Laplace Transform Integration of a Baroclinic Model

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Summary

The Laplace transform (LT) scheme provides an attractive alternative to the popular semi-implicit (SI) scheme. LT is more accurate than SI for both linear and nonlinear terms. Numerical experiments confirm its superior performance.

Problem and Solution

The accuracy of the SI scheme decreases as the time step is increased. The LT scheme is more accurate than SI for both linear and nonlinear terms of the equations.

SI stabilizes the high-frequency components of the flow, enabling the use of a large time step. However, it also introduces phase errors.

The LT scheme eliminates high frequency components but maintains the accuracy of components that are retained because the time averaging of the linear terms in SI is not done in LT.

The Oscillation Equation

We examine the properties of the SI and LT schemes for a simple *oscillation equation*

$$\frac{dX}{dt} = i\omega X + N(X)$$

We assume that the nonlinear term is $N(t) = \overline{N}$. The exact solution X^+ at time $(n+1)\Delta t$ is

$$X^+ = \left[\exp(2i\theta)\right] X^- + \left[\frac{\exp(2i\theta)-1}{2i\theta}\right] 2\Delta t \overline{N}$$

The digital frequency is $\theta = \omega \Delta t$.

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Accuracy Analysis: SI

The SI approximation for the equation is

$$\frac{X^+ - X^-}{2\Delta t} = i\omega \frac{X^+ + X^-}{2} + \overline{N}$$

Both the linear and nonlinear components of the solution are misrepresented. The linear term has phase given by

$$\arg \rho = 2 \arctan \theta \,.$$

The exact phase is $\arg \rho = 2\theta$. The phase error increases with the digital frequency θ. Moreover, the SI scheme has both modulus and phase errors in the treatment of the nonlinear term.

Accuracy Analysis: LT

The LT of the oscillation equation is

$$s\widehat{X} - X^- = i\omega \widehat{X} + \frac{\overline{N}}{s}.$$

where s is the complex variable in the Laplace transform. Solving for $\widehat{X}$ gives

$$\widehat{X} = \frac{X^-}{s - i\omega} + \frac{\overline{N}}{s(s - i\omega)}.$$

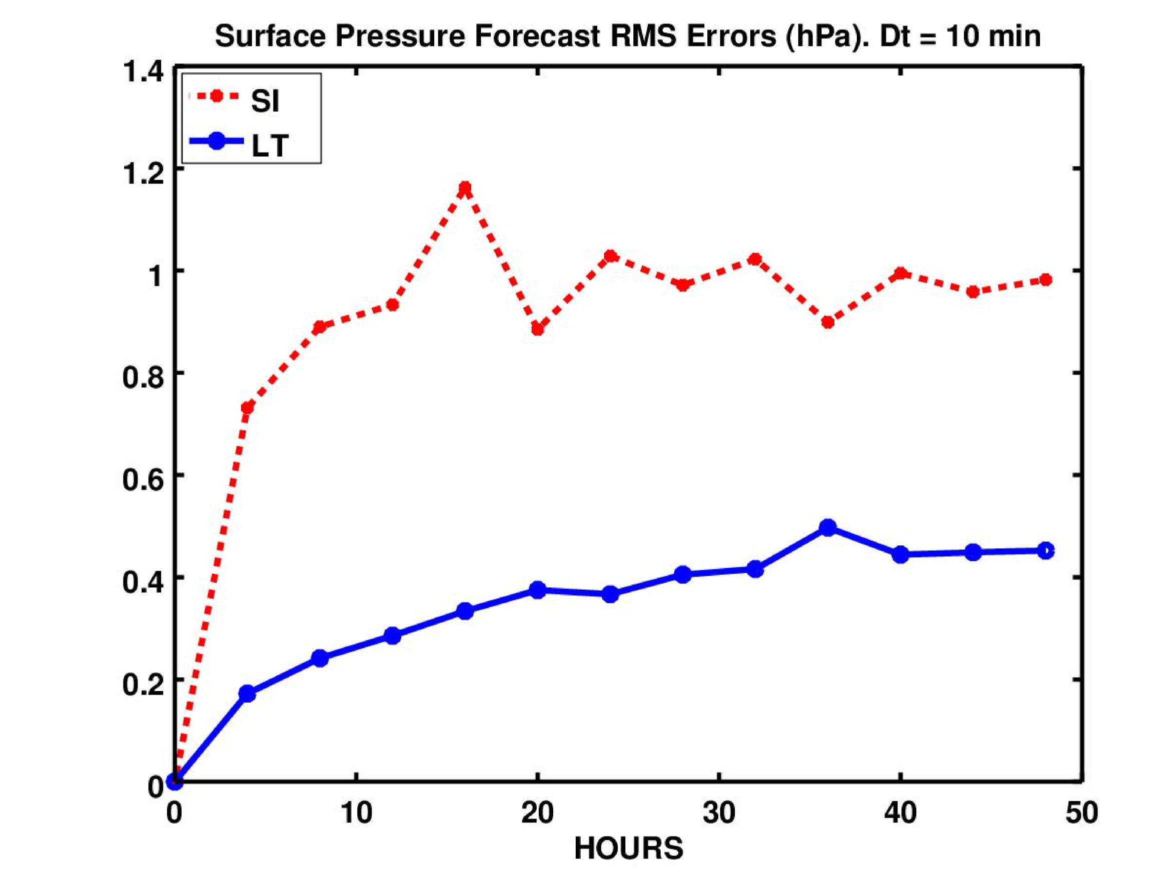
The inverse Laplace transform at time $2\Delta t$ gives the exact solution at that time.

Filtering with the LT Scheme

The LT scheme filters high frequency components by using a modified inversion operator $\mathfrak{L}^*$: this can be done numerically or analytically. The modified inversion operator is the composition of the filter $\mathcal{H}$ and the inverse Laplace transform: $\mathfrak{L}^* = \mathfrak{L}^{-1}\mathcal{H}$.

Recent Developments in Numerical Earth System Modelling

Scores for Real Data Case



Numerical Experiments

The LT scheme is compared to a reference SI scheme. SI and LT were run with 10 and 20 minute timesteps and scored against the reference (1 minute step).

Numerical tests were run with

- Kelvin Wave
- The Five-day Wave
- Rossby-Haurwitz Wave
- Baroclinic Development
- Real Analysis Data

In all cases, scores for LT were comparable to SI. In almost all cases LT was better.

The figure above shows the scores for the "real-data" case. LT is clearly superior to SI.

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LT of the Peak Equations

We apply the LT scheme to a three-dimensional forecasting model, Peak (Ehrendorfer, 2012):

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \boldsymbol{f}_{\boldsymbol{\eta}} - \varsigma \boldsymbol{\eta}$$

$$\frac{\partial \boldsymbol{\delta}}{\partial t} = \boldsymbol{f}_{\boldsymbol{\delta}} - \boldsymbol{\nabla}^2_{\sigma}(\overline{\boldsymbol{T}}\pi + \boldsymbol{\Phi}_* + \mathsf{G}\boldsymbol{T}) - \varsigma \boldsymbol{\delta}$$

$$\frac{\partial \boldsymbol{T}}{\partial t} = \boldsymbol{f}_{\boldsymbol{T}} - \mathsf{H}\boldsymbol{\delta} - \varsigma \boldsymbol{T}$$

$$\frac{\partial \pi}{\partial t} = f_{\pi} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\delta}$$

We now take the Laplace transform of these. We can eventually write the divergence as

$$\boldsymbol{\delta}^+ = \mathsf{P}_A\mathcal{A} + \mathsf{P}_B\mathcal{B} + \mathsf{P}_C\mathcal{C}$$

The P-matrices can be pre-computed and stored, since they do not depend on the model variables. The values of T^+, π^+ and η^+ are

$$\boldsymbol{T}^+ = \boldsymbol{T}^- + 2\Delta t(\boldsymbol{f}_{\boldsymbol{T}} - \varsigma \boldsymbol{T}^- - \mathsf{H}\tilde{\boldsymbol{\delta}}/2\Delta t)$$

$$\pi^+ = \pi^- + 2\Delta t(f_{\pi} - \boldsymbol{p}^{\mathrm{T}}\tilde{\boldsymbol{\delta}}/2\Delta t)$$

$$\boldsymbol{\eta}^+ = \boldsymbol{\eta}^- + 2\Delta t(\boldsymbol{f}_{\boldsymbol{\eta}} - \varsigma \boldsymbol{\eta}^-)\,.$$

Conclusions

The LT scheme provides an attractive alternative to the semi-implicit (SI) scheme, and it provides the possibility of improving weather forecasts at comparable computational cost.

References

- Ehrendorfer, Martin, 2012: *Spectral Numerical Weather Prediction Models.* Soc. Ind. Appl. Math. (SIAM).
- Harney, Eoghan, and Peter Lynch, 2018: Laplace Transform Integration of a Baroclinic Model. *Q.J.Roy.Met.Soc.*, **145**, 347–355.