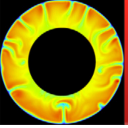

"Anti-Squeeze" for Mantle Convection Simulations in 2-D Spherical Geometries

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MOTIVATION

2-D spherical simulations of mantle convection are popular, either in spherical axisymmetric or spherical annulus geometry.
A problem is that the geometrical restriction forces a downwelling to deform as it sinks, whereas in 3D it can sink with no deformation. Basically, it is "squeezed" in the plane-perpendicular direction, forcing it to expand in the in-plane directions. A rigid/high viscosity downwelling resists this deformation, sinking with a greatly reduced and unrealistic velocity.
This can be solved by subtracting the geometrically-forced deformation ("squeezing") from the strain-rate tensor when calculating the stress tensor. Specifically, components of in-plane and plane-normal strain rate that are proportional to radial velocity are subtracted, a procedure that is here termed "anti-squeeze".
It is here demonstrated that this leads to realistic sinking velocities whereas without it, abnormal and unrealistic results can be obtained for high viscosity contrasts.
This correction has been used since 2010 in the the code StagYY for spherical annulus calculations (Tackley, PEPI 2008; Hernlund and Tackley, PEPI 2008).

Review:

Modeling mantle convection in the spherical annulus

John W. Hernlund[a,*], Paul J. Tackley[b]

Physics of the Earth and Planetary Interiors 171 (2008) 48–54

Conservation of mass is then given by,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^d}\frac{\partial}{\partial r}(r^d \rho v_r) + \frac{\partial}{\partial \phi}\left[\rho\left(\frac{v_\phi}{r}\right)\right] = 0.$$

The radial component of the momentum equation is,

$$\frac{1}{r^d}\frac{\partial}{\partial r}(r^d \tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\phi\phi} + \tau_{\theta\theta}}{r} - \frac{\partial p}{\partial r} - \rho g = 0,$$

while the angular component of momentum is,

$$\frac{1}{r^d}\frac{\partial}{\partial r}(r^d \tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} - \frac{1}{r}\frac{\partial p}{\partial \phi} = 0.$$

The deviatoric stresses τ are given by,

$$\tau_{rr} = 2\mu\left(\frac{\partial v_r}{\partial r}\right) - \left(k_0 + \frac{2\mu}{3}\right)\left[\frac{1}{r^d}\frac{\partial}{\partial r}(r^d v_r) + \frac{\partial}{\partial \phi}\left(\frac{v_\phi}{r}\right)\right], \quad (4)$$

$$\tau_{\phi\phi} = 2\mu\left[\frac{\partial}{\partial \phi}\left(\frac{v_\phi}{r}\right) + \frac{v_r}{r}\right] - \left(k_0 + \frac{2\mu}{3}\right)\left[\frac{1}{r^d}\frac{\partial}{\partial r}(r^d v_r) + \frac{\partial}{\partial \phi}\left(\frac{v_\phi}{r}\right)\right], \quad (5)$$

$$\tau_{\theta\theta} = (d-1)\frac{2\mu v_r}{r} - (d-1)\left(k_0 + \frac{2\mu}{3}\right)\left[\frac{1}{r^d}\frac{\partial}{\partial r}(r^d v_r) + \frac{\partial}{\partial \phi}\left(\frac{v_\phi}{r}\right)\right], \quad (6)$$

$$\tau_{r\phi} = \mu\left[\frac{1}{r}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]. \quad (7)$$

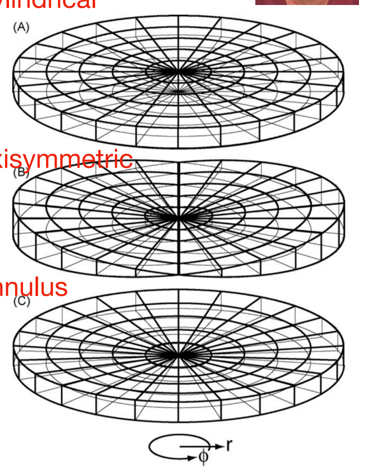
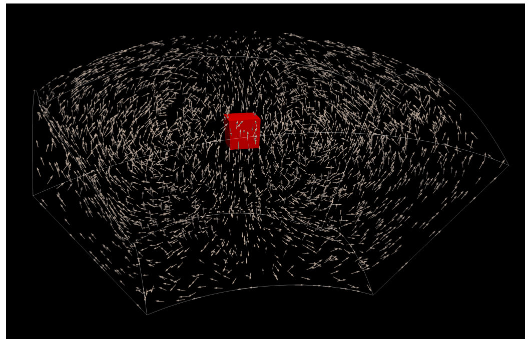


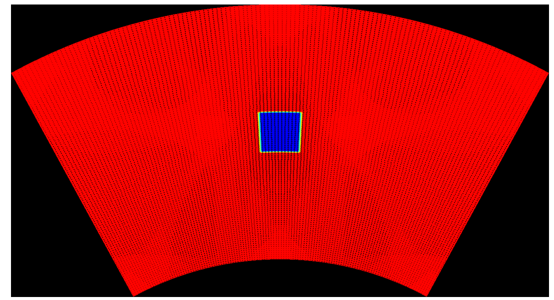
Fig. 2. Illustration of what is meant by the "virtual" thickness J/r of a 2D circular slice through a 3D grid. In the constant thickness case (A), representative of a cylindrical model with effective Jacobian $J = r$, the virtual thickness is constant everywhere. For a variable thickness in the angular direction (B), representative of a spherical axi-symmetric grid with effective Jacobian $J = r^2 \sin\phi$, the virtual thickess depends on the angular location in the grid and the radius. In the variable radial thickness case (C) with effective Jacobian $J = r^2$, the virtual thickness increases with distance from the center of the grid without any angular dependence.

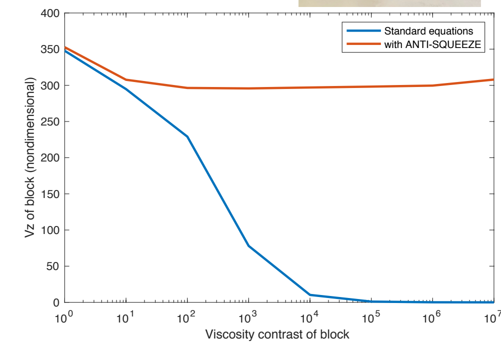
ANALYSIS

In 3-D geometry a rigid/high viscosity block can sink without deforming:



In 2-D spherical it must deform as it sinks, which causes it to sink very slowly (blue curves below) or get stuck (model car, right) unless the anti-squeeze correction given here is applied (red curves):





Incompressible spherical annulus geometry

The stress terms are:

$$\tau_{rr} = 2\eta\frac{\partial v_r}{\partial r} \qquad \tau_{\phi\phi} = 2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right) \qquad \tau_{\theta\theta} = 2\eta\frac{v_r}{r}$$

$$\tau_{r\phi} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) \qquad \tau_{r\theta} = 0 \qquad \tau_{\theta\phi} = 0$$

Note that the spherical annulus is the (r, phi) plane but $\tau_{\theta\theta}$ is not zero!
This can be a problem!
Radially-moving material is squeezed as it sinks, forced to deform as:

$$\dot{e}_{\theta\theta,forced} = \frac{v_r}{r} \quad ; \quad \dot{e}_{rr,forced} + \dot{e}_{\phi\phi,forced} = -\frac{v_r}{r}$$

High-viscosity material doesn't want to deform -> gets 'stuck'.

Solution: Subtract the squeeze!
Subtract forced strain-rates in the normal stress terms.
Assume equal deformation in the rr and theta-theta directions.
=> Anti-squeezed normal stresses:

$$\tau_{rr} = 2\eta\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{2r}\right) \qquad \tau_{\phi\phi} = 2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{3v_r}{2r}\right) \qquad \tau_{\theta\theta} = 0$$

It works! Sinking velocity now ~independent of viscosity (see tests).

Regarding spherical axisymmetric geometry

Stresses are:

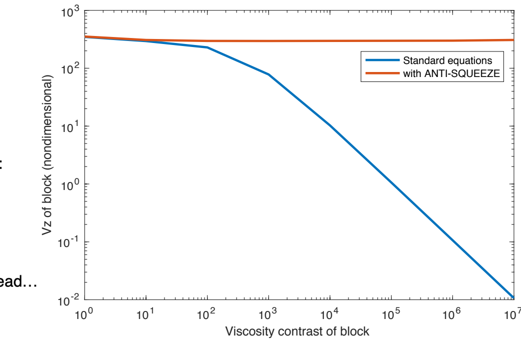
$$\tau_{rr} = 2\eta\frac{\partial v_r}{\partial r} \qquad \tau_{\theta\theta} = 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad \tau_{\phi\phi} = 2\eta\left(\frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)$$

$$\tau_{r\theta} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \qquad \tau_{r\phi} = 0 \qquad \tau_{\theta\phi} = 0$$

The out-of plane stress (phi-phi) is not 0!
Now forced deformation occurs for motion in both radial and theta directions:

$$\dot{e}_{\phi\phi,forced} = -\left(\dot{e}_{\theta\theta,forced} + \dot{e}_{rr,forced}\right) = \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}$$

Anti-squeeze could be used in both directions, or use spherical annulus instead…



Regarding compressibility

Now, normal stresses have an addition term subtracting the strain-rate due to velocity divergence (due to compression/decompression associated with increasing/decreasing pressure).

e.g.

$$\tau_{rr} = 2\eta\left(\frac{\partial v_r}{\partial r} - \frac{1}{3}\nabla\cdot\vec{v}\right)$$

In 2-D (Cartesian or spherical) geometries the factor should be 1/2 instead of 1/3, because the pressure-induced divergence can only be accommodated by strain in 2 dimensions. This is already known in continuum mechanics.

2-D version:

$$\tau_{rr} = 2\eta\left(\frac{\partial v_r}{\partial r} - \frac{1}{2}\nabla\cdot\vec{v}\right)$$

Recent results using annulus + anti-squeeze

Gondwana Research
Volume 73, September 2019, Pages 96-122

Growing primordial continental crust self-consistently in global mantle convection models

Charitra Jain[a, b], Antoine B. Rozel[a], Paul J. Tackley[a], Patrick Sanan[a], Taras V. Gerya[a]

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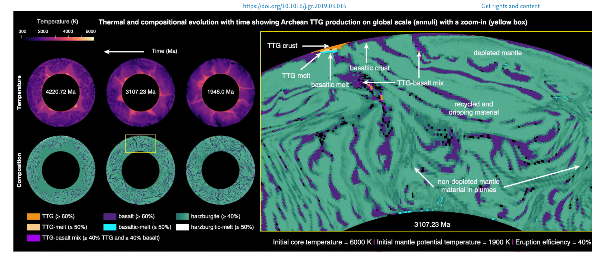

CONCLUSION

In 2-D spherical, use anti-squeeze!