Discussion on:

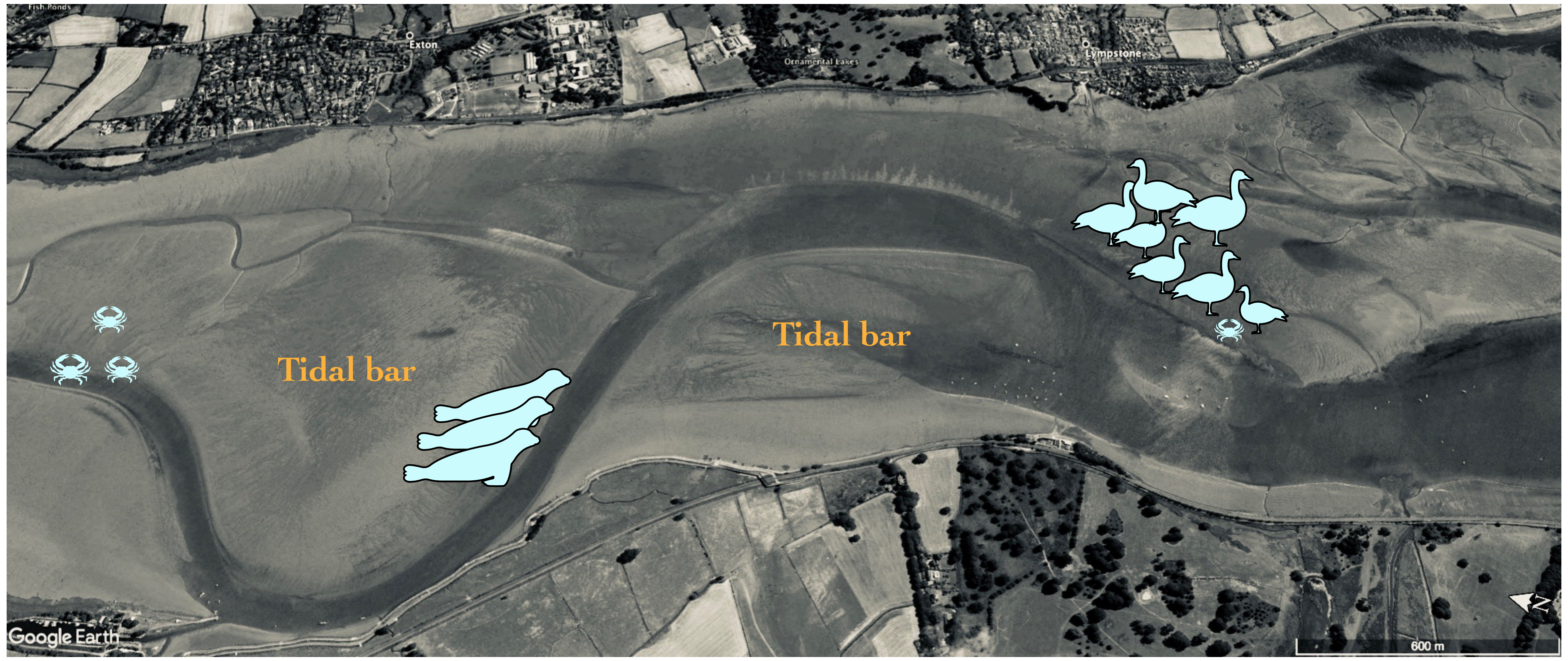
Data-driven reduced order modelling of sand bars in confined tidal channels

Tjebbe Hepkema, (t.m.hepkema@uu.nl)
Huib de Swart,
Henk Schuttelaars,
IMAU Institute for Marine and Atmospheric research Utrecht

Introduction

What are tidal bars?

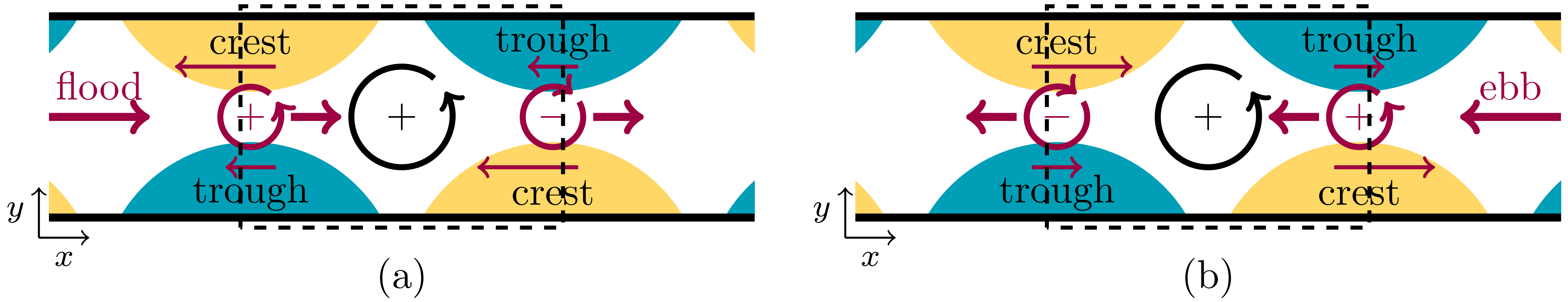
- Tidal bars are bed forms in tidal channels that have a wave-like structure in both the along-channel and cross-channel direction.
- Typically, tidal bars are several meters high, have wavelengths of 1-15 km and migration speeds of meters per day.
- Understanding their dynamics is important as they are invaluable for many living organisms (e.g., migrating birds) but they hamper marine traffic.



Exe estuary (England)

How do tidal bars form?

- Tidal bars owe their existence to the fact that a horizontal, flat bottom with a tidal current flowing over it is unstable.
- This instability can be understood by analysing the residual tidal vorticity induced by bottom perturbations.
- When these bars mature, their dynamics becomes nonlinear.



Generation of tidal residual vorticity cells that results in sediment transport from trough to crest (hence, growth). (a) Situation during flood. (b) Situation during ebb. Friction (thin red arrows) experienced by water column is larger on the crests than on the troughs and generates tidal vorticity (red circles). Both during ebb and flood, positive vorticity is transported into the dashed box and the negative vorticity is transported out, resulting in the build-up of positive vorticity (black circle).

From Hepkema et al. (2019) JGR Earth Surface.

Tjebbe Hepkema, (t.m.hepkema@uu.nl)
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Henk Schuttelaars,

Goal & Methods

Goal

Develop a numerical model and use its output to derive an interpretable low dimensional system of ODEs to analyse the nonlinear dynamics of tidal bars.

Sparse Identification of Nonlinear Dynamics (SINDy) (Brunton et al., 2016, PNAS)

- Project output of the numerical model on a basis with only a few elements (Fourier, certain eigenfunctions or POD or …).
- Come up with a library matrix with potential terms for the low dimensional system (e.g. polynomials) and fill the matrix.
- Approximate the time derivatives of the (POD/Fourier) modes.
- Find candidate model coefficients X with the sequential thresholded least-square algorithm or Lasso.
- Select a model that balances complexity (sparsity of X) and accuracy, using Akaike information criterion (Mangan et al., 2017, Proc. R. Soc. A.)

Numerical model

- 2D depth-averaged shallow water equations.
- Suspended sediment concentration equation.
- Bottom evolution (Exner) equation.
- Straight channel with uniform width and (initial) depth
- Two closed lateral boundaries and two open periodic boundaries.
- Solved by (explicit) finite differences schemes.
- Check stability of 'equilibria' with power method

$h(t,x,y)$ (output numerical model) $\xrightarrow{\text{Fourier or POD or…}}$ $(A_1(t), A_2(t), \dots, A_r(t))$

Calculate ↙ ↘ Approximate

$$t\downarrow\begin{pmatrix} | & | & & | & | & & | & \\ A_1 & A_2 & \cdots & A_1^2 & A_1\bar{A}_1 & \cdots & A_1^3 & \cdots \\ | & | & & | & | & & | & \end{pmatrix}\begin{pmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{N1} & \cdots & X_{Nr}\end{pmatrix} = \begin{pmatrix} | & | & & | \\ \dot{A}_1 & \dot{A}_2 & \cdots & \dot{A}_r \\ | & | & & | \end{pmatrix}$$

Library matrix

Solve for **sparse** X to obtain low (r-)dimensional model:

$$\begin{cases} \dot{A}_1 = X_{11}A_1 + X_{21}A_2 + \dots \\ \vdots \\ \dot{A}_r = X_{1r}A_1 + X_{2r}A_2 + \dots \end{cases}$$

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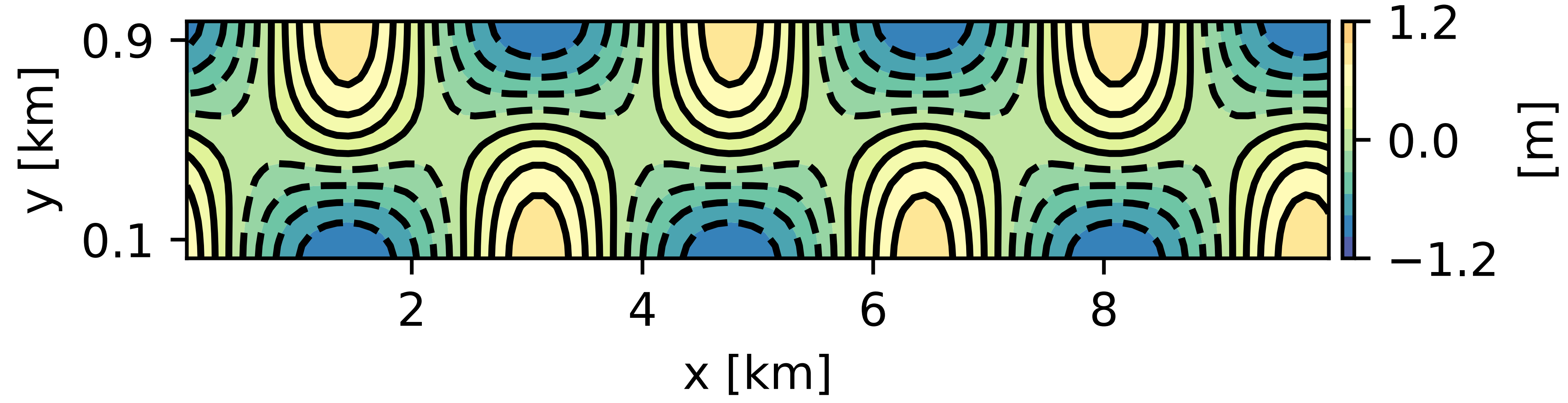
Results

Weakly nonlinear equilibrium



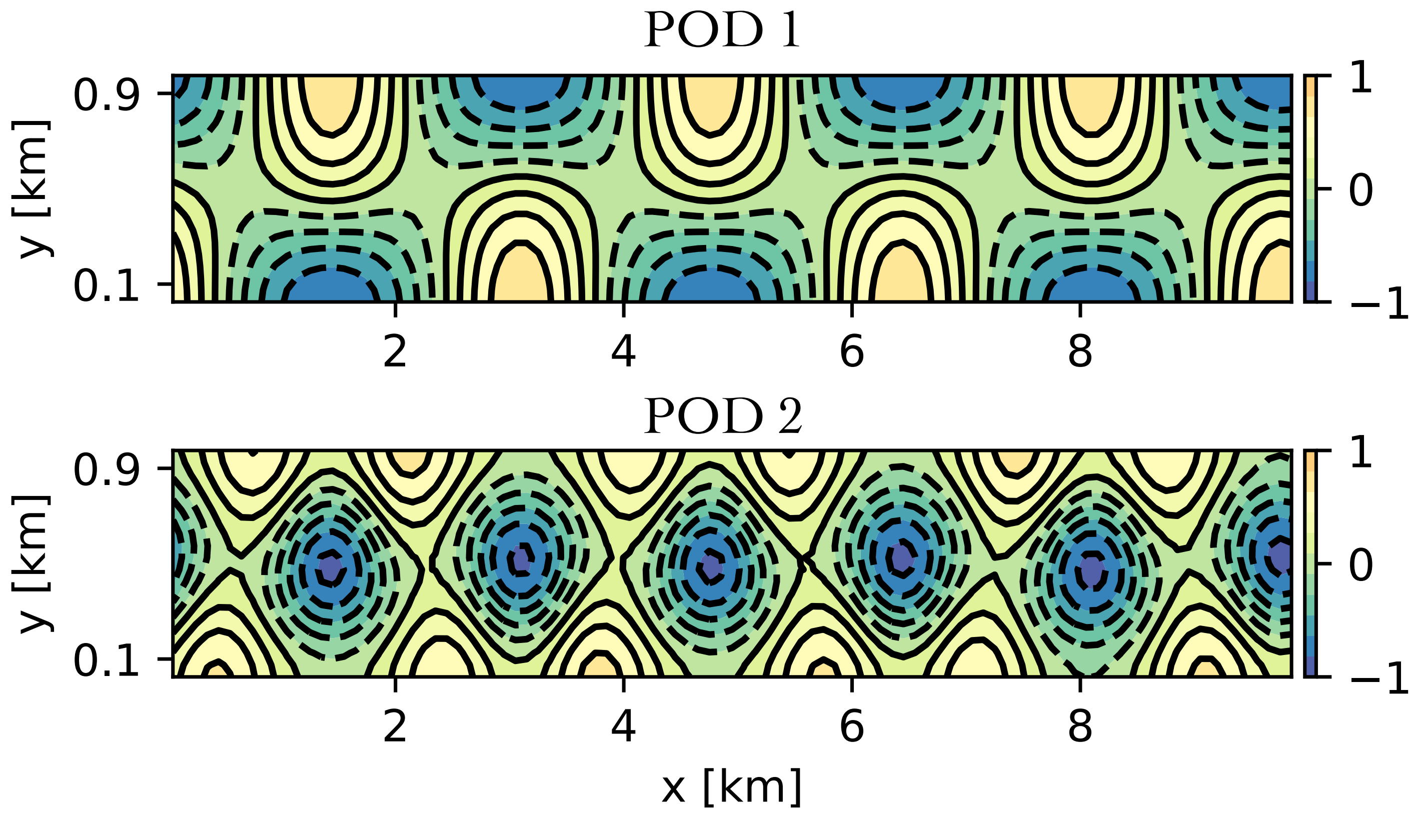
Example of nontrivial equilibrium of the numerical model for parameter values such that the dynamics is weakly nonlinear.

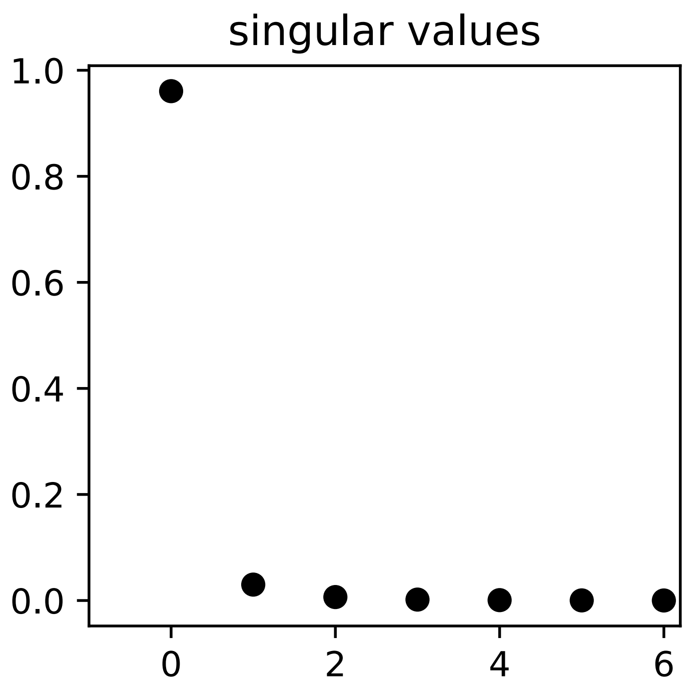
Reduced order model (ROM)

$$\dot{A}_1 = 0.0221A_1 - 0.0019A_1^2 + 0.0079A_1A_2 + 0.00004A_1^3$$
$$\dot{A}_2 = 0.0006A_1 + 0.0025A_1^2 - 0.0119A_1A_2 - 0.00013A_1^3$$

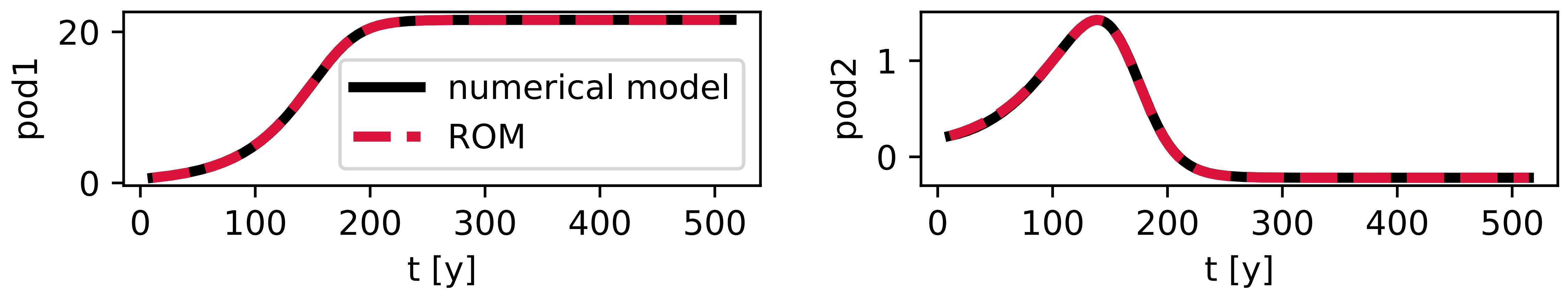
Here, A_1 is POD1 and A_2 is POD2. The dot represents time derivative.

PODs and singular values





Comparison time integration of ROM and numerical model



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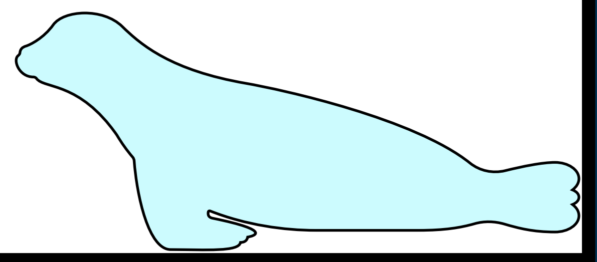
Discussion & Conclusions

Discussion points

- Projection on PODs works, but how to interpret the resulting low dimensional model?
- Projection on interpretable modes seems more difficult.
  - For example, in our case, we understand the physical mechanism behind the growth of Fourier modes on a flat bottom.
- How to choose the relevant (e.g., Fourier) modes?
- …

Discussion points (detail).

- In toy models, it seems that the accuracy of calculating the time derivatives of the PODs/Fourier modes matters. Is this expected/ seen elsewhere?



What can we do with a low dimensional model?

- It helps to understand the physical mechanisms of tidal bar dynamics by means of analysing the interactions of modes.
- It could help with discovering the basin of attraction of certain equilibria.
- It could suggest new equilibria/branches in the bifurcation diagram.

Conclusions

- After projecting the dynamics on PODs, SINDy identifies a low order system.
- After projecting the dynamics on Fourier modes, SINDy has trouble finding an accurate low dimensional system, possibly because not all relevant Fourier modes were taken into account.

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