Variational Water Wave Modelling: from Continuum to Experiment

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Introduction

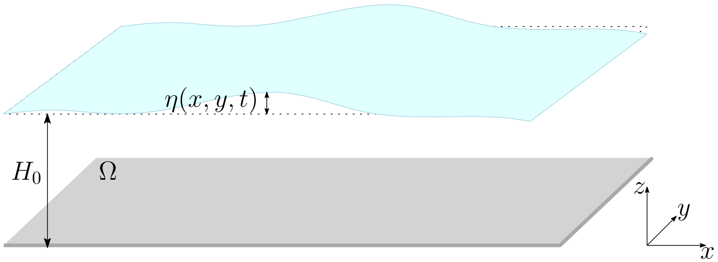
- Water waves: inviscid, incompressible and irrotational $\Rightarrow$ velocity potential, such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$.
- Variational and Hamiltonian dynamics.
- *Usual approach:* water wave problems governed by autonomous Hamiltonian dynamics.
- *Practically:* time-dependent internal or boundary conditions $\Rightarrow$ non-autonomous dynamics: explicit time-dependence via forcing or dissipation.

Objectives

- Derive reduced water wave model [Benney & Luke, 1964].
- Remain entirely within a variational framework.
- Obtain numerical simulations via finite element formulations.
- Compare against a soliton splash event in a wave channel with a removable sluice gate and a contraction [Bokhove et al., 2011].

Potential flow theory

- Velocity potential $\phi(x, y, z, t)$ and surface elevation $\eta(x, y, t)$



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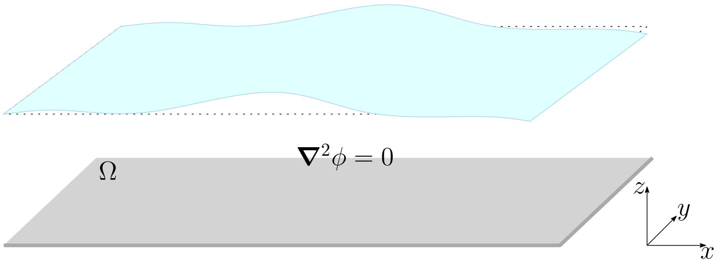

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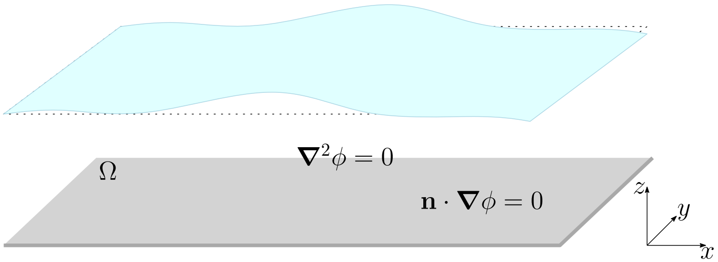

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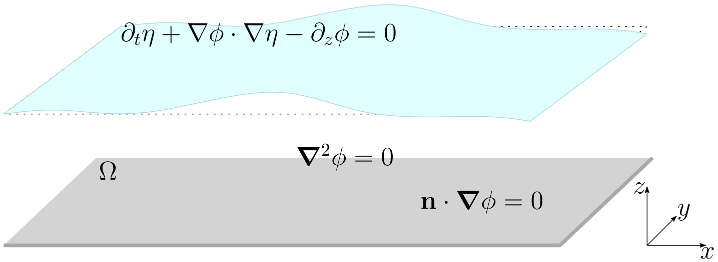

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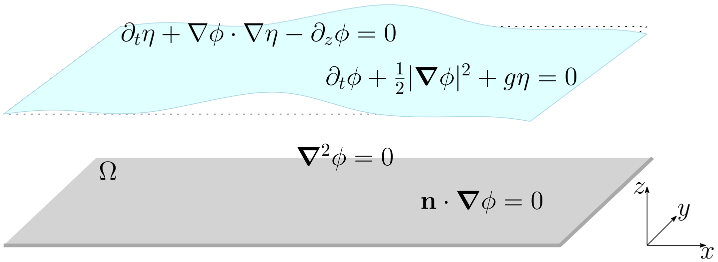

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Water wave equations

$$\begin{aligned}
\boldsymbol{\nabla}^2\phi &= 0 && \text{in} && \Omega \quad \& \quad 0 < z < H_0 + \eta \\
\mathbf{n} \cdot \boldsymbol{\nabla}\phi &= 0 && \text{at} && \partial\Omega \quad \& \quad z = 0 \\
\partial_t\eta + \nabla\phi \cdot \nabla\eta - \partial_z\phi &= 0 && \text{at} && z = H_0 + \eta \\
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- Can be derived from Luke's variational principle [Luke, 1967]

$$0 = \delta \int_0^T \iint_\Omega \int_0^{H_0+\eta} \partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g(z - H_0)\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t.$$

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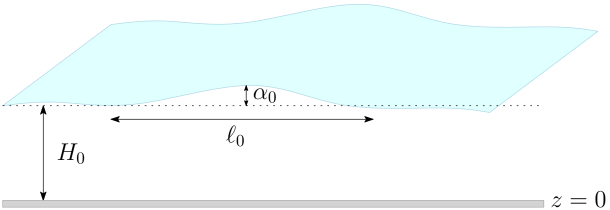
Scalings



3 length scales:

- α_0: typical amplitude
- ℓ_0: typical wave length
- H_0: water depth at rest

2 parameters:

- Small wave amplitude $\epsilon = \alpha_0/H_0 \ll 1$
- Long-waves in shallow water $\mu = (H_0/\ell_0)^2 \ll 1$

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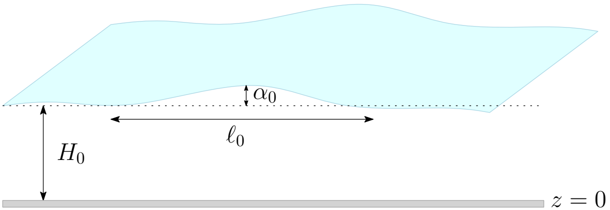

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Asymptotic expansion

- Rescale the problem by introducing non-dimensional variables.
- Potential at bottom

$$\Phi(x, y, t) = \phi(x, y, z = 0, t).$$

- Expand ϕ about Φ in powers of μ: [Pego & Quintero, 1999]

$$\phi = \Phi - \frac{\mu z^2}{2} \Delta \Phi + \frac{\mu^2 z^4}{24} \Delta^2 \Phi + \dots$$

- The VP can be integrated in z and up to $\mathcal{O}(\mu \epsilon^2)$ becomes

$$0 = \delta \int_0^T \iint_\Omega \eta \partial_t \Phi - \frac{\mu}{2} \eta \partial_t \Delta \Phi + \frac{1}{2}(1 + \epsilon \eta) |\nabla \Phi|^2 + \frac{\mu}{3} (\Delta \Phi)^2 + \frac{1}{2} \eta^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t.$$

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Benney-Luke equations

$$\partial_t \Phi - \frac{\mu}{2}\partial_t \Delta \Phi + \frac{\epsilon}{2}\left|\nabla \Phi\right|^2 + \eta = 0$$

$$\partial_t \eta - \frac{\mu}{2}\partial_t \Delta \eta + \nabla \cdot \big((1+\epsilon\eta)\nabla \Phi\big) - \frac{2\mu}{3}\Delta^2 \Phi = 0$$

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Challenges:

- Explicit time-dependence

Benney-Luke equations

$$\partial_t \Phi - \frac{\mu}{2}\partial_t \Delta \Phi + \frac{\epsilon}{2}\left|\nabla \Phi\right|^2 + \eta - \eta_R = 0$$
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Challenges:

- Explicit time-dependence $\Rightarrow$ gravitational potential $\eta_R(x,t)$

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Challenges:

- Explicit time-dependence $\Rightarrow$ gravitational potential $\eta_R(x,t)$
- Reduction of highest derivative order $\Rightarrow$ auxiliary variable $q = -\frac{2}{3}\Delta\Phi$

Modified Benney-Luke equations

$$
\begin{aligned}
\partial_t \Phi - \frac{\mu}{2}\partial_t \Delta \Phi + \frac{\epsilon}{2}\left|\nabla \Phi\right|^2 + \eta - \eta_R &= 0 \\
\partial_t \eta - \frac{\mu}{2}\partial_t \Delta \eta + \nabla \cdot \big((1+\epsilon\eta)\,\nabla\Phi\big) + \mu \Delta q &= 0 \\
q + \frac{2}{3}\Delta\Phi &= 0
\end{aligned}
$$

Challenges:

- Explicit time-dependence $\Rightarrow$ gravitational potential $\eta_R(x,t)$
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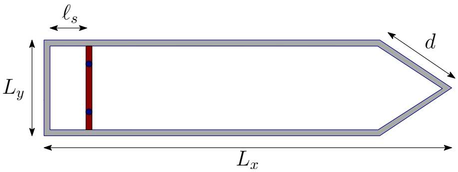
Soliton splash experiment



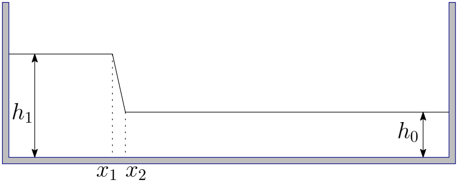
Wavetank length	$L_x = 43.63 \pm 0.1$ m
Wavetank width	$L_y = 2$ m
Wavetank height	$H = 1.2$ m
Contraction length	$d = 2.7$ m
Location of sluice gate	$\ell_s = 2.63$ m
Rest water level (high)	$h_1 = 0.9$ m
Rest water level (low)	$h_0 = 0.43$ m
Sluice gate release speed	$V_g \approx 2.5$ m/s
Sluice gate removal time	$T_s = h_1/V_g \approx 0.36$ s

Removable "sluice gate"

$$\eta_R(x,t) = \begin{cases} h_1 - H_0(t) & \text{if } \; x < x_1 \\ (h_1 - H_0(t))\left(1 - \frac{x-x_1}{x_2-x_1}\right) & \text{if } \; x_1 \leq x \leq x_2 \\ 0 & \text{if } \; x > x_2, \end{cases}$$

with

$$H_0(t) = \begin{cases} h_1 + (h_0 - h_1)\frac{(T_s - t)}{T_s} & \text{for } \; t < T_s \\ h_1 & \text{for } \; t \geq T_s. \end{cases}$$



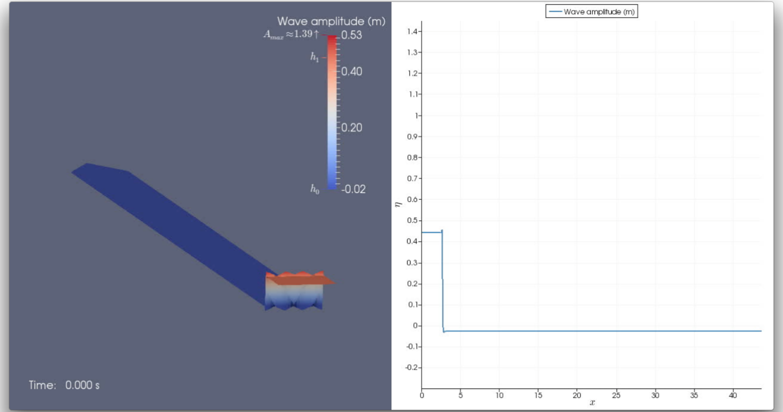
Numerical Implementation

Firedrake

An automated system for the solution of PDEs using the Finite Element Method (FEM).

- Continuous Galerkin Finite Element Method (CGFEM).
- Quadrilateral mesh with quadratic Lagrange polynomials.
- Symplectic 2nd- or 3rd-order time integrators.

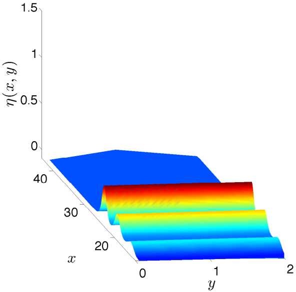
Numerical Implementation



Experiment vs. Simulation

$t = 8$ s

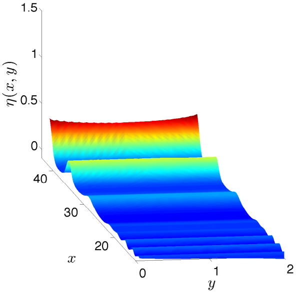
$t = 8.16$ s



Experiment vs. Simulation

$t = 14$ s

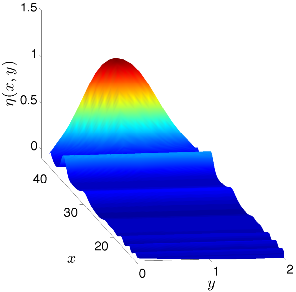
$t = 14.44$ s



Experiment vs. Simulation

$t = 15 \pm 0.5$ s

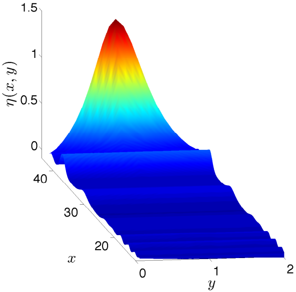
$t = 15.08$ s



Experiment vs. Simulation

$t = 15 \pm 0.5$ s

$t = 15.27$ s



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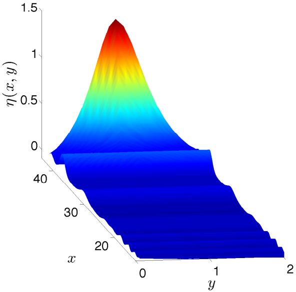


$\Rightarrow$Maximum wave amplitude $A_{max} \approx 4A_0$.

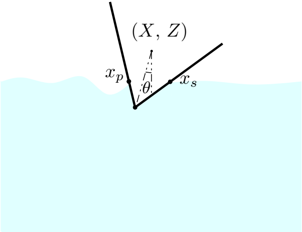
Bore-soliton splash Experiment

If $h_0 = 0.41$ m instead of $h_0 = 0.43$ m $\Rightarrow A_{max} \approx 10A_0!$

Future Work

- Apply Benney-Luke approximation to flows in vertical Hele-Shaw cells $\rightarrow$ damping, forcing and surface tension.

- Explore variational water wave methods for ships in modest to heavy seas $\rightarrow$ water wave dynamics coupled to water line and ship dynamics.



Conclusions

- Mathematical modelling of variational water wave dynamics.
- Reduced weakly nonlinear model for shallow and long waves.
- Time-dependent gravitational potential mimicking a removable "sluice gate".
- Discretised the model using finite element methods.
- Validation: soliton splash event in wave tank with a sluice gate.

[Bokhove O. and Kalogirou A., 2015. *Lecture Notes, London Mathematical Society.* (Submitted)]