Variational Model Reduction for Geophysical Flows with full Coriolis Force

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Introduction

L1 Balance Model

- It gives a simple relationship of velocity field with mass field i.e density.
- This model supplies a simple solution solving dynamical part of it because it filters out inertia gravity waves.
- It is obtained as an expansion of the Lagrangian and transformation for the system which has the balance in the leading order in $\varepsilon = Ro$.
- It keeps the regularity of potential vorticity inversion.
- The velocity field is determined by up to a constant of integration. It is separated as $\boldsymbol{u} = \hat{\boldsymbol{u}} + \bar{\boldsymbol{u}}$.
- $\hat{u}$ causes degeneracy on L1 and fixed via transformation vector.
- $\bar{u}$ is the mean velocity and found by the evolution of potential vorticity.

Aim

- To obtain the balance model using the full Coriolis force
- To show the non-hydrostatic effect on the 3D model

Axis of Rotation

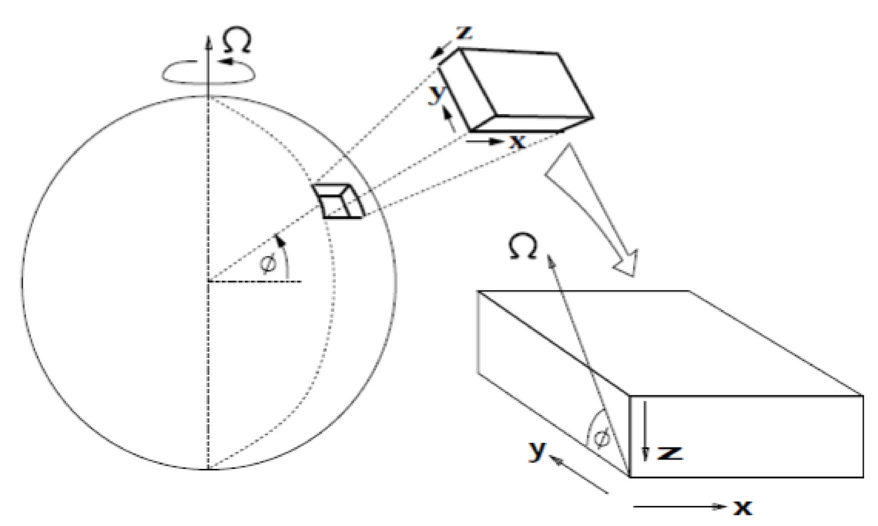


Figure 1: local f plane model positioned at latitude ϕ, [2].

The unit vector in the direction of the axis of rotation is

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 \\ \cos\phi / \sin\phi \\ 1 \end{pmatrix} \qquad (1)$$

Projectors along the rotation axis

$$\mathsf{Q} = \sin^2\phi\, \boldsymbol{\Omega}\boldsymbol{\Omega}^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^2\phi & \cos\phi\,\sin\phi \\ 0 & \cos\phi\,\sin\phi & \sin^2\phi \end{pmatrix} \quad \text{and} \quad \mathsf{P} = \mathsf{I} - \mathsf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\phi & -\cos\phi\,\sin\phi \\ 0 & -\cos\phi\,\sin\phi & \cos^2\phi \end{pmatrix} \qquad (2)$$

The family of characteristics is parameterized by $\boldsymbol{x}$. We can read as

$$\boldsymbol{\chi}(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x} + \boldsymbol{b}\,, \qquad (3)$$

where

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Omega_y \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = H\boldsymbol{\Omega}\,. \qquad (4)$$

Thermal Wind Relation

The fluid domain $\mathcal{D} = \mathbb{R}^2 \times [-H, 0]$. The thermal wind relation as

$$\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \hat{\boldsymbol{u}}_g = -\nabla^{\perp}\rho\,, \qquad (5)$$

where ρ is the density and the geostrophic velocity at any point along the characteristics is

$$\hat{u}_g \circ \boldsymbol{\chi} = -\int_{-H}^{0} \frac{z}{H} \nabla^{\perp}\rho \circ \boldsymbol{\chi}' - \int_{-H}^{z} \nabla^{\perp}\rho \circ \boldsymbol{\chi}' = \nabla^{\perp}\Theta \circ \boldsymbol{\chi}\,. \qquad (6)$$

L1 model for mid-latitude

Euler-Poincaré theorem for continua: $\boldsymbol{u}$ and ρ satisfy

$$\delta \int_{t_1}^{t_2} l(\boldsymbol{u}, \rho)\, dt = 0\,. \qquad (7)$$

L_1-type model which is the Lagrangian in the order of Rossby number is derived for mid-latitude,

$$\begin{aligned} L &= \int_{\mathcal{D}} \boldsymbol{u} \cdot \mathsf{J}\boldsymbol{x} - \rho\, z\, d\boldsymbol{x} + \varepsilon \int_{\mathcal{D}} \boldsymbol{u} \cdot \mathsf{J}\boldsymbol{v} + \frac{1}{2}|\boldsymbol{u}|^2 - \rho\, \boldsymbol{v} \cdot \boldsymbol{k}\, d\boldsymbol{x} \\ &= L_0 + \epsilon L_1\,. \end{aligned} \qquad (8)$$

We seek $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = \mathsf{P}\hat{\boldsymbol{v}} + \mathsf{Q}\hat{\boldsymbol{v}} + \bar{\boldsymbol{v}} \qquad (9)$$

with

$$\mathsf{P}\hat{\boldsymbol{v}} = -\frac{1}{2}\mathsf{J}^{\mathsf{T}}\hat{\boldsymbol{U}} + \lambda\, \mathsf{J}^{\mathsf{T}}\hat{\boldsymbol{U}}_g\,. \qquad (10)$$

In this expression, λ is a free parameter, J is the rotation matrix and

$$\boldsymbol{U}_g = \boldsymbol{u}_g + \boldsymbol{A}[\boldsymbol{u}_g]\,. \qquad (11)$$

We note that $\boldsymbol{U}_g$ takes the same form as $\boldsymbol{U}$, with $\boldsymbol{u}$ replaced by $\boldsymbol{u}_g$

$$L_1 = \int_{\mathcal{D}} \nu\, \hat{u} \cdot \hat{u}_g + \tfrac{1}{2}|\bar{u}|^2 - \frac{\lambda}{\hat{\Omega}_z}|\hat{u}_g|^2\, d\boldsymbol{x}\,, \qquad (12)$$

where $\nu = \lambda + 1/(2\hat{\Omega}_z)$.

Results

L1 Model and Potential vorticity

$$\tilde{\boldsymbol{\Omega}} \times \boldsymbol{u} + \rho\boldsymbol{k} + \varepsilon\,(\partial_t\boldsymbol{p} + (\boldsymbol{\nabla} \times \boldsymbol{p}) \times \boldsymbol{u} - \boldsymbol{\nabla}\rho\, \nabla^{\perp} \cdot B) = -\boldsymbol{\nabla}\phi\,. \qquad (13)$$

with incompressibility property

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,, \qquad (14)$$

where $\boldsymbol{p} = (\bar{\boldsymbol{u}} + \nu\hat{\boldsymbol{u}}_g, 0)$, $\boldsymbol{b} = \nu\hat{\boldsymbol{u}} - \frac{2}{\Omega_z}\hat{\boldsymbol{u}}_g$ and potential $\Phi = -\pi - \frac{1}{2}|\boldsymbol{u}|^2$ with the pressure π. B is the vertical anti-derivative of $\boldsymbol{b}$. The evolution equation for potential vorticity

$$\partial_t\omega + \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\omega = \boldsymbol{\zeta} \cdot \boldsymbol{\nabla}\bar{u}_3 - \overline{\nabla\rho \cdot \nabla^{\perp}\nabla^{\perp} \cdot B} - \nu\,\overline{(\partial_z\nabla\Theta \cdot \nabla u_3 - \Delta\Theta\,\partial_z u_3 + \boldsymbol{u} \cdot \boldsymbol{\nabla}\Delta\Theta)}\,, \qquad (15)$$

where $\boldsymbol{\zeta} = (\partial_z\bar{u}^{\perp}, \nabla^{\perp} \cdot \bar{u})$. The balance relation for $\hat{\boldsymbol{U}}$

$$L\hat{\boldsymbol{U}} = F\hat{\boldsymbol{U}} + G(\bar{\boldsymbol{U}}, \Theta)\,, \qquad (16)$$

where L is the elliptic operator, F is the linear operator. L is elliptic as long as the fluid is stably stratified and $\nabla\rho$ is not too large.

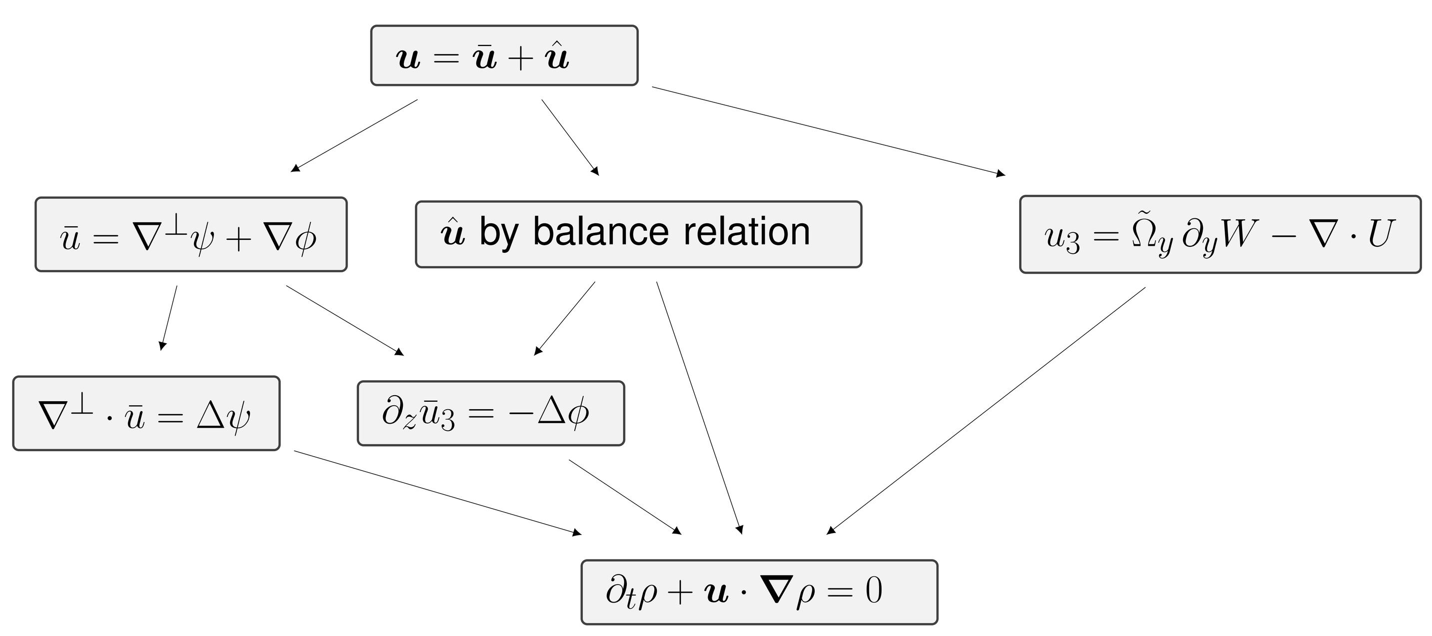
The potential vorticity is

$$q = \begin{vmatrix} -\varepsilon\nu\Delta\Theta - (1 + \varepsilon\omega) & \nabla(\tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{\nabla})\Theta \\ (\tilde{\Omega} + \varepsilon\partial_z\bar{u}^{\perp}) - \varepsilon\nu\partial_z\nabla\Theta & \partial_z(\tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{\nabla})\Theta \end{vmatrix}\,, \qquad (17)$$

which has the same ellipticity conditions with balance relation (16).

General Solution

Solution strategy: $L\hat{U} = F\hat{U} + G$



Ideas for future work

- The numerical solution is expected.
- Validity of variational principle for anisotrophic scaling for the equatorial long wave scaling is under progress.

References

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