Towards tackling ice sheets-ocean interactions with Finite Element Methods

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What we do

- Methods to study dynamics at interface between ice sheets and ocean

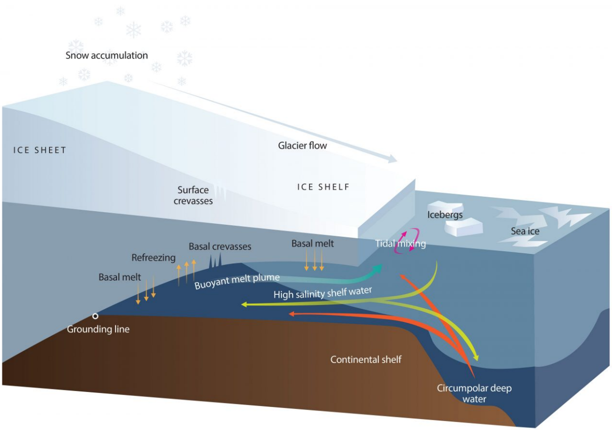


Image credit: Helen Amanda Fricker (Scripps Institution of Oceanography, UC San Diego).

- Project in collaboration with I. Koszalka, J. Wiskandt (Meterology department, Stockholm University)

What we do

Goal: couple ice and ocean in *one* numerical framework, avoid current sub-optimal splitted models.

Current models shortcomings:

- bouncing data back and forth between separate codes; models precision and time scales are quite different
- difficult to have ice that moves
- space discretization technique falls short on thin water columns

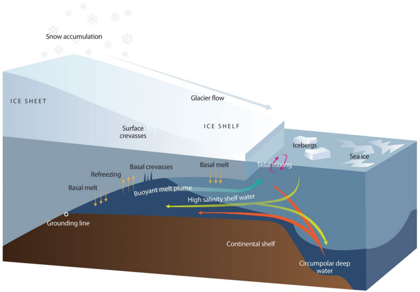


Image credit: Helen Amanda Fricker.

Why we care

- Climate change repercussions: e.g. currents and ocean circulation (even far away) are affected by fresh meltwater; feedback loops with melting/currents change
- Discover unknown phenomena: e.g. ice sheet accelerates when it gets in touch with ocean

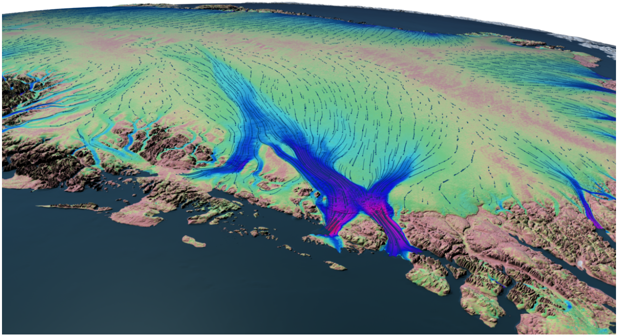
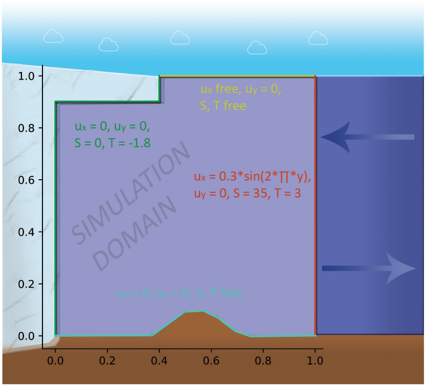


FIGURE: Velocity field; Greenland 79N glacier (https://go.nasa.gov/2VLHBuP)

Intro

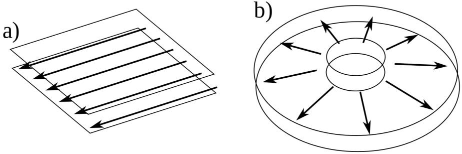
- Four quantities to simulate:
  $\underbrace{\text{velocity } \mathbf{u}}_{\text{Vector valued}}$ $\underbrace{\text{pressure } p\text{, temperature } T\text{, salinity } S}_{\text{Scalar valued}}$
- Start from the ocean alone, with boundary conditions resembling a real scenario



Navier-Stokes Equations

Fluid dynamics $\rightarrow$ Navier-Stokes Equations
(think about Newton's $\mathbf{F} = m\mathbf{a}$, but with body forces)

$$\rho \frac{D\mathbf{u}}{Dt} = \underbrace{-\nabla p}_{Normal\ \ stress\ \ (b)} + \overbrace{\nabla \cdot \tau}^{Tangential\ \ stress\ \ (a)} + \underbrace{f}_{Other\ \ forces}$$



The *Tangential stress* term can be rewritten into a viscosity term; *Other forces* are made up gravity only mostly.

Navier-Stokes Equations

Fluid dynamics → Navier-Stokes Equations
(think about Newton's $\mathbf{F} = m\mathbf{a}$, but with body forces)

$$\rho \frac{D\mathbf{u}}{Dt} = \underbrace{-\nabla p}_{Normal\ \ stress} + \overbrace{\mu\nabla^2\mathbf{u}}^{Viscosity\ \ term} + \underbrace{\rho\, g}_{Gravitational\ \ force}$$

Where the viscosity μ controls how much neighboring particles feel the velocity change.

Navier-Stokes Equations

Fluid dynamics → Navier-Stokes Equations
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Two unknowns → One more equation needed: incompressibility condition

$$\begin{cases} \rho\, \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\, g \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

(Notice: The equation will be stated in weak form, so $\mathbf{u}$ needs not to be twice differentiable at all points.)

Temperature and salinity

- How do temperature and salinity spread?

$$\frac{\partial T}{\partial t} = \underbrace{\mu \nabla^2 T}_{\text{"Temp. conductivity"}} - \overbrace{\nabla \cdot (\mathbf{u} T)}^{\text{Temp. carried by flowing particles}}$$

- T and S are used to compute the density ρ, which is itself a function of position:

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0))$$

- Bousinessq approx. $\rightarrow$ density only variable in gravity term

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho\, g$$

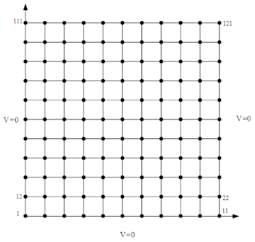
- All together, 4 differential equations for $\mathbf{u}, p, T, S$. Repeatedly solved with slowly advancing timesteps.

Intro

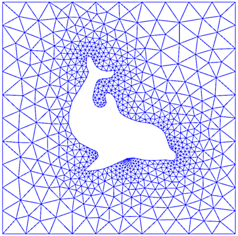
- A differential equation is an ∞-dimensional problem
- Discretization needed, both in time and in space
  Time is easy; space has several options

The FEniCS project is used as computing platform
https://fenicsproject.org/

Finite Differences vs Finite Elements



(A) Finite differences mesh



(B) Finite elements mesh

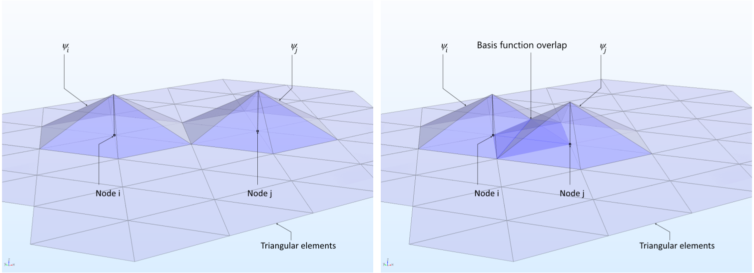
Finite Elements:

- are much more flexible, especially for complex geometries
- allow for easy and specific mesh refinement
- are based on functional analysis $\rightarrow$ solid math analysis
- benefit from weak form

$\rightarrow$ Finite differences used for time; finite elements for space.

Hat functions

Finite Elements rely on hat functions as basis for the space. They are the elementary building blocks to approximate functions.



COMSOL — Multiphysics Cyclopedia

...remental Pressure-Correction Scheme

Navier-Stokes equation requires solving for both $\mathbf{u}$ and p. Solving the system directly is heavy. Anything better?

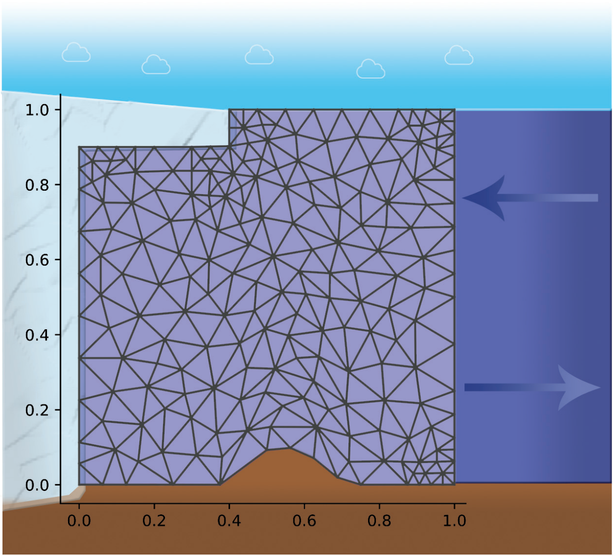
$$\begin{cases} \rho \, \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \, g \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Use a splitting scheme; Euler explicit to solve diff. problems!

1. Solve for $\mathbf{u}$ first, computing a *tentative velocity*, using $\mathbf{u}, p$ from previous iteration. Ignore divergence constraint.
2. Use divergence condition to obtain a tweaked equation that can be solved for p, using the tentative velocity
3. Correct the tentative velocity to get the final one

See *Guermond — An overview of projections methods for incompressible flows*.

Domain Mesh



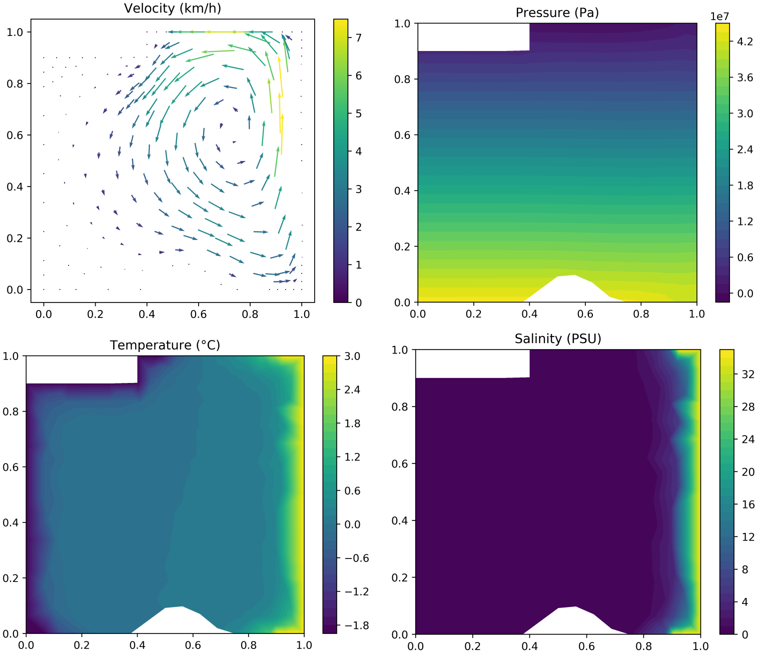


FIGURE: **Academic examples** — Vertexes: 218; Simulation time: 24 hours; Running time: 93 seconds

Future work

- Use real-world geometry relevant to fjords, drawing from real ice-sheets data
- Make a convergence test
- Move model and simulation to 3D
- Introduce ice sheet inside the domain