An N-wave with a leading depression entering a shoaling bay with a U-shaped cross section

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Introduction

The problem of a wave progressing in U-shaped inclined bay was solved by [1] using an hodograph transform. One peculiarity of this transform is that the width of the wave packet remains constant in the transformed space. Consequently the reflected wave has the same shape as the incident wave. That was not the case with Carrier-GreenSpan transform where the width of the wave packet increases with distorted time λ. Once the wave packet enters the inland intrusion it can be confined to the bay because of the multiple back reflections by the mouth of the bay. Because of the preservation of the width of the wave packet during its travel along the bay the maximum run-ups and run-downs will be produced at a very regular rate. In general the first the run-up is the largest because some the energy will be transmitted to the open ocean during the back reflections.

During the back reflection into the bay the polarity of the waves changes. At the instant of the back-reflection of the front of the N -wave, the tail of the N-wave may enter into the bay then the superposition of the two waves of the same parity will produce a larger wave leading to even larger run-ups.

If the aspect ratio of the bay is small then at the early stages of the formation of the standing waves in the bay , the magnitude scattered wave into the open sea is much smaller that of the incident wave.This is so because before the fully resonant regime develops in the bay, the amplitude of the incident wave from the open ocean and that of the standing wave are of the same order of magnitude. At each oscillation of the standing wave in the bay the scattered wave transmits to the open sea only a small fraction of the energy of the standing waves. That is why the amplitude of the scattered wave is much smaller that of the incident wave. Ignoring the scattered one can impose the Dirichlet boundary condition at the mouth of the bay. This Dirichlet condition stipulates that the wave height at the mouth of the bay is equal to the twice of the height of the incident wave. This simplification renders the problem one dimensional. Because tsunamis are transient phenomenon, there is not enough time for the resonant regime to set in an the simplified one dimensional model based on the Dirichlet boundary condition leads to an accurate prediction of the maximum run-up for slender bays (see figure 2 where comparisons between two and one dimensional models are made).

Model

An inclined bay with parabolic cross section is considered. The bottom of the bay is given by

$$z = -\alpha\left(x - x_0(\frac{y}{y_0})^2\right) \tag{1}$$

Here $z = 0$ is the undisturbed free surface, x_0 the length of the bay, and y_0 is its half width at $x = x_0$ (the mouth of the bay). The bay opens to a semi-infinite sea of uniform depth $H_0 = \alpha x_0$ Well inside the bay ($x_0 - x >> y_0$) the waves are one dimensional and in the linear limit they are given by

$$\eta = \tilde{r}(\omega) j_0\left(\frac{\omega}{\sqrt{g\alpha}}\sqrt{6x}\right)\exp(i\omega t) \tag{2}$$

according to [1]. Here η is the free surface disturbance associated with the wave, j_0 is the spherical Bessel function of the first kind , and, $\tilde{r}(\omega)$ Fourier transform of run-up as $j_0(0)$ is 1.

In the vicinity the mouth of the bay ($x_0 - x$ is of order of y_0), the waves feel the effect of geometrical spreading in the open sea, therefore two dimensional solution must be used in this region. In this region the relative change of depth in x direction is small accordingly we will be using the solutions of the linear shallow water equation in the infinite channel with parabolic cross section.

Because of the bathimetry is independent of x coordinate for the infinite channel, x dependent part of the solution of linear shallow water equation is an eigenfunction of the differential operator $\frac{d^2}{dx^2}$. We may propose for the waves in the infinite channel

$$\eta = \sum_p B_p \exp(\kappa_p(\omega)x - \phi_p) f_p(y,\omega)\exp(i\omega t)\,. \tag{3}$$

The expression above is indeed a solution of the shallow water equation in the infinite channel if function $f_p(y,\omega)$ satisfies the ordinary differential equation

$$\begin{aligned}&-\omega^2 f_p(y,\omega) - g\kappa_p^2\left\{\alpha x_0(1-\frac{y^2}{y_0^2})f_p(y)\right\}\\&-g\frac{d}{dy}\left[\alpha x_0(1-\frac{y^2}{y_0^2})\frac{d}{dy}f_p(y,\omega)\right] = 0\,.\end{aligned} \tag{4}$$

The equation above has two singular points at $y = \pm y_0$ and there are solutions regular at both singular points if continuous parameter κ is equal to some discrete values $\kappa_0, \kappa_1, \ldots$. To compute these discrete values of κ and obtain functions $f_p(y,\omega)$ the matrix of the ordinary differential equation given by (4) must be written in the basis of even Legendre polynomials (a solution that is symmetrical about $y = 0$ is sought). The eigenvalues and the eigenvectors of the resulting matrix are obtained using algebraic methods. The allowed values of κ^2 may be written in ascending order as

$$\kappa_0^2 < 0 < \kappa_1^2 < \kappa_2^2 < ..$$

if $\omega y_0/\sqrt{gH_0}$ is smaller than 2.5 there will only one κ with κ^2 negative (see Figure 1). Phase factor ϕ_0 in (3) will be chosen in a way to insure a smooth transition between our oscillatory solution $\exp(\kappa_0(\omega x - \phi_0)$ and $j_0(\omega\sqrt{6x}/\sqrt{g\alpha})$ (see [1]). Therefore ϕ_0 must be the root of the wronskian of these two functions for $x = x_0$. It is convenient to take $\phi_p = \kappa_p x_0$ for $p = 1, 2..$ so that the exponentially decaying functions $\exp(\kappa_p x - \phi_p)$ take the value 1 at the mouth of the bay. The scattered wave into open sea together with the incident wave and the wave reflected by the coastline at $x = x_0$ reads

$$\begin{aligned}\eta(t, x > x_0, y) = {}& 2\tilde{I}_0(\omega)\cos\left(\frac{\omega}{\sqrt{gH_0}}x\right)\exp(i\omega t)\\&+\exp(i\omega t)\int_{-y_0}^{y_0} dy'' S(y'',\omega)\frac{\omega}{2gH_0}H_0^{(2)}\left(\frac{\omega}{\sqrt{gH_0}}|y - y''|\right)\end{aligned} \tag{5}$$

for $\omega > 0$. Here $H_0^{(2)}$ is Hankel function of the second kind order 0, $\tilde{I}_o(\omega)$ is the amplitude of the incident wave, and function S is the virtual source distribution at the mouth of the bay. The continuity of depth integrated velocity in x direction at $x = x_0$ requires that

$$-\frac{gH_0}{i\omega}\left(1. - y^2/y_0^2\right)\left(B_0 i|\kappa_0(\omega)|\exp\left(i|\kappa_0(\omega)|x_0 + \phi_0\right)f_0(y,\omega) + \sum_{p=1}^{M-1} B_p|\kappa_p(\omega)|f_p(y,\omega)\right) = S(y,\omega)\,. \tag{6}$$

From equation above source distribution S is related to coefficients $B_0(\omega), B_1(\omega), \ldots$. The continuity of η across the mouth of the bay will lead to an integral equation for the coefficients $B_0(\omega), B_1(\omega), \ldots$. Continuity of η requires that (3) must be equal to (5) for $x \to x_0$. For each frequency the unknown coefficients can be determined minimising the penalty integral

$$\int_{-y_0}^{y_0} dy\,|\lim_{x\to x_0^+}\eta(x,y,\omega) - \lim_{x\to x_0^-}\eta(x,y,\omega)|^2 \tag{7}$$

with respect to $B_0(\omega), B_1(\omega), \ldots$. Such minimisation lead to algebraic equations for coefficients $B_0(\omega), B_1(\omega), \ldots$.

Simple solutions where the scattered wave is neglected

When the aspect ratio of bay is small the waves are effectively trapped inside the bay and the radiation of the waves from the mouth to the open sea is a slow process (it takes far longer than $L/\sqrt{gH_0}$(here L is the width of the incident wave packet). That makes the amplitude of the scattered wave negligible. The wave height at the mouth of the bay is then twice the height of the incident wave. The easiest way of solving this Dirichlet problem using Fast Fourier Transform is to consider an infinite channel of width $2y_0$ and uniform depth D connected U-shaped bay.The initial condition is a wave packet in the infinite channel progressing toward the U-shaped bay. If one allow D to become infinite than the amplitude the wave radiating from the mouth of the U-shaped bay into the infinite channel becomes zero. This problem can be easily solved in Fourier domain taking into account the flux continuity and continuity of η at the mouth of the bay.The run-up is then

$$r(t) = \lim_{D\to\infty}\int_{-\infty}^{\infty} d\omega \frac{2iD\tilde{I}(\omega)\exp(i\omega t)}{-\sqrt{2/3}j_1\left(\frac{\omega}{\sqrt{g\alpha}}\sqrt{6x_0}\right) + iDj_0\left(\frac{\omega}{\sqrt{g\alpha}}\sqrt{6x_0}\right)}\,.$$

Here $\tilde{I}(\omega)$ is the Fourier transform of the incident wave with respect to time at $x = x_0$.

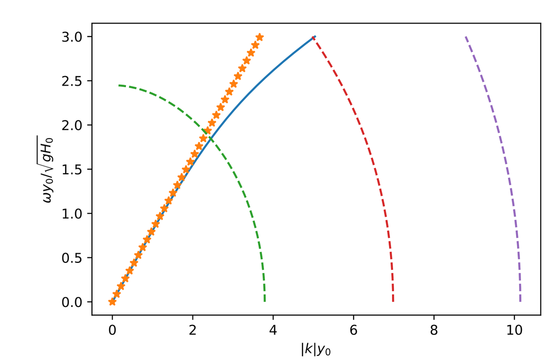

Fig. 1: The dispersion relation in infinite channel with parabolic cross-section. Broken curves means that wave vector is purely imaginary (associated modes grow exponentially in $+x$ direction). The stars are the dispersion relation given by $\omega = \sqrt{g2H_0/3}k$ where $2H_0/3$ is the averaged depth of the parabolic channel.

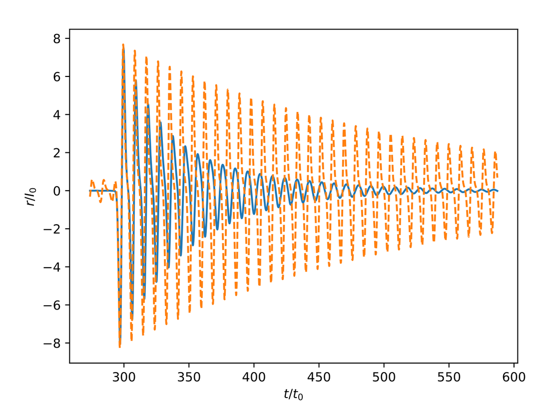

Fig. 2: Continuous curve is run-up as function of time. The aspect ratio of the U-shaped inclined bay is 10 (length divided by the maximum width). The incident wave packet is a gaussian depression given by $-I_0\exp(-(x+\sqrt{gH_0}t)^2/x_0^2)$. Parameter t_0 is $\sqrt{6x_0/(\alpha g)}$ (travel time of waves over the U-shape bay). In the broken curve the semi-infinite sea has been replaced by a semi-infinite channel of depth $D = 40\alpha x_0$ and width $2y_0$ ($2y_0$ is the maximum width of the U-shaped bay).

References

[1] Ira Didenkulova and Efim Pelinovsky. "Nonlinear wave evolution and runup in an inclined channel of a parabolic cross-section". In: *Physics of Fluids* 23.8 (2011), p. 086602. DOI: 10.1063/1.3623467. eprint: https://doi.org/10.1063/1.3623467. URL: https://doi.org/10.1063/1.3623467.