Stochastic Parameterisation of Uncertainty in Primitive Equation Ocean Models

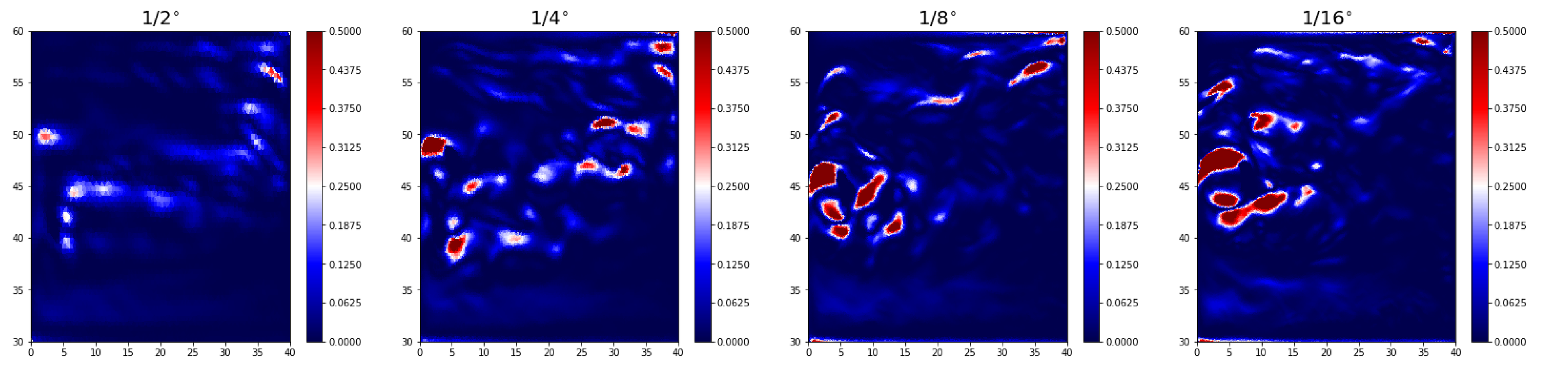
Stuart Patching
Imperial College London

Background

- Global ocean models incur exceptional computational expense.
- Therefore models must be run at low resolutions, and effects of small scales parameterised.
- We want to use a stochastic parameterisation added in such a way that does not destroy the fundamental physics of fluid dynamics.
- SALT (stochastic advection by Lie transport) is a method that does this (see [Hol15], [SC20]).
- We consider the ocean model FESOM2.0, see [DSWJ16].

Differences in flow solution as we change resolution

- One field in which we can see clear differences when we change the resolution is kinetic energy. elow we plot kinetic energy field snapshots for resolutions of $1/2°$, $1/4°$, $1/8°$ and $1/16°$.



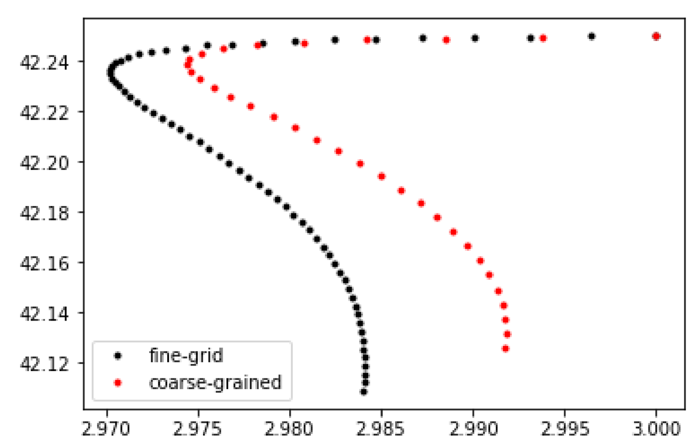
- As we increase the resolution, we see a stronger kinetic energy jet emerging from the western boundary. This kind of feature is what an effective parameterisation scheme should capture.

Lagrangian Paths

- A fluid is made up of a continuum of particles, each following a trajectory determined by the velocity field. To capture uncertainty we add a stochastic part to the particle trajectory:

$$d_t x_t = u(x_t, t)dt + \sum_i \xi_i(x_t) \circ dW_t^i \quad (1)$$

- We determine the ξ_i by taking the velocity field of a simulation on the fine grid, u_f. We then coarse-grain this solution by filtering, to get a smooth field $\overline{u}$. We then calculate the Lagrangian trajectories for each of the fields and look at the difference (see [CCH+18]):



- We use these trajectories to calculate:

$$\Delta X = x_f(t) - \overline{x}(t) \approx \sum_n u(x_f^n, t_n)\Delta t - \sum_m \overline{u}(\overline{x}^m, T_m)\Delta T \quad (2)$$

- The ξ_i are calculated as re-scaled EOFs of the field ΔX.
- An alternative simple approach is to take the difference in velocity fields:

$$\Delta X = (u_f(x,t) - \overline{u}(x,t))\, \Delta T \quad (3)$$

Variational Principles for stochastic Fluids

To use the stochastic Lagrangian paths in a way that preserves the properties of the flow, we use a variational principle involving the following action:

$$S = \int_0^T \underbrace{l(u, D, T)}\, dt + \underbrace{\langle d_t P, D - 1 \rangle}_{\text{incompressibility}} + \underbrace{\left\langle \pi, dx_t - u(x_t, t)dt - \sum_i \xi_i(x_t) \circ dW_t^i \right\rangle}_{\text{Lagrangian particle paths}} \quad (4)$$

The Lagrangian l contains information about the kinetic and potential energy of the system. For the primitive equations it is given by:

$$l(u, D, T) = \int_V \left(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{R}\cdot\mathbf{u} - \int_{z_0}^{z} (1 + B(T(\mathbf{x}, z, t), z'))\, dz' \right) d^3x \quad (5)$$

This consists of kinetic energy $\frac{1}{2}|\mathbf{u}|^2$, rotation $\mathbf{R}\cdot\mathbf{u}$ and potential energy $\int_{z_0}^{z}(1 + B(T(\mathbf{x}, z, t), z'))\, dz'$, where T is potential temperature and B comes from the equation of state, $\rho'/\rho_0 = B(T, z)$.
Deriving the equations in this way preserves important physical properties such as circulation and potential vorticity:

$$d_t \oint_{C(t)} (\mathbf{u} + \mathbf{R}) \cdot dx = (g/\rho_0) \iint_{S(t)} \hat{\mathbf{k}} \times \nabla\rho' \cdot d\mathbf{S}dt \quad (6)$$

$$d_t q + dx_t \cdot \nabla q = 0 \qquad q = \nabla T \cdot (\mathrm{curl}\, \mathbf{u} + f\hat{\mathbf{k}}) \quad (7)$$

Primitive Equations with SALT

The primitive equations with the SALT method are given by:

$$d_t\mathbf{u} + \left[u_3 \cdot \nabla_3 \mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} + \nabla p\right] dt + \sum_i \mathbf{G}_i \circ dW_t^i = (\boldsymbol{\tau} + \mathbf{D}_u)\, dt \quad (8a)$$

$$\nabla_3 \cdot \mathbf{u}_3 = 0 \quad (8b)$$

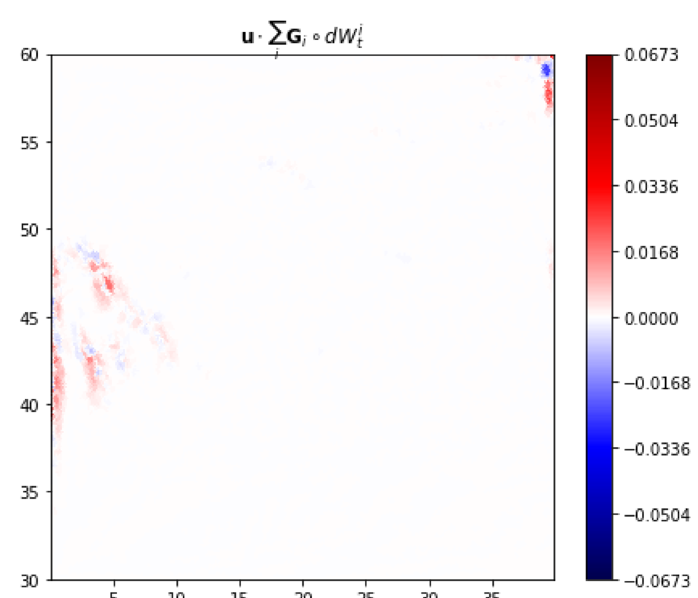
$$\frac{\partial p}{\partial z} = -g(1 + \rho'/\rho_0) \quad (8c)$$

$$d_t T + u_3 \cdot \nabla_3 T dt + \sum_i \xi_i \cdot \nabla_3 T \circ dW_t^i = F_T \quad (8d)$$

$\mathbf{G}_u$ is the stochastic forcing given by:

$$\mathbf{G}_i = \xi_i \cdot \nabla_3 \mathbf{u} + f\hat{\mathbf{k}} \times \xi_i + \nabla \boldsymbol{\xi}_i \cdot \mathbf{u} + \nabla \int_z^0 \frac{\partial \boldsymbol{\xi}_i}{\partial z} \cdot \mathbf{u} dz' \quad (9)$$

This forcing adds kinetic energy to the system at a rate given by $\mathbf{u}\cdot\sum_i \mathbf{G}_i \circ dW_t^i$. We plot this below for a simulation on $1/4°$ grid with ξ_i calculated from a $1/8°$ simulation:



We see that the forcing is acting where needed to increase the kinetic energy.

References

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