Spectral evolution of weakly nonlinear random waves: kinetic description vs direct numerical simulations

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motivation

Most existing approaches to the modelling of long-term evolution of random weakly nonlinear waves are based on the wave kinetic equation (KE). In the water wave context, this equation is referred to as the Hasselmann equation (Hasselmann 1962).

$$\frac{dn(\mathbf{k},\mathbf{x},t)}{dt} = S_{input} + S_{diss} + S_{nl}$$

where $n(\mathbf{k},t)$ is the 2D wave action spectrum. The interaction term S_{nl}, dominant for energy carrying waves, is derived from first principles employing an asymptotic procedure based upon smallness of nonlinearity parameter ε and a number of additional assumptions:

$$S_{nl} = 4\pi \int T^2_{0123} f_{0123} \delta_{0+1-2-3} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\mathbf{k}_{123}, \quad (1)$$

where $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, $n_i \equiv n(\mathbf{k}_i)$, $\delta_{0+1-2-3} \equiv \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$ and T_{0123} is given by an explicit but long expression.

motivation

The KE (Hasselmann equation) is based on two major assumptions:

- quasi-gaussianity, implied by the statistical closure
- quasi-stationarity, implied by the large-time limit

Quasi-stationarity means that the Hasselmann equation is not applicable to the situations with rapid changes of the environment, such as wind gusts. Due to the lack of alternatives, this fact is usually ignored, and the Hasselmann theory is used to model the response to an instant and sharp increase or decrease of wind (e.g. Young & van Agthoven 1997).
Quasi-gaussianity can be violated during a rapid transformation of the spectrum, or even for parts of a quasi-stationary spectrum where the growth rates are high (e.g. on the spectral front).
A clarification of the role of both assumptions is important and relevant within and beyond the water wave context.

generalised kinetic equation (gKE)

The gKE is derived using the same statistical closure as the KE, but without the assumption of quasi-stationarity. In the derivation of the kinetic theory, we have the equation for the spectrum in terms of the higher-order cumulant $J^{(1)}_{0123}$

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im} \int T_{0123} J^{(1)}_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123},$$

and the equation for the cumulant

$$\left(\mathrm{i} \frac{\partial}{\partial t} + \Delta\omega \right) J^{(1)}_{0123} = -2 T_{0123} f_{0123},$$

where $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$. Classic KE derivation drops $\partial/\partial t$ and leads to the approximate solution for large time in terms of generalised functions

$$J^{(1)}_{0123}(t) = -2 T_{0123} \left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega) \right] f_{0123}(t),$$

where P is "principal value", δ is Dirac δ-function.

generalised kinetic equation (gKE)

The gKE is derived using the exact solution of the differential equation for the cumulant (Annenkov & Shrira 2006 JFM vol 561). The resulting equation (gKE) has the form

$$\begin{aligned}\frac{\partial n_0}{\partial t} &= 4\mathrm{Re}\int\Bigg\{T^2_{0123}\left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau\right] \\ &\quad -\frac{\mathrm{i}}{2}T_{0123}J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\Bigg\}\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123} + S_{inp/diss}\,.\end{aligned}$$

The gKE is nonlocal in time: evolution of the spectrum depends on the previous history of evolution, starting from the initial moment when the value of cumulant $J^{(1)}_{0123}(0)$ is prescribed as the initial condition.
However, the gKE can be solved iteratively. On each time step, the value of $J^{(1)}_{0123}$ is computed and taken as the new initial condition, so that the 'internal' time integration is performed over one timestep only.
Details of the algorithm: Annenkov & Shrira, Modelling transient sea states with the generalised kinetic equation, In: *Rogue and Shock Waves in Nonlinear Dispersive Media*, M.Onorato et al (eds), Springer, 2016.

direct numerical simulation (DNS-ZE)

Direct numerical simulation

- is based on the Zakharov integrodifferential equation for water waves
- does not depend on any statistical assumptions
- since the Zakharov equation plays the role of the primitive equation of the theory of wave turbulence, we refer to this model as direct numerical simulation of spectral evolution (DNS-ZE)
- at present, this is the only DNS algorithm that allows to trace the evolution of wave spectra up to $O(10^4)$ periods
- details of the algorithm – e.g. Annenkov & Shrira (2013) JFM **726** 517–546
- averaging in this study is over 100 realisations

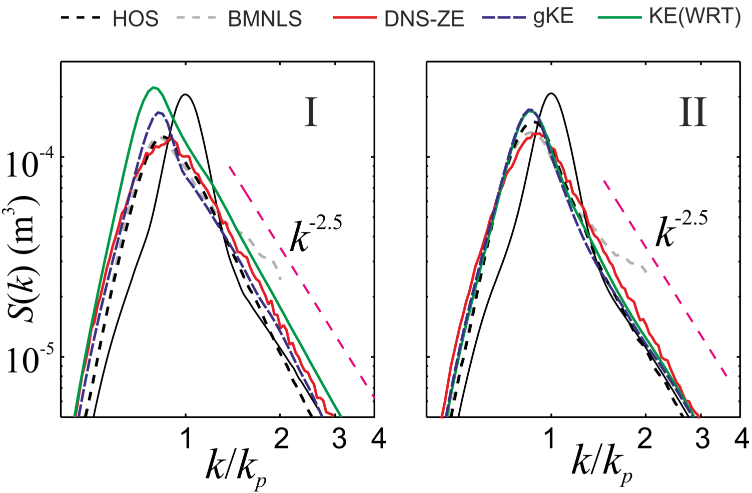
initial conditions

As initial conditions, we consider two JONSWAP spectra with the same frequency distribution ($H_s = 0.08$ m, $T_p = 1$ s, and $\gamma = 6$), different only in the initial directional distribution.

- Spectrum I ("narrow") – corresponds to $N = 840$ in the $\cos^N$ model
- Spectrum II ("wide") – corresponds to $N = 24$

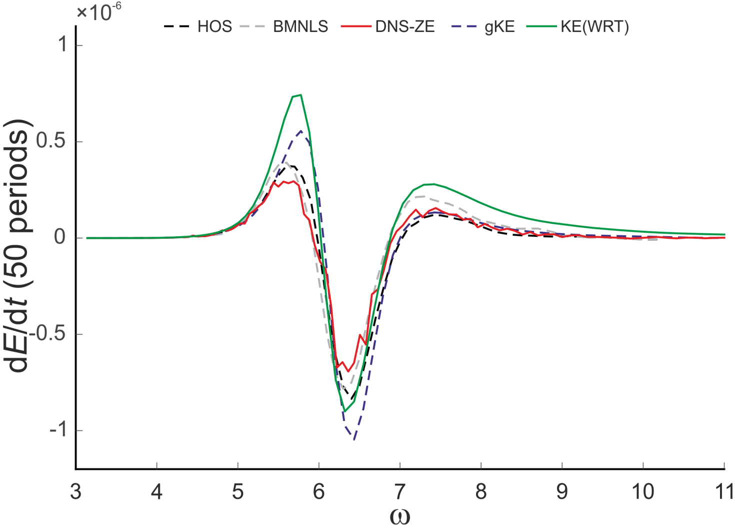
The same spectra were used as initial conditions in the experimental study by Onorato et al (2009) and numerical studies by Toffoli et al (2010) and Xiao et al (2013).
In particular, Xiao et al (2013) performed numerical simulations of the evolution (only about 150 periods) of the same initial spectra using higher-order spectral method (HOS) and broadband NLS (Dysthe equation, BMNLS).
Thus, we can consider the short-term evolution of these spectra (without wind forcing) with five different approaches, based on different sets of assumptions, and use the results for comparison and validation of the new algorithms. The KE (Hasselmann) equation is simulated using the standard WRT algorithm (code provided by Gerbrandt van Vledder)

first 150 periods



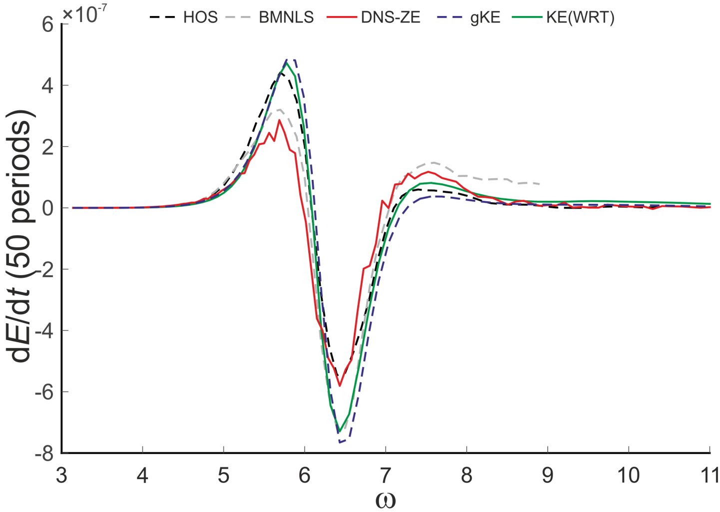
Evolution (first 150 periods) of spectrum I (narrow in angle) and II (wider in angle), with a direct comparison of 5 approaches (modified from figure 7*a,b* of Xiao et al 2013)

growth rates over first 50 periods, spectrum I



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with 5 approaches (values for HOS and BMNLS taken from figure 7 of Xiao et al 2013). Initial peak is at $\omega = 2\pi$

growth rates over first 50 periods, spectrum II



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with 5 approaches (values for HOS and BMNLS taken from figure 7 of Xiao et al 2013). Initial peak is at $\omega = 2\pi$

Angular broadening

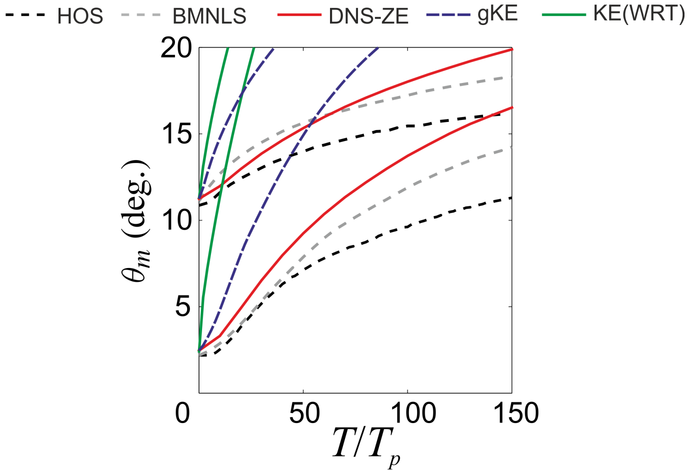
As a measure of the angular width of the spectrum, it is convenient to use the average of the second-order moment of directional distribution, defined as

$$\theta_m = \overline{\theta_2(k)},$$

$$\theta_2(k) = \left(\int_0^{\pi/2} \theta^2 D(k,\theta)\,\mathrm{d}\theta \right)^{1/2} \left(\int_0^{\pi/2} D(k,\theta)\,\mathrm{d}\theta \right)^{-1/2},$$

where $D(k,\theta)$ is the angular distribution function of the spectrum (Hwang *et al* 2000).

evolution of mean directional width



Evolution (first 150 periods) of the averaged angular spread θ_m of spectra I and II, with a direct comparison of 5 approaches (modified from figure 7*c* of Xiao et al 2013)

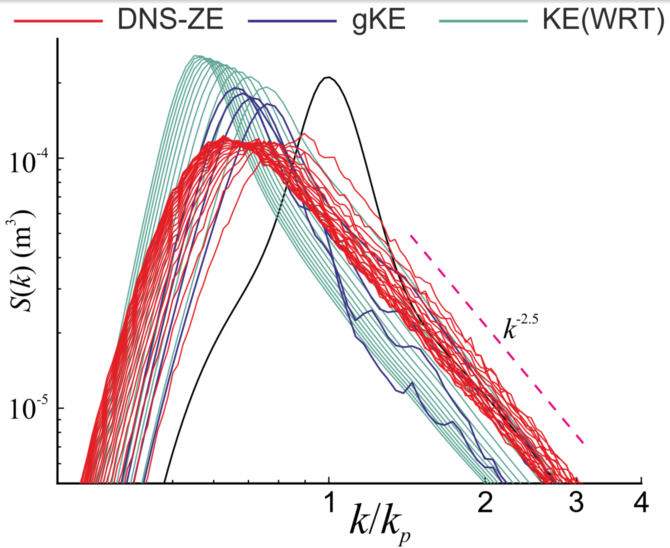
short-term evolution – discussion

Direct comparison of DNS-ZE with HOS and BMNLS, and of the two kinetic equations shows that

- KE and gKE results coincide in the wider case II
- in the narrow case I, the KE overestimates the amplitude of the spectral peak
- DNS-ZE, HOS and BMNLS are consistent with each other, but different from both kinetic equations
- the kinetic equations show more narrow spectra, with a pronounced overshoot, while the DNS algorithms give wider spectra with lower amplitude of the peak
- there is a dramatic difference in the rate of angular broadening, which is consistent between DNS-ZE, HOS and BMNLS, much higher for gKE, and even higher for the KE
- growth rates over the first 50 periods are higher for the kinetic equations than for the DNS algorithms

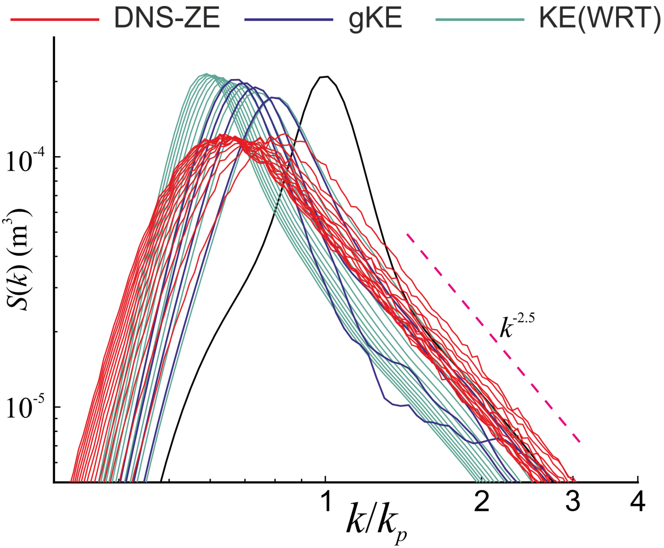
This validates both the gKE and the DNS-ZE approaches in the short term. Now we can proceed with studying the long-term evolution

long-term evolution, spectrum I



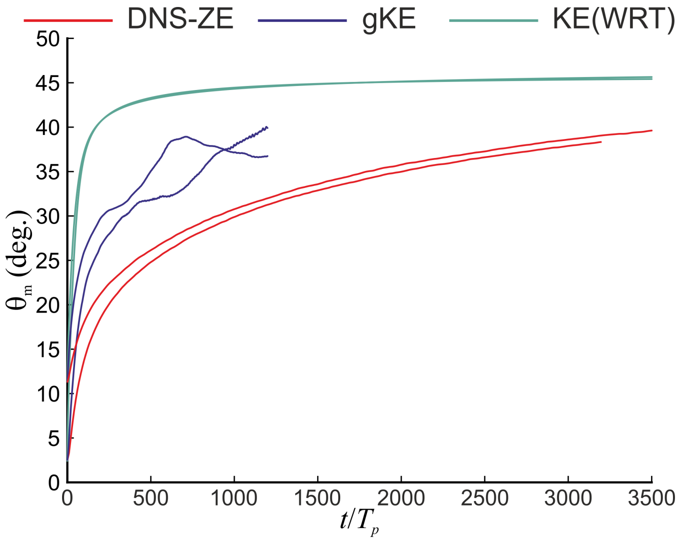
Long-term spectral evolution for spectrum I, with the comparison of DNS-ZE and both kinetic equations (KE and gKE). Spectra are plotted every 300 periods

long-term evolution, spectrum II

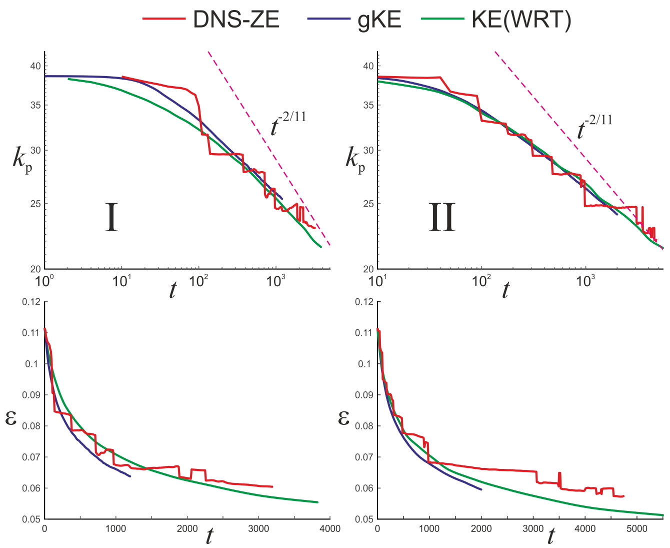


Long-term spectral evolution for spectrum II

long-term evolution of mean directional spreading

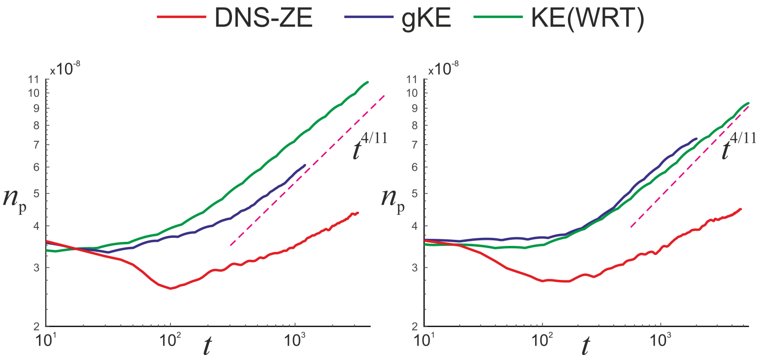


peak wavenumber and wave steepness



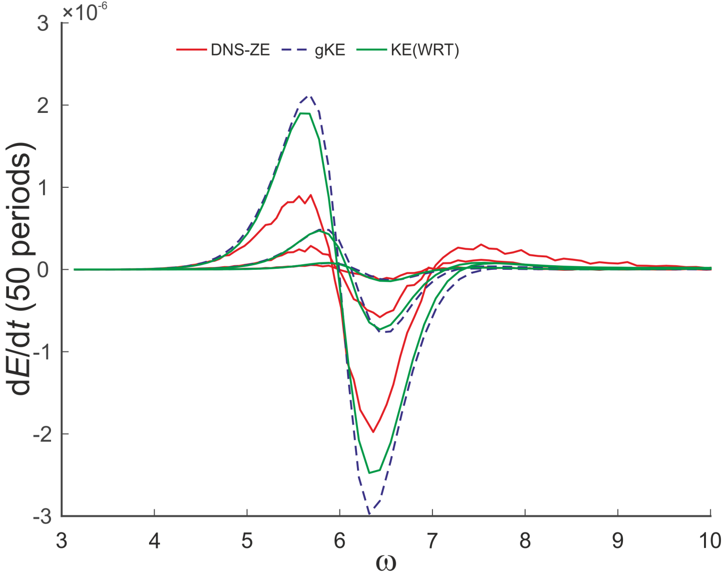
Evolution of the wavenumber of the spectral peak (with theoretical asymptotic $\sim t^{-2/11}$) and wave steepness for spectra I and II

peak amplitude



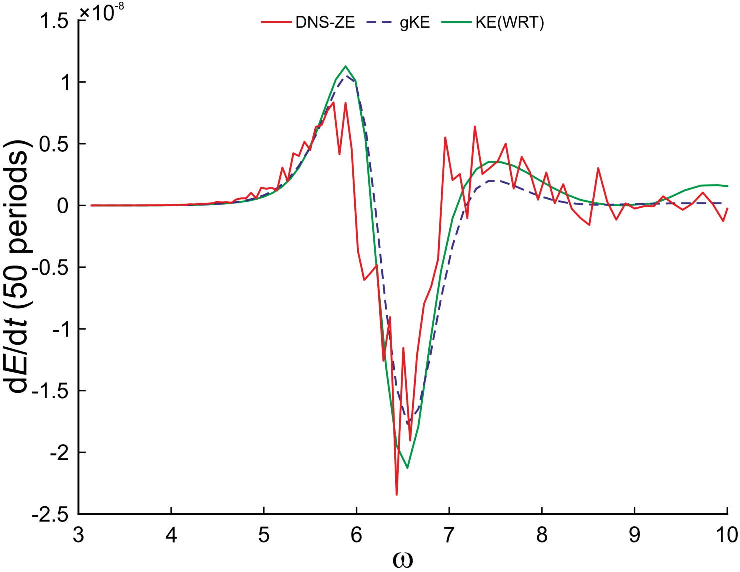
Evolution of the amplitude of the spectral peak (with theoretical asymptotic $\sim t^{4/11}$) for spectra I and II

growth rates for different amplitudes, spectrum II



Growth rates $\mathrm{d}E(\omega,t)/\mathrm{d}t$ over first 50 periods of evolution, with DNS, KE and gKE, for amplitude multiplied by $1/\sqrt{2}$, 1 and $\sqrt{2}$

growth rates for small amplitude, spectrum II



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with DNS, KE and gKE, for half amplitude

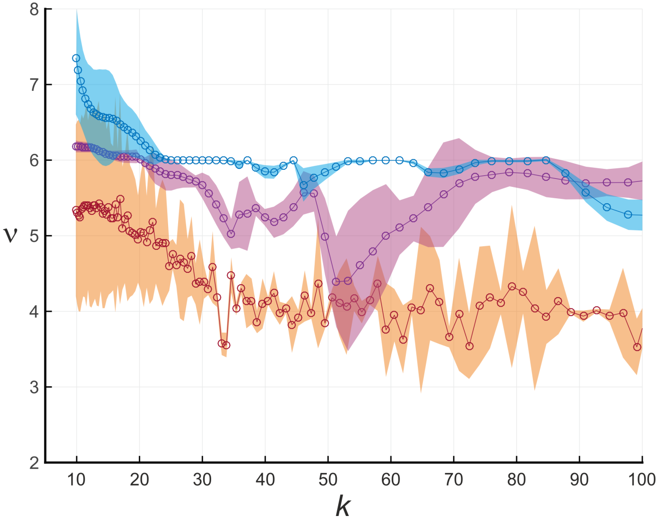
scaling of growth rates

In order to understand how the growth rates of wave action $n(k,t)$ scale with nonlinearity within different approaches, we find the maximum value of $\mathrm{d}n/\mathrm{d}t$ and perform a numerical fit

$$\log\max \mathrm{d}n/\mathrm{d}t = \nu \log\varepsilon + \beta$$

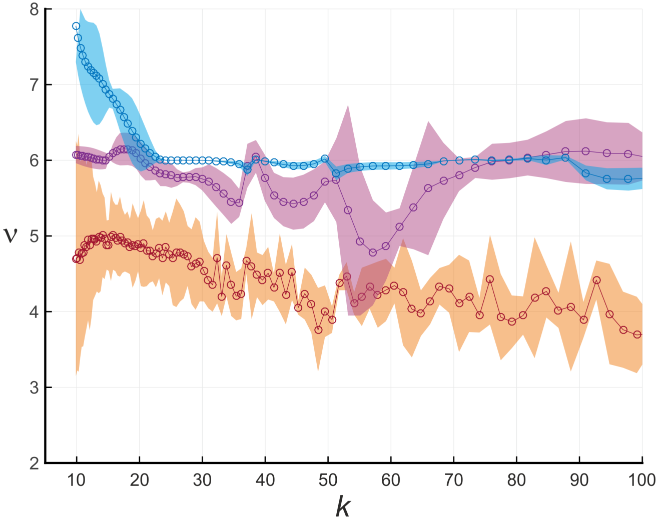
over 5 different amplitudes (different from the initial one by 0.5, $1/\sqrt{2}$, 1, $\sqrt{2}$ and 2). Thus, we draw a straight line through 5 points by least squares, find the coefficient ν and the 95% confidence bounds for it. We know *a priori* that the KE, being an equation in real variables, has the strict $\nu = 6$ scaling (that is, $\mathrm{d}n/\mathrm{d}t \sim \varepsilon^6$). The values of ν for other approaches are to be found numerically. The DNS can be expected to give $\nu = 4$ (the dynamic scaling of the growth rate, rather than the statistical one)

scaling for maximum growth rates, spectrum I



Exponent ν of the scaling as ε^{ν} for maximum growth rates for KE, gKE and DNS-ZE, and its 95% confidence bounds. The initial spectral peak is at $k = 4\pi^2 \approx 39.5$. Blue: KE(WRT), purple: gKE, orange: DNS-ZE

scaling for maximum growth rates, spectrum II



Exponent ν of the scaling for maximum growth rates for KE, gKE and DNS-ZE, and its 95% confidence bounds. The initial spectral peak is at $k = 4\pi^2 \approx 39.5$. Blue: KE(WRT), purple: gKE, orange: DNS-ZE

conclusions

We have considered the short- and long-term evolution of narrow spectra without wind forcing, using three different models, employing different sets of assumptions. Two of these models are new (gKE and DNS-ZE). The gKE employs the statistical closure, but is free of quasi-stationarity assumption. DNS-ZE does not depend on any statistical assumptions.

- in the short term, DNS-ZE results show good agreement with DNS simulations by Xiao et al (2013), both for the evolution of frequency spectra and for the directional spreading
- the gKE agrees with the classic KE (WRT algorithm) for the evolution of frequency spectra, unless the initial spectrum is very narrow in angle
- gKE and DNS-ZE allow long-term simulations of spectra, which is not possible with other existing alternatives to the KE
- in the long term, all three approaches demonstrate very close evolution of integral characteristics of spectra, approaching for large time the theoretical asymptotes of the self-similar stage of evolution

conclusions continued

- there is a striking difference for the rate of angular broadening, which is much larger for the gKE and especially for the KE, than for the DNS-ZE
- DNS-ZE results show considerably wider spectra with less pronounced peak
- the rates of change of the spectra obtained with the DNS-ZE are proportional to $O(\varepsilon^4)$, corresponding to the dynamical timescale of evolution
- the gKE scaling of growth rates is close to the strict statistical $O(\varepsilon^6)$ scaling of the KE, but slightly and distinctively less
- despite the different scaling, the growth rates are close for small nonlinearity ($\varepsilon \leq 0.05$) and diverge for $\varepsilon = O(0.1)$ (that is, ε^6 becomes larger than ε^4, although ε is small). This indicates the presence of a certain large parameter
- the difference of growth rate scaling in the presence of self-similarity can be responsible for the difference in spectral shapes and rates of angular broadening