Numerical solution of the mass continuity equation for snowpack modeling on moving meshes: Coupling between mechanical settling and water (vapor) transport

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Motivation

Well known: The processes **mechanical settling** and **phase change** are coupled in snowpacks

Alpine snowpacks

Polar snowpacks

Snow avalanche risk assessment

Climate models

Current snowpack models lack a sound mathematical coupling of both processes

With such a model we could:

- assess competing effects from **mechanical settling** and **phase change** in the snowpack
- improve representation of snow properties
Ice Mass Balance – Starting Point for a Flexible Snowpack Model Including Settling

The ice phase evolves due to coupled mechanical and metamorphic phase change processes

\[ \partial_t \phi + \partial_z (\phi \cdot v) = \frac{1}{\rho_i} c \]

Microscale processes are captured in macroscale properties

- **Ice volume fraction** \( \phi \): ice volume per total volume
- **Phase change rate** \( c \): loss or gain of ice mass in a specific volume per time
- **Settling velocity** \( v \): settlement due to mechanical strain per time
The snowpack evolves due to coupled mechanical and metamorphic phase change processes

\[ \partial_t \phi + \partial_z (\phi \cdot v) = \frac{1}{\rho_i} c \]

ic volume fraction \( \phi \)
settling velocity \( v \)
phase change rate \( c \)

Challenge:

- **Flexible solution technique** – Solve the ice mass balance for ice volume fraction in a way that can be applied to generic settling velocities and phase change processes.

- **Settling velocity** – Parametrize the settling velocity in a physically consistent way.

- **Metamorphic phase change** – Couple settling to complex phase change operators that result from established snow and firn models.
Flexible Solution Technique – General Idea

The snowpack evolves due to coupled mechanical and metamorphic phase change processes

\[ \partial_t \phi + \partial_z (\phi \cdot v) = \frac{1}{\rho_i} c \]

**Two-step approach:**

**Step 1:** Determine phase change rate from a conventional process model in a **Eulerian** reference frame, e.g. solution for dry snow in model from Hansen and Foslien (2015) (or other process models such as Calonne et al. (2014))

**Step 2:** Use phase change rate to solve ice mass balance for ice volume fraction based on a settling velocity in a **Lagrangian** reference frame, e.g. a mesh strain based on the method of characteristics.

Note: Due to Step 2 the mesh will be distorted, hence Step 1 has to be solved on a non-uniform grid!
Flexible Solution Technique - Mixed Eulerian Lagrangian Solution Method

Method of Characteristics (MOC) to solve non-linear Advection Equation with Source Term

Ice Mass Balance

\[ \partial_t \phi + \partial_z (\phi \cdot v) = \frac{1}{\rho_i} c \]

Apply MOC

Update of ice volume fraction with coupled coordinate update

\[ \partial_t z = v \]
\[ \partial_t \phi = \frac{1}{\rho_i} c + \partial_z v \cdot \phi \]

Analytical

For \( c = \text{const} \) and \( v = \text{const} \) exists an analytical solution

\[ \phi = \frac{1}{\rho_i} c \cdot t + \phi^0 \]
\[ z = v \cdot t + z^0 \]

because \( \partial_z v(\phi, z, \eta) = 0 \)

Numerical

Let \( \phi_k^n := (t_n, z_k) \) \( t_n, n \in \{0, ..., N\} \) be a discretization of time axis \( t \) and \( z_k, k \in \{0, ..., K\} \) be a spatial discretization. Then \( \phi_k^n := \phi(t_n, z_k) \)

\[ \phi_k^{n+1} = \phi_k^n + \Delta t \cdot \left( \frac{1}{\rho_i} c_k^n + \partial_z v \cdot \phi_k^n \right) \]
\[ z_k^{n+1} = z_k^n + \Delta t \cdot v_k^n \]

Flexibility: arbitrary settling velocity closures

Simple constant settling velocity closure only

From the perspective of common snowpack models:
Consider MOC as extension of the „layer boundary motion scheme“ that combines settling with the source term
Settling Velocity – Simple Constant Velocity Closure Leads to Non-realistic Results

Constant settling velocity yields linear advection equation

\[ \partial_t \phi + v \partial_z (\phi) = c \]

Initial ice volume fraction:
\[ \phi (t_0, z) = \phi^0 \]

Snowpack penetrates ground:
\[ v = \text{const} \] does not comply with reality!

Physical constraints have to be incorporated!

Legend
Settling process in a snowpack model has to comply with the following physical constraints:

1) **Non-penetration of the ground**, hence the settling velocity has to vanish at height zero

2) **Incompressibility of ice**, or rather compressibility is only due to a change in volume fraction, such that the snowpack can densify only up to a maximum given value

3) **Self-consolidation**, velocity at location $z$ is dependent on all strain below $z$, hence the settling velocity is given by the integral of the local strain rate

**Reflect physical constraints in settling velocity equations!**
Settling velocity is the integral of the local strain rate:

\[ \dot{\varepsilon}(z) : \text{Local Strain rate} \quad \text{[s}^{-1}] \]

\[ \dot{\varepsilon}(z) = \partial_z v(z) \]

\[ v(z) : \text{Local Settling Velocity} \quad \text{[ms}^{-1}] \]

\[ v(z) = \int_0^z \dot{\varepsilon}(\tilde{z}) \, d\tilde{z} \]

**Observation:**
Settling velocity as integrated from the local strain rate, is inherently non-penetrating, hence complies with constraint 1)

**Strategy:**
Test Mixed Eulerian Lagrangian solution method for a number of strain rate closures

1) Test concept with several strain rates and 2) use them in equation for settling velocity
Settling Velocity – Hierarchy of Test Cases

Strain rates $\dot{\varepsilon}$ of increasing complexity are integrated to settling velocities $v$

Depth dependent

$$\dot{\varepsilon}(z) = D_c(z)$$

$$v(z) = \int_0^z \dot{\varepsilon}(\tilde{z}) \, d\tilde{z}$$

- Strain rate is depth dependent
- $D_c$ : strain rate coefficient
- Velocity independent of ice volume fraction
- Used for numerical benchmark

Ice volume dependent

$$\dot{\varepsilon}(\phi(z)) = D_c \cdot \left(1 - \frac{\phi(z)}{\phi_{\text{max}}(z)}\right)$$

$$v(\phi(z)) = \int_0^z \dot{\varepsilon}(\phi(\tilde{z})) \, d\tilde{z}$$

- Strain rate depends on ice volume fraction $\phi$
- $D_c$ : strain rate coefficient
- Commonly used in firn models

Stress dependent

$$\dot{\varepsilon}(\sigma(z), \eta(T)) = \frac{1}{\eta(T)} \sigma(\phi, z)$$

$$v(\sigma(\phi, z), \eta(T)) = \int_0^z \dot{\varepsilon}(...) \, d\tilde{z}$$

- Strain rate depends on stress $\sigma$ and temperature dependent viscosity $\eta$
- Derived from constitutive equation for mechanics
- Commonly used in snow models

Legend

Run simulations with strain rate/settling velocity pairs

Initial ice volume fract. $\phi^0$ : 2 layer snowpack

Neglect phase change rate $c = 0$
Simulation Results – Depth Dependent Settling

Constant strain rate coefficient $D_c = 10^{-5} \text{s}^{-1}$

We observe that:

- **Snow height:**
  decreases continuously

- **Settling velocity:**
  linear and decreases with time

- **Layer thickness:**
  upper layer decreases faster

- **Ice volume fraction:**
  increases to non-physical value above 1

- Non-penetration
- Incompressibility
- Self-consolidation

1 Intermediate value of fast (>10^{-4} \text{s}^{-1}) and slow (<10^{-6} \text{s}^{-1}) strain rates (Johnson (2011))
Simulation Results – Ice Volume Fraction Dependent Strain Rate

Maximum ice volume fraction $\phi_{\text{max}} = 1$

We observe that:

- **Snow height**: decreases with realistic asymptote
- **Settling velocity**: piecewise linear and decreasing
- **Layer thickness**: Lower layer decreases faster
- **Ice volume fraction**: increases to $\phi_{\text{max}}$

 ✓ Non-penetration
 ✓ Incompressibility
 ✓ Self-consolidation

![Graph showing simulation results for snow height, settling velocity, layer thickness, and ice volume fraction.](image-url)
Simulation Results – Ice Volume Fraction Dependent Strain Rate

Our approach allows a flexible depth dependent definition of $\phi_{\text{max}}(z)$

- $\phi_{\text{max}} = 0.9 \gg \phi^0$
- $\phi_{\text{max}} = 0.35 > \phi^0$
- $\phi_{\text{max}} = 0.25 < \phi^0$
- $\phi_{\text{max}}(z) = \frac{0.4 - 0.9}{H} \cdot z + 0.9 > \phi^0$

![Simulation Results Diagram](image)
Settling Velocity – Stress controlled Strain Rate

Constant snow viscosity $\eta = 355211162 \text{ Pas}^1$

$\eta \to \infty$ for $\phi > 0.95$ to constrain ice volume fraction to values lower than 1

We observe that:

- **Snow height**: decreases with realistic asymptote
- **Settling velocity**: non-linear, decreases with time
- **Layer thickness**: lower layer decreases faster
- **Ice volume fraction**: increases to maximum value 0.95

According to snow viscosity formulation from Vionnet et al. (2012) for $T = 263K$ and $\phi = 0.16$

✓ Non-penetration
✓ Incompressibility
✓ Self-consolidation
Phase Change Term – Determined from a Conventional Process Model

Any type of continuum mechanical process model, that allows for non-uniform grids can be coupled to MOC

Here, we test the coupling with the model from Hansen and Foslien (2015)\(^1\)

**Assumptions:**

- Phase change covers water vapor and ice, so deposition and sublimation only
  
  Referred to as **condensation rate** \(c\) in the following: deposition \(+c\) and sublimation \(−c\)

- Water vapor is always at saturation density

- Mechanical settling neglected

**Mathematical model:**

- Conservation equations for temperature, phase change, ice mass and energy

**Results:**

- Profiles for temperature and condensation rate

**Extend mathematical model for our purposes:** Add settling velocity

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\(^1\) More results of coupled computation can also be read in Simson (2019)
Adjust mathematical model for coupled phase change and settling

(1) Ice mass balance
\[ \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z} (\phi \cdot v) = \frac{c}{\rho_i} \]

(2) Water vapor mass balance
\[ (1 - \phi) \cdot \frac{d \rho_v^{eq}}{dT} \frac{dT}{dt} - \frac{\partial}{\partial z} \left( D_{eff} \cdot \frac{d \rho_v^{eq}}{dT} \frac{dT}{dz} \right) + \rho_v^{eq} \cdot \frac{\partial}{\partial z} (\phi \cdot v) = -c \]

(3) Temperature equation
\[ \left( (\rho C)_eff + (1 - \phi) \cdot \frac{d \rho_v^{eq}}{dT} \cdot L \right) \frac{\partial T}{\partial t} - \frac{\partial}{\partial z} \left( \left( k_{eff} + L \cdot D_{eff} \cdot \frac{d \rho_v^{eq}}{dT} \right) \frac{\partial T}{\partial z} \right) = -L \cdot \rho_v^{eq} \cdot \frac{\partial}{\partial z} (\phi \cdot v) \]

Computational workflow of the coupled system

Simulation run for the coupled system

Legend

1 Derivation in Appendix
We observe that:

- **Snow height & layer thickness:**
  similar to settling only

- **Condensation rate:**
  Deposition in vicinity of layer boundary clearly visible

**Open question:**
Can competing effects be determined from ice volume fraction?
Simulation Results – Settling Velocity Coupled to Process Model

Assess competing effects after 5 days

Settling only vs. Coupled system

- Coupling yields increase in ice volume fraction

Condensation rate only vs. Coupled system

- Coupling yields increased deposition close to layer boundary
Conclusion and Outlook

The introduced model …

- is flexible: can be used with arbitrary strain rate formulations from snow and firn models
- is modular: competing effects of different processes can easily be tested
- combines advantages of Lagrangian and Eulerian formulations, e.g. preserving layer transitions in the ice phase while being coupled to Eulerian formulations of the vapor/water phase
- can be applied to arbitrary, continuous density profiles (does not rely on layers)

Future potential:

- Integrate further processes, e.g. evolution of specific surface area
- Use for model selection, to find dominant processes in common snow regimes
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References


**Legend**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
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<tbody>
<tr>
<td>$\rho_i$</td>
<td>Ice density</td>
<td>$[\frac{kg}{m^3}]$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Ice volume fraction</td>
<td>[-]</td>
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<tr>
<td>$c$</td>
<td>Phase change rate</td>
<td>$[\frac{kg}{m^3 \cdot s}]$</td>
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<tr>
<td>$\rho_{veq}$</td>
<td>Water vapor density at saturation</td>
<td>$[\frac{kg}{m^3}]$</td>
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<td>$D_{eff}$</td>
<td>Effective diffusion coefficient</td>
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<tr>
<td>$z$</td>
<td>Depth coordinates</td>
<td>[m]</td>
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<tr>
<td>$H$</td>
<td>Total height</td>
<td>[m]</td>
</tr>
<tr>
<td>$L$</td>
<td>Latent heat of ice</td>
<td>$[\frac{J}{kg}]$</td>
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<td>$k_{eff}$</td>
<td>Effective thermal conductivity</td>
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<td>$v$</td>
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<td>$D_c$</td>
<td>Strain rate coefficient</td>
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<td>$\sigma$</td>
<td>Stress from overburdened mass</td>
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<tr>
<td>$\eta$</td>
<td>Snow viscosity</td>
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<tr>
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<tr>
<td>$T$</td>
<td>Temperature</td>
<td>[K]</td>
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</table>
Extended Mathematical Model – Water Vapor Mass Balance

Here \( \frac{d\rho_v^{eq}}{dT} \frac{\partial T}{\partial t} \) is equivalent to \( \partial_t \rho_v \) and \( \frac{d\rho_v^{eq}}{dT} \frac{\partial T}{\partial z} \) is equivalent to \( \partial_z \rho_v \)

Ice mass balance: \( \rho_i \cdot \partial_t \phi + \rho_i \cdot \partial_z (\phi \cdot v) = c \)

Water vapor mass balance: \( (1 - \phi) \cdot \partial_t \rho_v - \partial_z (D_{eff} \partial_z \rho_v) - \rho_v \partial_t \phi = -c \)

\( (1 - \phi) \cdot \partial_t \rho_v - \partial_z (D_{eff} \partial_z \rho_v) = -c + \rho_v \partial_t \phi \cdot \frac{\rho_i}{\rho_i} \)

Now, add terms on both sides to prepare substitution of second term on RHS

\( (1 - \phi) \cdot \partial_t \rho_v - \partial_z (D_{eff} \partial_z \rho_v) + \rho_i \cdot \partial_z (\phi \cdot v) \cdot \frac{\rho_v}{\rho_i} = -c + \rho_i \partial_t \phi \cdot \frac{\rho_v}{\rho_i} + \rho_i \cdot \partial_z (\phi \cdot v) \cdot \frac{\rho_v}{\rho_i} \)

Now, substitute second and third term on RHS with ice mass balance

\( (1 - \phi) \cdot \partial_t \rho_v - \partial_z (D_{eff} \partial_z \rho_v) + \rho_v \cdot \partial_z (\phi \cdot v) = c \cdot \left( \frac{\rho_v}{\rho_i} - 1 \right) \)

Assume \( \frac{\rho_v}{\rho_i} \approx 0 \)

\( (1 - \phi) \cdot \partial_t \rho_v - \partial_z (D_{eff} \partial_z \rho_v) + \rho_v \cdot \partial_z (\phi \cdot v) = -c \)