitely long. To make up for this, we found a new, additional equation,

$$\zeta_t u_x = u_t\zeta_x. \quad (10)$$

INTERPRETATION

Open boundary

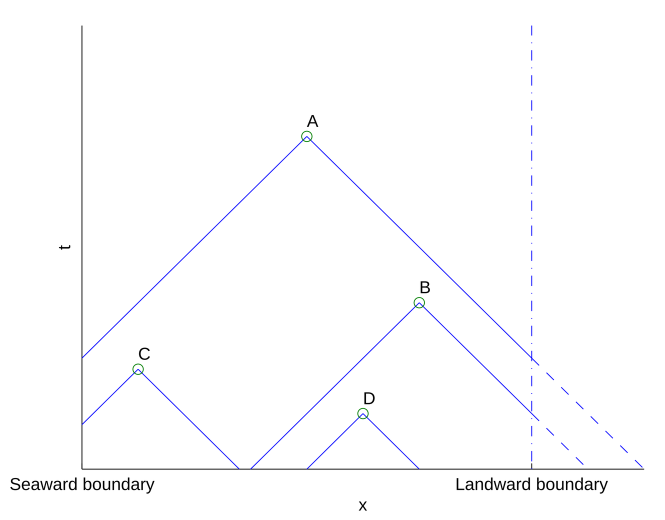

Figure 3: Each invariant trajectory (which depends on h and u) leads back to an initial condition or a boundary condition. A part of the trajectory of R_2 lies outside the domain (the dashed lines). For a 'perfect' channel this does not matter, since the value of R_2 does not change along its trajectory (equation (19)). For a 'non-perfect' channel, the value of R_2 depends on the local solution of the water depth h and the velocity u (equation (5)), which lies outside the domain.

Domain-splitting

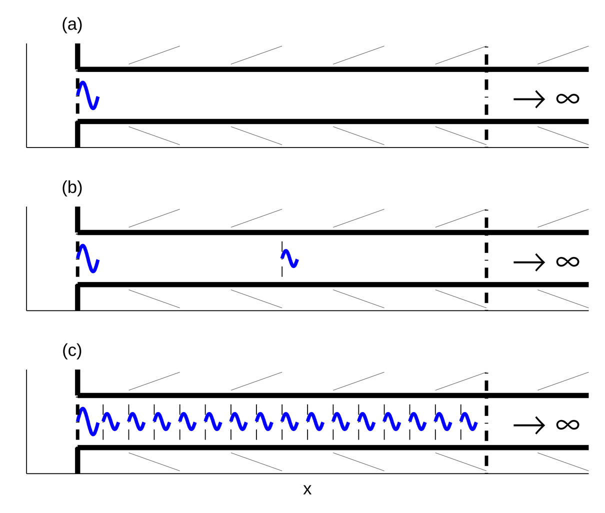

Figure 4: The same simulation is displayed on a domain with: (a) a single open boundary; (b) two open boundaries; or (c) a large number (theoretically infinite) of open boundaries. Each internal open boundary should contain the same information. This means that the single domain can be described by an infinite amount of domains. The boundary condition, which applies locally, becomes an additional equation valid at each location in the domain.

APPLICATIONS

Reconstructing the velocity

$$f_1 u^2 + f_2 u + f_3 = 0, \quad (26)$$

$$f_1 = \zeta_x(h_x - \beta h \pm f_W h), \quad f_2 = h_t h_x + h_t\zeta_x - h_t\beta h,$$
$$f_3 = {h_t}^2 - g h{\zeta_x}^2, \quad f_W = \frac{W}{u|u|}.$$

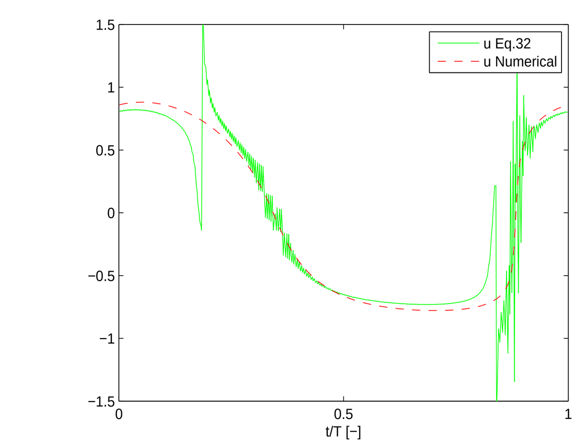

Figure 5: The resulting velocity from equation (26) is compared with the numerical velocity, with $\eta_0 = 1.2$ m, $\overline{h}_0 = 28$ m, $b = 280$ km, $d = 1111$ km, $K = 50\ m^{1/3}s^{-1}$.

Estimation of friction

$$W = \frac{f_4 - f_5 f_6}{f_7}, \quad (27)$$

$$f_4 = g(h_t)^2 - h(u_t)^2, \quad f_5 = u(Z_x + \beta h),$$
$$f_6 = \left(\frac{u^2}{2} + g\,h\right)_t, \quad f_7 = (u\,h)_t.$$

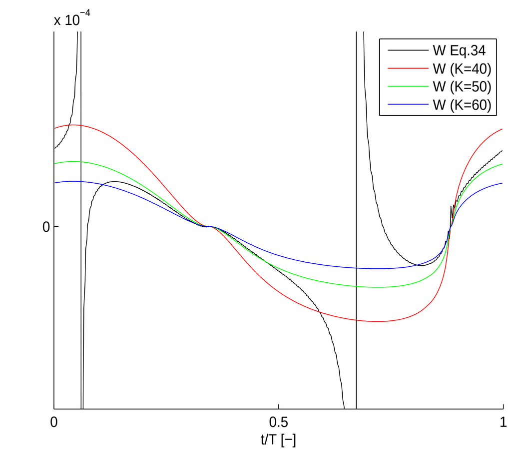

Figure 6: Equation (27) is compared with three estimates of the friction term W using equation (9). The parameters are from the same simulation as in Figure 5.

OSP

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