

European Geosciences Union General Assembly 2017 Vienna | Austria | 23–28 April 2017

Formulas of Gravitational Curvatures of Tesseroid Both in Spherical and Cartesian

Integral Kernels

Xiao-Le Deng¹ (<u>xldeng@whu.edu.cn</u>) and Wen-Bin Shen^{1, 2} (<u>wbshen@sgg.whu.edu.cn</u>)

1 School of Geodesy and Geomatics, Wuhan University, Wuhan, China

EGU² State Key Laboratory of Information Engineering in Surveying, Mapping and Remote Sensing, Wuhan University, Wuhan, China





Content

Introduction

2 Method

1

3 Experiments

4 Conclusion

1 Introduction

➢ Gravitational Curvatures (GC)

(Hamáčková et al. 2016; Šprlák et al. 2016; Šprlák and Novák 2016a)

- Spectral domain: Tóth and Földvàry (2005); Tóth (2005); Casotto and Fantino (2009); Fantino and Casotto (2009) ; Fukushima (2012a; 2012b; 2012c); Hamácková et al. (2016) ; Šprlák et al. (2016).
- **Spatial domain**: *Šprlák and Novák (2015)*



1 Introduction

Gravity Effects of the Tesseroid

(Gravitational Potential; Gravity Vector; Gravity Gradient Tensor)

GP&GV&GGT

- Kuhn (2003);
- Heck and Seitz (2007);
- Asgharzadeh et al. (2007);
- Wild-Pfeiffer (2008);
- Tsoulis et al. (2009);
- Grombein et al. (2013; 2014; 2016);
- Deng et al. (2016);
- Shen and Deng (2016);
- Kuhn and Hirt (2016)



2 Method

Formulas of Gravitational Curvatures of the Tesseroid in Spherical Integral Kernels



(Tóth 2005; Tóth and Földváry 2005; Casotto and Fantino 2009; Šprlák and Novák 2015; Šprlák et al. 2016; Šprlák and Novák 2016a, 2016b)

$$D_{xxx}^{T3D} = \frac{1}{r_p^3 \cos^3 \theta_p} \left(V_{\lambda_p \lambda_p \lambda_p} - 2V_{\lambda_p} \right) + \frac{3}{r_p^2 \cos \theta_p} \left(V_{\lambda_p r_p} - \tan \theta_p \frac{V_{\lambda_p \theta_p}}{r_p} \right)$$
(1)

$$D_{xxy}^{T3D} = \frac{V_{\lambda_p \lambda_p \theta_p} + 2 \tan \theta_p V_{\lambda_p \lambda_p} - V_{\theta_p}}{r_p^3 \cos^2 \theta_p} - \frac{\tan \theta_p}{r_p^3} V_{\theta_p \theta_p} + \frac{1}{r_p^2} V_{\theta_p r_p}$$
(2)

$$D_{xxz}^{T3D} = \frac{1}{r_p^2 \cos^2 \theta_p} \left(V_{\lambda_p \lambda_p r_p} - \frac{2}{r_p} V_{\lambda_p \lambda_p} \right) - \frac{\tan \theta_p}{r_p^2} V_{\theta_p r_p} + \frac{1}{r_p} V_{r_p r_p} + \frac{2 \tan \theta_p}{r_p^3} V_{\theta_p} - \frac{1}{r_p^2} V_{r_p}$$
(3)
$$D_{xyz}^{T3D} = \frac{1}{r_p^2 \cos \theta_p} \left(V_{\lambda_p \theta_p r_p} - \frac{2 \tan \theta_p}{r_p} V_{\lambda_p} + \tan \theta_p V_{\lambda_p r_p} - \frac{2}{r_p} V_{\lambda_p \theta_p} \right)$$
(4)

$$D_{yyx}^{T3D} = \frac{1}{r_p^3 \cos \theta_p} \left(V_{\theta_p \theta_p \lambda_p} + 2 \tan \theta_p V_{\lambda_p \theta_p} + r_p V_{\lambda_p r_p} + 2 \tan^2 \theta_p V_{\lambda_p} \right)$$
(5)

2 Method

Formulas of Gravitational Curvatures of the Tesseroid in Cartesian Integral Kernels

2 Method

- > Different numerical approaches both in 3D and 2D forms
 - Taylor Series Expansion approach (TSE) (Kuhn 2003; Heck and Seitz 2007; Wild-Pfeiffer 2008; Deng et al. 2016; Shen and Deng 2016; Grombein et al. 2013, 2016);
- Gauss-Legendre Quadrature approach (GLQ) (Asgharzadeh et al. 2007; Wild-Pfeiffer 2008; Li et al. 2011; Hirt et al. 2011; Du et al. 2015; Roussel et al. 2015; Rexer and Hirt 2015; Uieda et al. 2016);
- Newton-Cotes Quadrature approach (NCQ) (Wild-Pfeiffer 2008);

$$F_{m} = \sum_{i,j,k} X_{ijk} \frac{(\Delta \lambda)^{i+1} (\Delta \theta)^{j+1} (\Delta r)^{k+1}}{2^{i+j+k} (i+1)! (j+1)! (k+1)!}$$
(6)
$$H_{n} = \sum_{i,j} Y_{ij} \frac{(\Delta \lambda)^{i+1} (\Delta \theta)^{j+1}}{2^{i+j} (i+1)! (j+1)!}$$
(7)

$$F_{GLQ}^{3D} \approx G\rho_{S} \frac{(\lambda_{2} - \lambda_{1})(\theta_{2} - \theta_{1})(r_{2} - r_{1})}{8} \sum_{k=1}^{N^{r}} \sum_{j=1}^{N^{\theta}} \sum_{i=1}^{N^{\lambda}} W_{i}^{\lambda} W_{j}^{\theta} W_{k}^{r} I\left(\lambda_{i}, \theta_{j}, r_{k}\right) (8)$$

$$F_{GLQ}^{2D} \approx G\rho_{S} \frac{(\lambda_{2} - \lambda_{1})(\theta_{2} - \theta_{1})}{4} \sum_{j=1}^{N^{\theta}} \sum_{i=1}^{N^{\lambda}} W_{i}^{\lambda} W_{j}^{\theta} J\left(\lambda_{i}, \theta_{j}\right)$$

$$(9)$$

$$F_{NCQ}^{3D} \approx G\rho_S \sum_{k=0}^{N^r} \sum_{j=0}^{N^{\theta}} \sum_{i=0}^{N^{\lambda}} W_i^{\lambda} W_j^{\theta} W_k^r I(\lambda_i, \theta_j, r_k)$$
(10)

$$F_{NCQ}^{2D} \approx G\rho_S \sum_{j=0}^{N^{\theta}} \sum_{i=0}^{N^{\lambda}} W_i^{\lambda} W_j^{\theta} J(\lambda_i, \theta_j)$$
(11)

> Analytical GC formulas of a homogeneous spherical shell

Fig.2 The spherical shell

Comparison of time cost between Spherical and Cartesian Integral Kernels

Fig.3 Relative time histogram of GC (Dxxx for **a**; Dxxy for **b**; Dxxz for **c**; Dxyz for **d**; Dyyx for **e**; Dyyy for **f**; Dyyz for **g**; Dzzx for **h**; Dzzy for **i**; Dzzz for **j**) for Spherical (blue column) and Cartesian (red column) Integral Kernels with 3D TSE zero-order, second-order and fourth-order approach.

Comparison of time cost between Spherical and Cartesian Integral Kernels

Table 1 Comparison of computation time cost (%) using 3D TSE approach with different order (Zero, Second and Fourth) for Spherical and Cartesian Integral Kernels to evaluate the average time cost information of GC, which are listed as t (D_{ijk}), respectively. All values are in percentage form with respect to the computation time cost (unit: s) of the second-order spherical integral kernels.

Name	Tess	seroid (Spheri	ical)	Tesseroid (Cartesian)						
	Zero	Second	Fourth	Zero	Second	Fourth				
$t(D_{ijk})$	5	100	1094	2	46	566				

Comparison of precision and time cost for Cartesian integral kernels with different numerical approaches (3D/2D TSE, GLQ, NCQ: Closed NCQ and Open NCQ)

Fig.4 a GC approximation errors in Log_{10} form with 3D/2D TSE (blue triangle), GLQ (red circle), CNCQ (green square) and ONCQ (black pentagrams), the unit of GC is m⁻¹ s⁻²; **b** GC histogram of CPU time percentage by 3D/2D TSE (blue column), GLQ (red column), CNCQ (green column) and ONCO (black column) with respect to 3D TSE second-order GC formulas

Fig.5 a Graphic of approximation errors in Log10 of GP, GV, GGT and GC by 3D TSE zero-order approach for
Spherical Integral Kernels with latitude variation from equator to North Pole;
b 3D TSE second-order approach for Spherical Integral Kernels;

Fig.5 c 3D TSE zero-order approach for Cartesian Integral Kernels; d 3D TSE second-order approach for Cartesian Integral Kernels

Fig.5 e 3D GLQ with nodes (1, 1, 1) approach for **Cartesian Integral Kernels**; **f 3D GLQ with nodes (2, 2, 2)** approach for **Cartesian Integral Kernels**;

Fig.5 g 3D CNCQ with nodes (2, 2, 2) approach for **Cartesian Integral Kernels**; **h 3D CNCQ with nodes (3, 3, 3)** approach for **Cartesian Integral Kernels**;

Fig.5 i 3D ONCQ with nodes (2, 2, 2) approach for **Cartesian Integral Kernels**; **j 3D ONCQ with nodes (3, 3, 3)** approach for **Cartesian Integral Kernels**;

Table 2. The approximation errors ranges (from minimum to maximum) and standard deviation (S.D) in Log10 form for GP, GV, GGT and GC in Fig. 5, where the unit of GP, GV, GGT and GC are $m^2 s^{-2}$, $m s^{-2}$, s^{-2} and $m^{-1} s^{-2}$, respectively.

Name	GP(δV)		$\mathrm{GV}(\delta g_z)$		GGT (δM_{xx})		GGT (δM_{yy})		GGT (δM_{zz})		GC (δD_{xxz})		GC (δD_{yyz})		GC (δD_{zzz})	
	Range	S.D	Range	S.D	Range	S.D	Range	S.D	Range	S.D	Range	S.D	Range	S.D	Range	S.D
Fig. 5a	[-2.1, -0.8]	0.3	[-9.2, -6.3]	0.7			[-16.4, -11.7]	1.2	[-17.6, -11.4]	1.3			[-22.5, -16.6]	1.4	[-23.0, -16.3]	1.5
Fig. 5b	[-9.6, -4.2]	1.2	[-16.4, -9.2]	1.7		/	[-22.0, -14.3]	1.8	[-22.1, -14.0]	1.8			[-28.3, -19.0]	2.3	[-28.2, -18.7]	2.2
Fig. 5c	[-2.1, -0.8]	0.3	[-9.2, -6.3]	0.7	[-16.0, -11.7]	1.1	[-16.4, -11.7]	1.2	[-17.6, -11.4]	1.3	[-22.7, -16.6]	1.4	[-22.5, -16.6]	1.4	[-23.0, -16.3]	1.5
Fig. 5d	[-9.6, -4.2]	1.2	[-16.4, -9.2]	1.7	[-23.2, -14.3]	2.0	[-22.0, -14.3]	1.8	[-22.1, -14.0]	1.8	[-29.7, -19.0]	2.5	[-28.3, -19.0]	2.3	[-28.2, -18.7]	2.2
Fig. 5e	[-2.1, -0.8]	0.3	[-9.2, -6.3]	0.7	[-16.0, -11.7]	1.1	[-16.4, -11.7]	1.2	[-17.6, -11.4]	1.3	[-22.7, -16.6]	1.4	[-22.5, -16.6]	1.4	[-23.0, -16.3]	1.5
Fig. 5f	[-9.9, -4.6]	1.2	[-16.2, -9.5]	1.6	[-22.8, -14.6]	1.9	[-22.8, -14.6]	1.9	[-22.4, -14.3]	1.7	[-28.8, -19.3]	2.0	[-29.2, -19.3]	2.2	[-27.6, -19.0]	1.7
Fig. 5g	[-1.8, -0.5]	0.3	[-8.9, -6.0]	0.7	[-15.7, -11.4]	1.1	[-16.1, -11.4]	1.2	[-17.3, -11.1]	1.3	[-22.4, -16.3]	1.4	[-22.2, -16.3]	1.4	[-22.7, -16.0]	1.5
Fig. 5h	[-9.8, -4.4]	1.2	[-16.6, -9.3]	1.7	[-24.5, -14.4]	2.0	[-22.4, -14.4]	1.8	[-23.0, -14.1]	1.9	[-29.5, -19.2]	2.3	[-29.1, -19.2]	2.2	[-29.0, -18.9]	2.1
Fig. 5i	[-2.7, -1.5]	0.3	[-9.8, -6.9]	0.7	[-16.6, -12.3]	1.1	[-17.0, -12.3]	1.2	[-18.2, -12.0]	1.3	[-23.3, -17.2]	1.4	[-23.1, -17.2]	1.4	[-23.6, -16.9]	1.5
Fig. 5j	[-10.2, -4.8]	1.2	[-16.8, -9.7]	1.7	[-24.0, -14.9]	2.1	[-23.0, -14.9]	1.8	[-24.4, -14.6]	2.0	[-30.5, -19.6]	2.3	[-29.1, -19.6]	2.2	[-28.3, -19.3]	1.9

• The **GC formulas** of the tesseroid both in Spherical and Cartesian Integral Kernels are derived in spatial domain.

• The advantages of GC formulas in **Cartesian** Integral Kernels are **concise** in forms and **efficient** in numerical calculations; moreover, they can **avoid** the polar singularity problem.

• GLQ approach is recommended for the practical application of GC.

Thanks!

- Asgharzadeh MF, Von Frese RRB, Kim HR, Leftwich TE, Kim JW. (2007). Spherical prism gravity effects by Gauss-Legendre quadrature integration. Geophysical Journal International, 169, 1-11.
- Casotto S, Fantino E. (2009). Gravitational gradients by tensor analysis with application to spherical coordinates. Journal of Geodesy, 83, 621-634.
- Deng XL, Grombein T, Shen WB, Heck B, Seitz K. (2016). Corrections to "A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling" (Heck and Seitz, 2007) and "Optimized formulas for the gravitational field of a tesseroid" (Grombein et al., 2013). Journal of Geodesy, 90, 585-587. doi: 10.1007/s00190-016-0907-8.
- Du J, Chen C, Lesur V, Lane R, Wang H. (2015). Magnetic potential, vector and gradient tensor fields of a tesseroid in a geocentric spherical coordinate system. Geophysical Journal International, 201, 1977-2007.
- Fantino E, Casotto S (2009) Methods of harmonic synthesis for global geopotential models and their first-, second- and third-order gradients. Journal of Geodesy, 83, 595-619. doi:10.1007/s00190-008-0275-0.
- Fukushima T. (2012a). Recursive computation of finite difference of associated Legendre functions. Journal of Geodesy, 86, 745-754. doi: 10.1007/s00190-012-0553-8.
- Fukushima T. (2012b). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers. Journal of Geodesy, 86, 271-285. doi: 10.1007/s00190-011-0519-2.
- Fukushima T. (2012c). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers: II first-, second-, and third-order derivatives. Journal of Geodesy, 86, 1019-1028. doi: 10.1007/s00190-012-0561-8.
- Grombein T, Seitz K, Heck B. (2013). Optimized formulas for the gravitational field of a tesseroid. Journal of Geodesy, 87, 645-660. doi: 10.1007/s00190-013-0636-1.
- Grombein T, Luo X, Seitz K, Heck B. (2014). A Wavelet-Based Assessment of Topographic-Isostatic Reductions for GOCE Gravity Gradients. Surveys in Geophysics, 35, 959-982.

Reference

- Grombein T, Seitz K, Heck B. (2016). The Rock–Water–Ice Topographic Gravity Field Model RWI_TOPO_2015 and Its Comparison to a Conventional Rock-Equivalent Version. Surveys in Geophysics, 1-40. doi: 10.1007/s10712-016-9376-0.
- Hamáčková E, Šprlák M, Pitoňák M, Novák P. (2017). Non-singular expressions for the spherical harmonic synthesis of gravitational curvatures in a local north-oriented reference frame. Computers & Geosciences, 88, 152-162. doi: 10.1016/j.cageo.2015.12.011
- Heck B, Seitz K. (2007). A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling. Journal of Geodesy, 81, 121-136.
- Hirt C, Featherstone WE, Claessens SJ. (2011). On the accurate numerical evaluation of geodetic convolution integrals. Journal of Geodesy, 85, 519-538.
- Kuhn M. (2003). Geoid determination with density hypotheses from isostatic models and geological information. Journal of Geodesy, 77, 50-65
- Kuhn M, Hirt C. (2016). Topographic gravitational potential up to second-order derivatives: an examination of approximation errors caused by rock-equivalent topography (RET). Journal of Geodesy, 90, 883-902. doi: 10.1007/s00190-016-0917-6.
- Li Z, Hao T, Xu Y, Xu Y. (2011). An efficient and adaptive approach for modeling gravity effects in spherical coordinates. Journal of Applied Geophysics, 73, 221-231.
- Roussel C, Verdun J, Cali J, Masson F. (2015). Complete gravity field of an ellipsoidal prism by Gauss-Legendre quadrature. Geophysical Journal International, 203, 2220-2236.
- Rexer M, Hirt C. (2015). Ultra-high-Degree Surface spherical Harmonic Analysis Using the Gauss–Legendre and the Driscoll/Healy Quadrature Theorem and Application to Planetary Topography Models of Earth, Mars and Moon. Surveys in Geophysics, 36, 803-830.
- Šprlák M, Novák P. (2015). Integral formulas for computing a third-order gravitational tensor from volumetric mass density, disturbing gravitational potential, gravity anomaly and gravity disturbance. Journal of Geodesy, 89, 141-157. doi: 10.1007/s00190-014-0767-z.
- Shen WB, Deng XL. (2016). Evaluation of the fourth-order tesseroid formula and new combination approach to precisely determine gravitational potential. Studia Geophysica et Geodaetica, 1-25. doi: 10.1007/s11200-016-0402-y.

Reference

- Šprlák M, Novák P. (2016a). spherical gravitational curvature boundary-value problem. Journal of Geodesy, 1-13. doi: 10.1007/s00190-016-0905-x.
- Šprlák M, Novák P. (2016b). Spherical integral transforms of second-order gravitational tensor components onto third-order gravitational tensor components. Journal of Geodesy, 1-28. doi: 10.1007/s00190-016-0951-4.
- Šprlák M, Novák P, Pitoňák M. (2016). spherical Harmonic Analysis of Gravitational Curvatures and Its Implications for Future Satellite Missions. Surveys in Geophysics, 37, 681-700. doi: 10.1007/s10712-016-9368-0.
- Tóth G. (2005). The gradiometric-geodynamic boundary value problem. In: Jekeli C, Bastos L, Fernandes J (eds), Gravity, Geoid and Space Missions: GGSM 2004 IAG International Symposium Porto, Portugal August 30 September 3, 2004. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 352-357.
- Tóth G, Földváry L. (2005). Effect of geopotential model errors on the projection of GOCE gradiometer observables. In: Jekeli C, Bastos L, Fernandes J (eds), Gravity, Geoid and Space Missions: GGSM 2004 IAG International Symposium Porto, Portugal August 30 September 3, 2004. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 72-76.
- Tsoulis D, Novak P, Kadlec M. (2009). Evaluation of precise terrain effects using high-resolution digital elevation models. Journal of Geophysical Research: Solid Earth, 114,294–386. doi:10.1029/2008JB005639.
- Uieda L, Barbosa V, and Braitenberg C. (2016). Tesseroids: Forward-modeling gravitational fields in spherical coordinates, Geophysics, 81(5), F41-F48
- Wild-Pfeiffer F. (2008). A comparison of different mass elements for use in gravity gradiometry. Journal of Geodesy, 82, 637-653.

This study is supported by National 973 Project China (grant Nos. 2013CB733301 and 2013CB733305), NSFCs (grant Nos. 41174011, 41429401, 41210006, 41128003, 41021061), and Key Laboratory of GEGME fund (grant No. 16-02-02).

