On the “Optimal” Choice of Trial Functions for Modelling Potential Fields

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to outline

Outline

Trial Functions

Examples of trial functions which are available on the sphere are

- Spherical harmonics (ideal frequency localization, no space localization) click for details
- Slepian functions click for details
- Radial basis functions (aka reproducing kernel based functions) click for details
- Locally supported functions (very low frequency localization, high space localization) click for details

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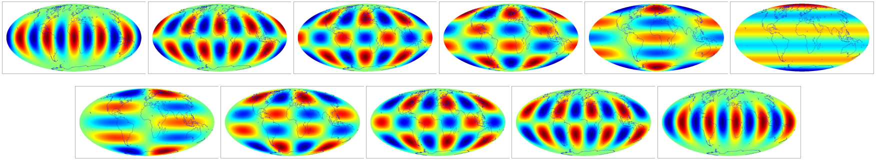
Spherical Harmonics



Pros:

- Established system with known physical interpretation
- Efficient numerical codes available

Cons:

- Instabilities for irregular data grids or regional approximation
- Maximum degree is the only parameter to control the resolution
- Large data sets cause large systems of equations
- Local noise becomes global example

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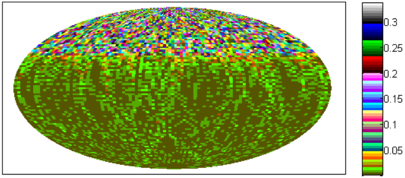
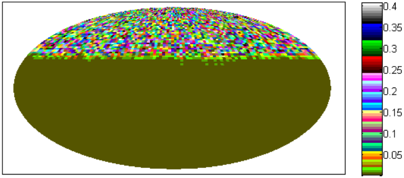
Local noise becomes global

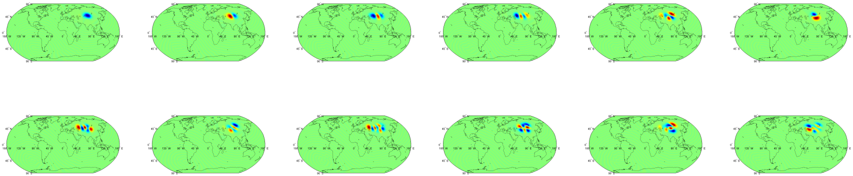
Data are locally perturbed at a cap in the North and we interpolate the data by spherical harmonics (left-hand) and by radial basis functions (right-hand). The plots show the approximation errors.





return to spherical harmonics — return to trial functions — return to outline — back to beginning

Slepian Functions (due to Simons et al.)



Pros:

- Functions can (theoretically) be calculated for every arbitrary region
- Analysis of regional effects is easily possible

Cons:

- Functions are bandlimited, i.e. polynomials
- Resolution of an eigenvalue problem required (but with stable approach for simple geometries)

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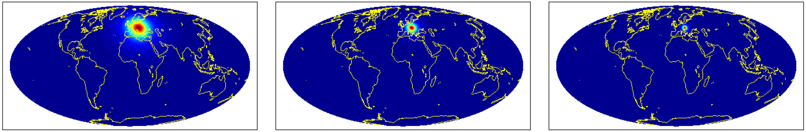
Radial Basis Functions

$$K(\xi,\eta) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} k_n \, Y_{n,j}(\xi) \, Y_{n,j}(\eta) = \sum_{n=0}^{\infty} k_n \, \frac{2n+1}{4\pi} \, P_n(\xi \cdot \eta), \; \xi,\eta \in \Omega.$$



Pros:

- Local noise remains local.
- Resolution can be locally controlled.
- Irregular data sets can be handled (by using splines).
- Multiresolution analysis is possible (by using wavelets).

Cons:

- Splines: large data sets require solution of large systems of equations.
- Wavelets: quadrature rule needed, but difficult for irregular data sets.

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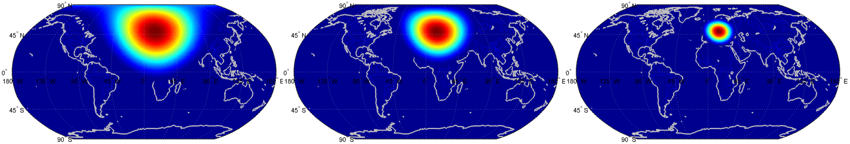
Locally Supported Functions

$$B_{h,k}(\xi) = \frac{(\xi\cdot\eta-h)^k}{(1-h)^k} \cdot \chi_{[h,1]}(\xi\cdot\eta),\ \xi \in \Omega \text{ variable},\ \eta \in \Omega \text{ fixed centre}$$



Pros:

- Space-limited functions can e.g. fade out certain areas
- Numerically very easy to implement
- Smoothness can be controlled

Cons:

- Spectral properties are complicated (way out: up-functions by Schreiner)
- No eigenfunctions of typical geodetic/geophysical equations

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RFMP – A Best Basis Algorithm for Inverse Problems

We search a function F such that $\mathcal{F}F = y$, where $y \in \mathbb{R}^l$ is given.

- Choose a so-called dictionary $\mathcal{D}$ of trial functions that might be useful
- Iteratively construct an expansion of the unknown signal F as follows:
  - If
  $$F_n = \sum_{k=1}^{n} \alpha_k \, d_k$$
  with $\alpha_k \in \mathbb{R}$ and $d_k \in \mathcal{D}$ has already been constructed, then add another summand
  $$F_{n+1} = F_n + \alpha_{n+1} \, d_{n+1}$$
  such that the (regularized) data misfit
  $$\|y - \mathcal{F}F\|^2_{\mathbb{R}^l} + \lambda \|F\|^2_{\mathcal{H}(\Omega)}$$
  is minimized.

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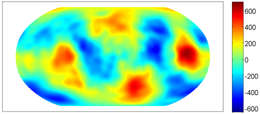
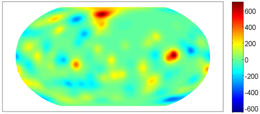
Multiscale Approximation of the Potential

We approximate the EGM 2008 potential from irregularly distributed samples (more data on the continents) with the RFMP (30 000 iterations). The dictionary contains spherical harmonics and localized trial functions (scaling function and wavelets at regular grid of centres). The approximation is finally split up into the contributions of different trial functions (for details, see Michel and Telschow 2014).

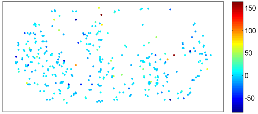
coarse to fine approximation



added details of scale $J = 0, 1, 2$



centres of added wavelets (dots) and coefficients (colour)



for the results, click here — Example 2 — Example 3 — return to outline —

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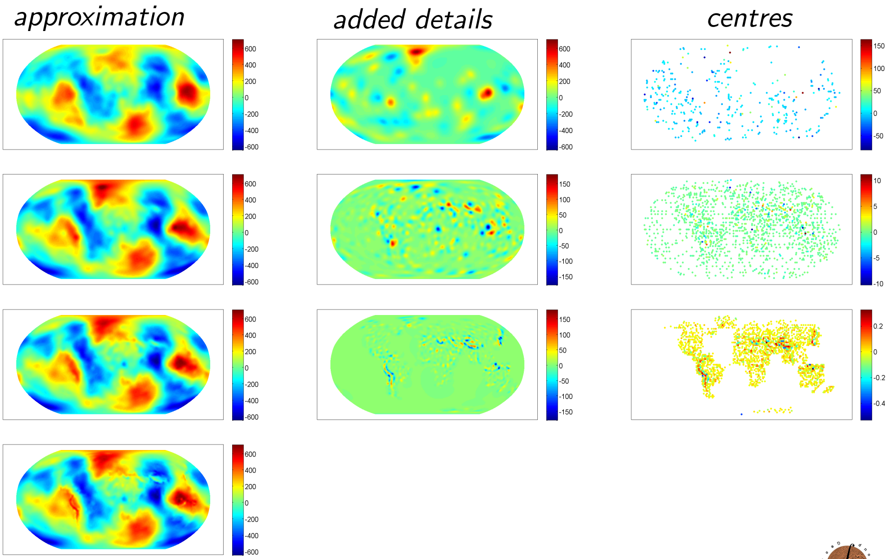
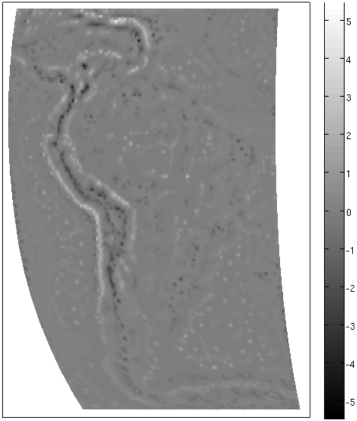


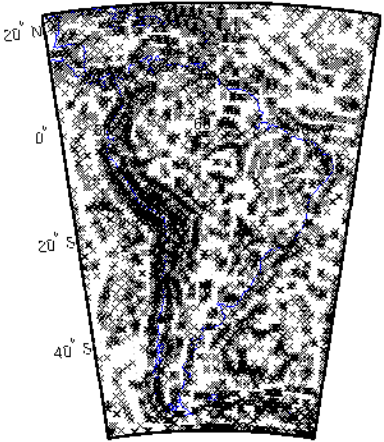
We can handle large, irregular data grids <u>and</u> we obtain a multiresolution analysis!

for explanations — outline — beginning

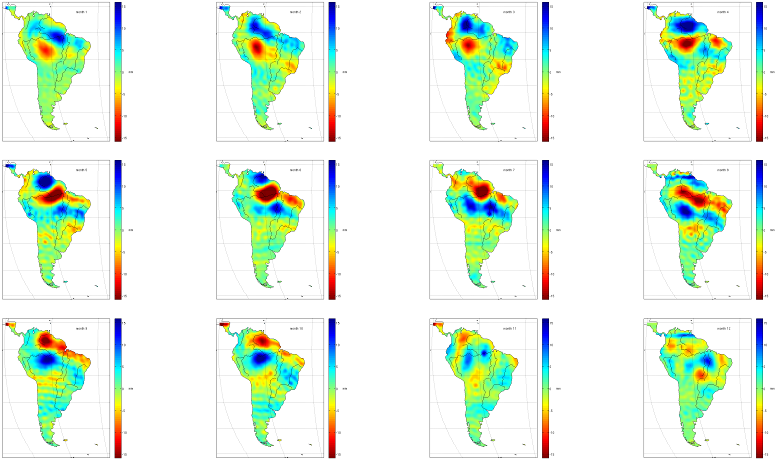
Near-surface mass anomalies from EGM2008

density approximation (from degree 3) and centres of chosen RBFs





Water mass transport in the Amazon area in 2008



Example 1 — Example 2 — outline — beginning

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