Turbulence can cause anisotropy

SA and BM anisotropies in 3 cases \((h, p, f)\)

Main equations and parameters

Comparison of different anisotropies’ effects
All kinds of anisotropy with greater diffusivity in vertical direction facilitate convection

There are also anisotropies which inhibit convection in comparison with isotropic case

Anisotropies, in particular the strong ones, allow the preference (or/and existence) of some modes, what is not possible in the isotropic case

anisotropy = anisotropic diffusion, anisotropic diffusive coefficients, anisotropic diffusivities
Anisotropic diffusivities effects in Rotating Magnetoconvection study

Enrico Filippi¹ (enrico.filippi@gmail.com) Jozef Brestensky¹ Tomáš Šoltis²

¹ FMPI, Comenius University, Bratislava, Slovakia ² Earth Science Institute, SAV, Bratislava, Slovakia

There is strong belief that the Earth’s Core is in turbulent state

The Earth’s core is driven into motion by buoyancy forces so strong that the flow and field are turbulent, fluctuating on every length and time scale, as it is accepted by the most of geophysicists. Traditional approach to turbulence (see, e.g., Krause and Rädler, 1980)

\[ u = \bar{u} + u', \quad B = \bar{B} + B' \]

the mean IE is:

\[ \frac{\partial \bar{B}}{\partial t} = \eta_0 \nabla^2 \bar{B} + \nabla \times (\bar{u} \times \bar{B}) + \nabla \times E, \quad E = u' \times B' \]

\[ E = u' \times B' \]

within suitable approximations and conditions we can say that \( \nabla \times E = -\nabla \times (\beta \nabla \times \bar{B}) \), therefore

\[ \frac{\partial \bar{B}}{\partial t} = \eta_0 \nabla^2 \bar{B} + \nabla \times (\bar{u} \times \bar{B}) + \beta \nabla^2 \bar{B}, \quad "beta-effect" \]

In general, due to turbulence the diffusive coefficients are anisotropic; for instance the buoyancy has a preferred direction \( \Rightarrow \) local turbulence may be significantly anisotropic with respect to gravity direction

\[ \nabla \times E = -\nabla \times (\beta \nabla \times \bar{B}) \rightarrow \nabla \times E = -\nabla \times (\beta \nabla \times \bar{B}) = \nabla \cdot (\beta \nabla \bar{B}) \]

where \( \beta \) is a tensor quantity, therefore we can speak about an "anisotropic beta-effect" or anisotropic magnetic diffusivity, anisotropic \( \eta \) tensor (\( \eta = \eta_0 + \beta \))
Anisotropy in viscosity and thermal diffusivity

Turbulence can cause anisotropy also in viscosity and thermal diffusivity (see, e.g., Fearn and Roberts, 2007), therefore it is worth to introduce anisotropy in the rotating magnetoconvection models; this was done for the first time in the model by Šoltis and Brestenský (2010) and recently it was advanced by Filippi et al. (2019). These models deals about Rotating Magnetoconvection in horizontal plane layer rotating about vertical axis and permeated by homogeneous horizontal magnetic field (Roberts and Jones, 2000), influenced by anisotropic diffusivities, viscosity and thermal diffusivity.
Anisotropic diffusive coefficients in SA and BM anisotropy (partial and full)

Partial anisotropy model by Šoltis and Brestenský (2010) with $\nu$, $\kappa$ anisotropic tensors and $\eta_0$ isotropic tensor, where

$$\nu = \begin{pmatrix} \nu_{xx} & 0 & 0 \\ 0 & \nu_{yy} & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \eta_0 & 0 & 0 \\ 0 & \eta_0 & 0 \\ 0 & 0 & \eta_0 \end{pmatrix}.$$  

Magnetic diffusion also due to turbulence inspired a full anisotropy model (Filippi et al., 2019) with anisotropic $\nu$, $\kappa$, $\eta$, where $\eta_{xx} = \eta_0 + \beta_{xx}$, $\eta_{yy} = \eta_0 + \beta_{yy}$, $\eta_{zz} = \eta_0 + \beta_{zz}$.

Later, we will speak about another kind of anisotropy, heat transport anisotropy, a partial anisotropy with $\nu$ also isotropic.

SA, Stratification anisotropy:
$\nu_{xx} = \nu_{yy} \neq \nu_{zz}$, $\kappa_{xx} = \kappa_{yy} \neq \kappa_{zz}$, $\eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0$ and $\eta_{xx} = \eta_{yy} \neq \eta_{zz}$.

Gravity or/and the Archimedean buoyancy force are dominant

BM by Braginsky and Meytlis (1990):
$\nu_{xx} < \nu_{yy} = \nu_{zz}$, $\kappa_{xx} < \kappa_{yy} = \kappa_{zz}$, $\eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0$ and $\eta_{xx} < \eta_{yy} = \eta_{zz}$.

Rotation and magnetic field are dominant

There is “horizontal isotropy” in SA, but not in BM
Donald and Roberts (2004) developed a dynamo model which introduces anisotropy only in the thermal diffusivity. Inspired by them we began to study also another case of anisotropy in rotating magnetoconvection, we call it Heat transport anisotropy.

\[ \nu = \begin{pmatrix} \nu_0 & 0 & 0 \\ 0 & \nu_0 & 0 \\ 0 & 0 & \nu_0 \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \eta_0 & 0 & 0 \\ 0 & \eta_0 & 0 \\ 0 & 0 & \eta_0 \end{pmatrix}. \]

This new simplified model is developed for wider chance to compare the effects of more anisotropic models.
Anisotropic diffusivities effects in Rotating Magnetoconvection study

Enrico Filippi\textsuperscript{1} (enrico.filippi@gmail.com) Jozef Brestenský\textsuperscript{1} Tomáš Šoltis\textsuperscript{2}

\textsuperscript{1} FMPI, Comenius University, Bratislava, Slovakia \textsuperscript{2} Earth Science Institute, SAV, Bratislava, Slovakia

**Outline**

- Conclusion
- Anisotropic diffusion
- SA and BM
- Main equations
- Compared anisotropies

**Dimensionless governing equations and parameters (partial and full)** 1/2

So, in case of partial and full anisotropy we get the following main dimensionless equations, after using standard procedures

\[
R_o \partial_t \mathbf{u} + \hat{z} \times \mathbf{u} = -\nabla p + \Lambda (\nabla \times \mathbf{b}) \times \hat{y} + R \vartheta \hat{z} + E_z \nabla^2 \nu \mathbf{u}
\]

\[
\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \hat{y}) + \nabla^2 \mathbf{b}, \quad \partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \hat{y}) + \nabla^2 \eta \mathbf{b}
\]

\[
\frac{1}{q_z} \partial_t \vartheta = \hat{z} \cdot \mathbf{u} + \nabla^2 \nu \vartheta,
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0
\]

where $\tilde{z} = dz$, $\tilde{\mathbf{b}} = B_M \mathbf{b}$ and

\[
\tilde{\mathbf{u}} = U \mathbf{u} = \frac{\eta_0}{d} \mathbf{u}, \quad \tilde{t} = \frac{d}{U} = \frac{d^2}{\eta_0} t, \quad \tilde{\rho} = 2\Omega_0 \eta_0 \rho_0 p, \quad \tilde{\vartheta} = \frac{\eta_0 \Delta T}{\kappa_{zz}} \vartheta,
\]

\[
\tilde{\mathbf{u}} = U \mathbf{u} = \frac{\eta_{zz}}{d} \mathbf{u}, \quad \tilde{t} = \frac{d}{U} t = \frac{d^2}{\eta_{zz}} t, \quad \tilde{\rho} = 2\Omega_0 \eta_{zz} \rho_0 p, \quad \tilde{\vartheta} = \frac{\eta_{zz} \Delta T}{\kappa_{zz}} \vartheta
\]
Dimensionless governing equations and parameters (partial and full) 2/2

\[ R_o = \frac{\eta_0}{2\Omega_0 d^2}, \quad \Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \mu_0 \eta_0}, \quad E_z = \frac{\nu_{zz}}{2\Omega_0 d^2}, \quad R = \frac{\alpha T g \Delta T d}{2\Omega_0 \kappa_{zz}}, \quad q_z = \frac{\kappa_{zz}}{\eta_0}, \]

\[ R_o = \frac{\eta_{zz}}{2\Omega_0 d^2}, \quad \Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \mu_0 \eta_{zz}}, \quad q_z = \frac{\kappa_{zz}}{\eta_{zz}}, \]

\[ \alpha_{\nu} = \frac{\nu_{xx}}{\nu_{zz}}, \quad \alpha_{\kappa} = \frac{\kappa_{xx}}{\kappa_{zz}}, \quad \alpha_{\eta} = \frac{\eta_{xx}}{\eta_{zz}} \]

SA anisotropy:

\[ E_z \nabla^2_{\nu} u = E_z[(1 - \alpha_{\nu}) \partial_{zz} + \alpha_{\nu} \nabla^2] u, \quad \nabla^2_{\kappa} \vartheta = [(1 - \alpha_{\kappa}) \partial_{zz} + \alpha_{\kappa} \nabla^2] \vartheta, \]

\[ \nabla^2_{\eta} b = [(1 - \alpha_{\eta}) \partial_{zz} + \alpha_{\eta} \nabla^2] b \]

BM anisotropy:

\[ E_z \nabla^2_{\nu} u = E_z[(\alpha_{\nu} - 1) \partial_{xx} + \nabla^2] u, \quad \nabla^2_{\kappa} \vartheta = [(\alpha_{\kappa} - 1) \partial_{xx} + \nabla^2] \vartheta, \]

\[ \nabla^2_{\eta} b = [(\alpha_{\eta} - 1) \partial_{xx} + \nabla^2] b \]
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Enrico Filippi\(^1\) (enrico.filippi@gmail.com) Jozef Brestenský\(^1\) Tomáš Šoltis\(^2\)
\(^1\)FMPI, Comenius University, Bratislava, Slovakia \(^2\)Earth Science Institute, SAV, Bratislava, Slovakia

Method of solution in partial and full anisotropy

Further, for simplicity we assume that in partial and full anisotropy:
\[
\alpha_\kappa = \alpha_\nu = \alpha, \quad \alpha_\kappa = \alpha_\nu = \alpha_\eta = \alpha
\]
solution by some methods developed, e.g., in Chandrasekhar (1961):
\[
\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{u} = a^{-2}[\nabla \times (\nabla \times w \hat{z}) + \nabla \times \omega \hat{z}] \quad \text{and} \quad \mathbf{b} = a^{-2}[\nabla \times (\nabla \times b \hat{z}) + \nabla \times j \hat{z}]
\]

All perturbations (\(w, \omega, b, j\) and \(\vartheta\)) have a form like
\[
\varphi(x, y, z, t) = \Re \{ F(z) \exp(ilx + imy) \exp(\lambda t) \} \quad \text{with} \quad a = \sqrt{l^2 + m^2}, \quad \lambda = i\sigma \in \mathbb{C},
\]
\(F(z) = W(z), \ \Omega(z), \ B(z), \ J(z), \ \text{and} \ \Theta(z)\)

If we take some curls of momentum, induction and heat equations, introduce inverted magnetic Prandtl number
\(p = \eta_{zz}/\nu_{zz} = R_o/E_z\) and we consider the simplest boundary conditions at \(z = \pm 1/2\), i.e.:
- stress free boundaries: \(W = D^2W = D\Omega = 0\)
- perfectly thermally conducting boundaries: \(\Theta = 0\)
- perfectly electrically conducting boundaries: \(B = DJ = 0\).

\[
K^2 = \pi^2 + a^2, \quad K_\alpha^2 = \begin{cases} \pi^2 + \alpha l^2 + m^2 & \text{in case of BM anisotropy} \\ \pi^2 + \alpha a^2 & \text{SA anisotropy} \end{cases}
\]

\[
R \frac{q_z(K^2 + \lambda)}{q_zK_\alpha^2 + \lambda} a^2 = \frac{K^2[E_z(K^2 + \lambda)(K_\alpha^2 + p\lambda) + m^2\Lambda]^2 + \pi^2(K^2 + \lambda)^2}{E_z(K^2 + \lambda)(K_\alpha^2 + p\lambda) + m^2\Lambda}
\]

\[
R \frac{q_z(K_\alpha^2 + \lambda)}{q_zK_\alpha^2 + \lambda} a^2 = \frac{K^2[E_z(K_\alpha^2 + \lambda)(K_\alpha^2 + p\lambda) + m^2\Lambda]^2 + \pi^2(K_\alpha^2 + \lambda)^2}{E_z(K_\alpha^2 + \lambda)(K_\alpha^2 + p\lambda) + m^2\Lambda}.
\]
Critical Rayleigh numbers in stationary SA (partial, full) anisotropy

\[ \lambda = 0, \quad 1_{\alpha} = \left( K_{\alpha}/K \right)^2 \]

\[ R_s^p = \frac{\pi (\pi^2 + a^2)^{1/2} K_{\alpha}^2}{a^2} \left( C_p + \frac{1}{C_p} \right), \quad R_s = \frac{\pi (\pi^2 + a^2)^{3/2}}{a^2} \left( C_{\alpha} + \frac{1}{C_{\alpha}} \right). \]

where \( C_p = \frac{E_z K^2 K_{\alpha}^2 + \Lambda m^2}{\pi K} \) and \( C_{\alpha} = \frac{E_z K_{\alpha}^2 + \Lambda m^2}{\pi K} \)

in SA \( K_{\alpha}^2 = \pi^2 + \alpha a^2 \) and

\[ C_p = \frac{E_z (\pi^2 + a^2)(\pi^2 + \alpha a^2) + \Lambda m^2}{\pi (\pi^2 + a^2)^{1/2}}, \quad C_{\alpha} = \frac{E_z (\pi^2 + \alpha a^2)^2 + \Lambda m^2}{\pi (\pi^2 + a^2)^{1/2}} \]  \( \tag{1} \)

\[ \frac{\partial R_s^p}{\partial C_p} = 0, \quad \frac{\partial R_s^p}{\partial a^2} = 0 \quad \text{and} \quad \frac{\partial R_s}{\partial C_{\alpha}} = 0, \quad \frac{\partial R_s}{\partial a^2} = 0 \]

we have critical \( C_p, a^2 \) and \( C_{\alpha}, a^2 \) in case of partial and full anisotropy respectively thus

\[ C_{pc} = 1, \quad a_c^2 = \pi^2 2_{\alpha} = \pi^2 \frac{1 + \sqrt{1 + 8\alpha}}{2\alpha} \quad \text{and} \quad C_{\alpha c} = 1_{\alpha}, \quad a_c^2 = \pi^2 2_{\alpha} \]  \( \tag{2} \)

by the (1) and (2) it is possible to obtain \( m_c \) and \( l_c \)
The orientation of the rolls of convection

Roberts and Jones (2000)

\[ \gamma = \arctan \left( \frac{m}{l} \right) \]

is the angle between the axis of rolls and the magnetic field \( B \).

\( l = 0 \Rightarrow \gamma = \pi/2 \) rolls \( \perp B \),

\( m = 0 \Rightarrow \gamma = 0 \) rolls \( \parallel B \)

graphical results about partial and full \( \text{Sa} \), \( \text{So} \) and \( \text{BM} \) anisotropy
Enrico Filippi\textsuperscript{1} (enrico.filippi@gmail.com) Jozef Brestenský\textsuperscript{1} Tomáš Šoltis\textsuperscript{2}
\textsuperscript{1} FMPI, Comenius University, Bratislava, Slovakia  \textsuperscript{2} Earth Science Institute, SAV, Bratislava, Slovakia

Comparison of three different anisotropic diffusions in SA and BM cases

heat transport \((h)\), partial \((p)\) and full \((f)\) anisotropies

\[
\begin{align*}
\text{h-case} \Rightarrow \quad R \frac{q_z(K^2 + \lambda)}{(q_z K_{\alpha}^2 + \lambda)} a^2 &= \frac{K^2 N_h^2 + \pi^2 (K^2 + \lambda)^2}{N_h} \\
N_h &= (K^2 + \lambda) (K^2 + p\lambda) E + \Lambda m^2
\end{align*}
\]

\[
\begin{align*}
\text{p-case} \Rightarrow \quad R \frac{q_z(K^2 + \lambda)}{(q_z K_{\alpha}^2 + \lambda)} a^2 &= \frac{K^2 N_p^2 + \pi^2 (K^2 + \lambda)^2}{N_p} \\
N_p &= (K^2 + \lambda) (K_{\alpha}^2 + p\lambda) E_z + \Lambda m^2
\end{align*}
\]

\[
\begin{align*}
\text{f-case} \Rightarrow \quad R \frac{q_z(K_{\alpha}^2 + \lambda)}{(q_z K_{\alpha}^2 + \lambda)} a^2 &= \frac{K^2 N_f^2 + \pi^2 (K_{\alpha}^2 + \lambda)^2}{N_f} \\
q_z &= \kappa_{zz} / \eta_{zz}
\end{align*}
\]

some regime diagrams in 3 cases of anisotropy

isotropic \quad anisotropic
\[
\begin{array}{c}
\nu, \eta \\
\kappa
\end{array}
\]

\[
\begin{array}{c}
\eta \\
\nu, \kappa
\end{array}
\]

\[
\begin{array}{c}
- \\
\nu, \kappa, \eta
\end{array}
\]
Anisotropic diffusivities effects in Rotating Magnetoconvection study

Enrico Filippi¹ (enrico.filippi@gmail.com) Jozef Brestensky¹ Tomáš Šoltis²
¹ FMPI, Comenius University, Bratislava, Slovakia ² Earth Science Institute, SAV, Bratislava, Slovakia

ΛEₜ Regime diagrams in SA anisotropy (α = 0.1, 10)

The ΛEₜ regime diagrams for steady convection in SA anisotropy in h, p and f cases compared with isotropy. In the left figure atmospheric anisotropy (Sa), in the right figure oceanic anisotropy (So).

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Bibliography


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Enrico Filippi\textsuperscript{1} (enrico.filippi@gmail.com) Jozef Brestenský\textsuperscript{1} Tomáš Šoltis\textsuperscript{2}

\textsuperscript{1}FMPI, Comenius University, Bratislava, Slovakia \textsuperscript{2}Earth Science Institute, SAV, Bratislava, Slovakia

Anisotropic diffusive coefficients in SA and BM anisotropy (partial and full)

Assumption 1: following anisotropy in the viscosity and thermal diffusivity holds
\[ \nu = \begin{pmatrix} \nu_{xx} & 0 & 0 \\ 0 & \nu_{yy} & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}. \]

Assumption 2: same diffusion in two directions, but a different diffusion in the third one. This is called \textit{partial anisotropy} for reasons which will be clearer later. However, in a more general case we need to deal also with the \textit{magnetic diffusion also due to turbulence}:
\[ \eta_{xx} = \eta_0 + \beta_{xx}, \quad \eta_{yy} = \eta_0 + \beta_{yy}, \quad \eta_{zz} = \eta_0 + \beta_{zz}. \]
We call this situation \textit{full anisotropy} (Filippi et al., 2019) considering magnetic diffusivity also anisotropic

SA, Stratification anisotropy:
\[ \nu_{xx} = \nu_{yy} \neq \nu_{zz}, \quad \kappa_{xx} = \kappa_{yy} \neq \kappa_{zz}, \quad \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} = \eta_{yy} \neq \eta_{zz}. \]
Gravity or/and the Archimedean buoyancy force lead the dynamics of turbulent eddies

BM by Braginsky and Meytlis (1990):
\[ \nu_{xx} < \nu_{yy} = \nu_{zz}, \quad \kappa_{xx} < \kappa_{yy} = \kappa_{zz}, \quad \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} < \eta_{yy} = \eta_{zz}. \]
Rotation and magnetic field lead the dynamics of turbulent eddies

There is “horizontal isotropy” in SA, but not in BM
Method of solution in partial and full anisotropy

In the next slides, for simplicity we assume that in partial and full anisotropy:

$$\alpha_\kappa = \alpha_\nu = \alpha, \quad \alpha_\kappa = \alpha_\nu = \alpha_\eta = \alpha$$

We look for a solution by using some methods developed, e.g., in Chandrasekhar (1961): $u$ and $B$ are divergenceless, therefore

$$u = a^{-2} [\nabla \times (\nabla \times w \hat{z}) + \nabla \times \omega \hat{z}] \quad \text{and} \quad b = a^{-2} [\nabla \times (\nabla \times b \hat{z}) + \nabla \times j \hat{z}]$$

All perturbations ($w, \omega, b, j$ and $\vartheta$) have a form

$$f(x, y, z, t) = \Re\{F(z) \exp(\im \sigma x + \im y) \exp(\lambda t)\}$$

$$a = \sqrt{l^2 + m^2}, \quad \lambda = \im \sigma \in \mathbb{C}, \quad F(z) = W(z), \Omega(z), B(z), J(z), \text{and } \Theta(z)$$

$$[E_z D_\alpha - R_\Omega \lambda] \Omega + DW + \im \Lambda J = 0,$$

$$(D^2 - a^2)[E_z D_\alpha - R_\Omega \lambda]W - D \Omega + \im \Lambda (D^2 - a^2)B = a^2 R \Theta,$$

$$(D^2 - a^2 - \lambda)J + \im \Omega = 0, \quad (D_\alpha - \lambda)J + \im \Omega = 0$$

$$(D^2 - a^2 - \lambda)B + \im W = 0, \quad (D_\alpha - \lambda)B + \im W = 0$$

$$(D_\alpha - \zeta \lambda) \Theta + W = 0.$$ 

where $\zeta = q_z^{-1}$, $D = d/dz$ and $D_\alpha$ is equal to $D^2 - \alpha l^2 - m^2$ and $D^2 - \alpha l^2 - \alpha m^2$ for BM and SA types of anisotropies, respectively.
Regime diagrams in $\alpha$ anisotropy, $\alpha < 1$

$\Lambda E_z$ diagrams for several cases of $\alpha$ anisotropy compared with isotropic case.

...back to rolls
Regime diagrams in So anisotropy, $\alpha > 1$

$\Lambda E_z$ diagrams for several cases of So anisotropy compared with isotropic case.

- Full So anisotropy $\alpha=1000$
- Partial So anisotropy $\alpha=1000$
- Full So anisotropy $\alpha=100$
- Partial So anisotropy $\alpha=100$
- Full So anisotropy $\alpha=10$
- Partial So anisotropy $\alpha=10$
- Isotropy
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Enrico Filippi\textsuperscript{1} (enrico.filippi@gmail.com) Jozef Brestenský\textsuperscript{1} Tomáš Šoltis\textsuperscript{2}

\textsuperscript{1} FMPI, Comenius University, Bratislava, Slovakia
\textsuperscript{2} Earth Science Institute, SAV, Bratislava, Slovakia

Λ$E_z$-Regime diagrams in BM anisotropy

The regime diagrams for steady convection in various cases of partial and full BM anisotropy in two different Λ-axis scales. For comparison cases of isotropy, $\alpha = 1$, and of strong Sa, $\alpha = 0.1$, are added; there are three asymptotes at $\Lambda \rightarrow \infty$, represented by the dotted vertical lines, for six SO/P lines for $\alpha = 0.1, 0.5, 0.9$. 

...back to rolls