Laplacian Structure, Solution
Domain Geometry and Successive Approximations in Gravity Field Studies

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1. Introduction

Green’s functions are an important tool in solving problems of mathematical physics. Equally this holds in gravity field studies.

- Green’s function is an integral kernel, which, convolved with input values, gives the solution of the particular problem considered.

- Regarding its construction, there exist elegant and powerful methods for one or two dimensional problems.

- However, only very few of these methods carried over to higher dimensions. The higher the dimension of the Euclidean space the simpler the boundary of the region of interest had to be.

- In order to preserve the benefit of the Green’s function method an approximation procedure is discussed. Our aim is to implement the procedure with the particular focus on the solution of the LGBVP (Linear gravimetric boundary value problem) and on the related functional analytic aspects.
2. Linear Gravimetric Boundary-Value Problem

For this problem the solution domain $\Omega$ is the exterior of the Earth and the problem means to find $T$ such that

$$\Delta T = \text{div} \ \text{grad} \ T = 0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial T}{\partial s} = \langle s, \text{grad} \ T \rangle = -\delta g \quad \text{on} \quad \partial \Omega,$$

where

$$s = -\frac{1}{\gamma} \ \text{grad} \ U,$$

$\langle , \rangle$ is the inner product, $\Delta$ means Laplace’s operator and $\partial \Omega$ is the boundary of $\Omega$.

Note that $T = W - U$ is the disturbing potential and $\delta g = g - \gamma$ the gravity disturbance, where with $W$ and $U$ we identify the gravity and a normal potential of the Earth.
To be more specific, we recall the classical theory of normal gravity referred to a level ellipsoid with semiaxes \( a \) and \( b \), \( a \geq b \) and the linear eccentricity \( E = \sqrt{a^2 - b^2} \).

- In this case we can introduce \textit{ellipsoidal coordinates} \( u, \beta, \lambda \) related to \textit{Cartesian coordinates} \( x_1, x_2, x_3 \) by the equations

  \[
  x_1 = \sqrt{u^2 + E^2} \cos \beta \cos \lambda , \quad x_2 = \sqrt{u^2 + E^2} \cos \beta \sin \lambda , \quad x_3 = u \sin \beta .
  \]

- In addition we will suppose that a \textit{function} \( h(\beta, \lambda) \) \textit{describes the boundary} \( \partial \Omega \) of \( \Omega \) with respect to the level ellipsoid \( u = b \), i.e. \( \partial \Omega \) is represented by

  \[
  x_1 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos \beta \cos \lambda , \quad x_2 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos \beta \sin \lambda , \quad x_3 = [b + h(\beta, \lambda)] \sin \beta .
  \]
Now we return to the **LGBVP (Linear Gravimetric Boundary-Value Problem)**. We can interpret the boundary condition

\[
\frac{\partial T}{\partial s} = \langle s, \text{grad } T \rangle = -\delta g \quad \text{on} \quad \partial \Omega
\]

in terms of a derivative of \( T \) with respect to \( u \), i.e.

\[
\frac{\partial T}{\partial u} = -w(b + h, \beta) \delta g \quad \text{on} \quad \partial \Omega,
\]

where

\[
w(u, \beta) = \sqrt{\frac{u^2 + E^2 \sin^2 \beta}{u^2 + E^2}}.
\]

In solving the **LGBVP** a **transformation of coordinates** will be applied.

This will open a way for an alternative between the **boundary complexity** and the **complexity of the coefficients** of the partial differential equation governing the solution.
3. Transformation of Coordinates and an Attenuation Function

Our starting point will be the mapping as above, i.e.

\[
\begin{align*}
x_1 &= \sqrt{u^2 + E^2} \cos \beta \cos \lambda, \\
x_2 &= \sqrt{u^2 + E^2} \cos \beta \sin \lambda, \\
x_3 &= u \sin \beta,
\end{align*}
\]

but with

\[u = z + \omega(z) h(\beta, \lambda),\]

where \(z\) is a new coordinate and \(\omega(z)\) is a \emph{twice continuously differentiable attenuation function} defined for \(z \in [b, \infty)\), such that
\[ \omega(z) h(\beta, \lambda) > - b \ , \]
\[ \omega(b) = 1 \ , \quad \frac{d\omega}{dz}(b) = 0 \]

and
\[ \omega(z) = 0 \quad \text{for} \quad z \in \left[ z_{ext}, \infty \right) \ , \quad \text{where} \quad b < z_{ext} \ . \]

**Stress** that the assumption concerning the continuity of \( \omega \) and its 1st and the 2nd derivatives implies
\[ \lim_{z \to z_{ext}^-} \omega(z) = 0 \ , \quad \lim_{z \to z_{ext}^-} \frac{d\omega(z)}{dz} = 0 \ , \quad \lim_{z \to z_{ext}^-} \frac{d^2\omega(z)}{dz^2} = 0 \quad \text{for} \quad z \to z_{ext}^- \ . \]

**Note:** \( z, \beta, \lambda \) form a system of new curvilinear coordinates and for
\[ \frac{du}{dz} = 1 + \frac{d\omega}{dz} h > 0 \ , \]
The transformation is a one-to-one mapping between $\Omega$ and the outer space $\Omega_{\text{ell}}$ of our oblate ellipsoid of revolution.

The construction of the attenuation function $\omega(z)$ in the interval $[b, z_{\text{ext}})$, i.e. for $b \leq z < z_{\text{ext}}$, deserves some attention. Here we give an example (applied in this work). We put

$$\omega(z) = \exp \left[ 2 - \frac{2(\Delta z)^2}{(\Delta z)^2 - (z-b)^2} \right], \quad \text{where} \quad \Delta z = z_{\text{ext}} - b,$$

i.e.

$$\omega(z) = e^{\left[ 2 - \frac{2(\Delta z)^2}{(\Delta z)^2 - (z-b)^2} \right]} \quad \text{with} \quad e \approx 2,71828.$$

The attenuation function and its derivatives are demonstrated on the following slide.
4. Transformation of the Boundary Condition

In the coordinates \( z, \beta, \lambda \) the boundary \( \partial \Omega \) is defined by \( z = b \) and its image \( \partial \Omega_{\text{ell}} \) coincides with our oblate ellipsoid.

In addition the transformation changes the formal representation of the LGBVP. Indeed, the boundary condition turns into

\[
\frac{\partial T}{\partial z} = -w\left[z + \omega(z) h(\beta, \lambda)\right] \delta g \quad \text{for} \quad z = b.
\]

Hence, denoting by \( \partial / \partial n \) the derivative in the direction of the unit (outer) normal \( n \) of \( \partial \Omega_{\text{ell}} \), we obtain

\[
\frac{\partial T}{\partial n} = -\sqrt{1 + \varepsilon} \delta g \quad \text{on} \quad \partial \Omega_{\text{ell}},
\]

where

\[
\varepsilon = \frac{E^2 (2bh + h^2) \cos^2 \beta}{(a^2 \sin^2 \beta + b^2 \cos^2 \beta) [(b + h)^2 + E^2]}
\]

may practically be neglected.
5. Laplacian and Topography-Dependent Coefficients

It is somewhat more complicated to express Laplace’s operator of $T$ in terms of the coordinates $z, \beta, \lambda$, which do not form an orthogonal system. We will use the tensor calculus. Recalling

$$\begin{align*}
x_1 &= \sqrt{\left[ z + \omega(z) h(\beta, \lambda) \right]^2 + E^2 \cos\beta \cos\lambda}, \\
x_2 &= \sqrt{\left[ z + \omega(z) h(\beta, \lambda) \right]^2 + E^2 \cos\beta \sin\lambda}, \\
x_3 &= \left[ z + \omega(z) h(\beta, \lambda) \right] \sin\beta
\end{align*}$$

and putting $y_1 = z$, $y_2 = \beta$, $y_3 = \lambda$, we easily deduce that the Jacobian

$$J = \left| \frac{\partial x_i}{\partial y_j} \right| = -\left(1 + \frac{d\omega}{dz}h\right)\left[ (z + \omega h)^2 + E^2 \sin^2 \beta \right] \cos\beta < 0,$$

apart from its zero values for $\beta = -\pi/2$ and $\pi/2$. Thus, the transformation is a one-to-one mapping.
Moreover, for Laplace’s operator $\Delta$ applied on $T$ we generally have

$$\Delta T = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_i} \left( \sqrt{g} g^{ij} \frac{\partial T}{\partial y_j} \right) = g^{ij} \frac{\partial^2 T}{\partial y_i \partial y_j} + \frac{1}{\sqrt{g}} \frac{\partial g^{ij}}{\partial y_i} \frac{\partial T}{\partial y_j}.$$  

Here $g = \left| g_{ij} \right| = J^2$ is the determinant related to the respective metric tensor $g_{ij}$ and $g^{ij}$ means the associated metric tensor.

After some algebra and neglecting the difference

$$w^2(z + \omega h, \beta) - w^2(z, \beta) \leq \frac{E^2}{z^2} \left[ 2\omega \frac{h}{z} + \left( \omega \frac{h}{z} \right)^2 \right] \cos^2 \beta,$$

we can deduce

$$\Delta T = \frac{z^2 + E^2 \sin^2 \beta}{(z + \omega h)^2 + E^2 \sin^2 \beta} \left[ \Delta_{\text{ell}} T - \delta(T, h) \right],$$
where
\[
\Delta_{\text{ell}} T = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[ \left( z^2 + E^2 \right) \frac{\partial^2 T}{\partial z^2} + 2z \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial \beta^2} - \frac{\sin \beta}{\cos \beta} \frac{\partial T}{\partial \beta} + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2)^2 \cos^2 \beta} \frac{\partial^2 T}{\partial \lambda^2} \right],
\]

\[
\delta (T, h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{w(z + \omega h, \beta)}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial^2 T}{\partial z \partial \lambda},
\]

and \( A_i \) are topography dependent coefficients given by

\[
A_1 = \left( 1 + \frac{d \omega}{dz} h \right)^{-1} \left[ 2 \left( \frac{d \omega}{dz} - \frac{\omega}{z} \right) \frac{zh}{z^2 + E^2 \sin^2 \beta} + \omega \Delta_E h \right] - 2 \left( 1 + \frac{d \omega}{dz} h \right)^{-2} \omega \frac{d \omega}{dz} |\text{grad}_E h|^2 + \left( 1 + \frac{d \omega}{dz} h \right)^{-3} \left[ \frac{(z + \omega h)^2 + E^2}{z^2 + E^2 \sin^2 \beta} + \omega^2 |\text{grad}_E h|^2 \right] \frac{d^2 \omega}{dz^2} h,
\]
\[ A_2 = \left(1 + \frac{d\omega}{dz} h\right)^{-2} \left\{ 2 \left( \frac{d\omega}{dz} - \frac{\omega z}{z^2 + E^2} \right) h + \left[ \left( \frac{d\omega}{dz} \right)^2 - \frac{\omega^2}{z^2 + E^2} \right] h^2 \right\} \frac{z^2 + E^2}{z^2 + E^2 \sin^2 \beta} - \left(1 + \frac{d\omega}{dz} h\right)^{-2} \omega^2 \left| \text{grad}_E h \right|^2 , \]

\[ A_3 = \left(1 + \frac{d\omega}{dz} h\right)^{-1} \frac{2\omega}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial h}{\partial \beta} , \quad A_4 = \left(1 + \frac{d\omega}{dz} h\right)^{-1} \frac{2\omega w(z + \omega h, \beta)}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial h}{\partial \lambda} \]

with

\[ \left| \text{grad}_E h \right|^2 = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[ \left( \frac{\partial h}{\partial \beta} \right)^2 + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2) \cos^2 \beta} \left( \frac{\partial h}{\partial \lambda} \right)^2 \right] \]

and

\[ \Delta_E h = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[ \frac{\partial^2 h}{\partial \beta^2} - \frac{\sin \beta}{\cos \beta} \frac{\partial h}{\partial \beta} + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2) \cos^2 \beta} \frac{\partial^2 h}{\partial \lambda^2} \right] \]

being the 1st and the 2nd Beltrami differential operators.
6. Linear GBVP and Neumann’s Function

The disturbing potential \( T \) is a harmonic function in the original solution domain \( \Omega \). In the space of the curvilinear coordinates \( z, \beta, \lambda \), therefore, \( T \) satisfies Laplace’s equation \( \Delta T = 0 \) for \( z > b \). This yields

\[
\Delta_{\text{ell}} T = \delta(T, h) \quad \text{for} \quad z > b ,
\]

where

\[
\delta(T, h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{w(z + \omega h, \beta)}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial^2 T}{\partial z \partial \lambda}.
\]

Hence the linear gravimetric boundary value problem attains the form

\[
\Delta_{\text{ell}} T = f \quad \text{in} \quad \Omega_{\text{ell}} ,
\]

\[
\frac{\partial T}{\partial n} = - \sqrt{1 + \varepsilon} \delta g \quad \text{on} \quad \partial \Omega_{\text{ell}} ,
\]

where \( f = \delta(T, h) \) and \( \varepsilon \) may be omitted.
Neglecting the fact that \( f = \delta(T, h) \) depends on \( T \), we can represent the solution of the problem formally by means of a classical apparatus of mathematical physics.

Indeed, we may construct the respective Green’s function of the second kind \( N(x, y) \), which solves Neumann’s boundary value problem for the domain \( \Omega_{\text{ell}} \). Then (formally)

\[
T(y) = \frac{1}{4\pi} \int_{\partial\Omega_{\text{ell}}} \delta g(x) N(x, y) d_x S - \frac{1}{4\pi} \int_{\Omega_{\text{ell}}} f(x) N(x, y) d_x V .
\]

7. Iteration Process

Nevertheless, Neumann’s function \( N(x, y) \) can also be used to solve the transformed LGBVP, where \( f = \delta(T, h) \) depends on \( T \). In this case the integral formula above represents an integro-differential equation for \( T \).

For clarity we can put
\begin{align*}
F(y) &= \frac{1}{4\pi} \int_{\partial \Omega_{\text{ell}}} \delta g(x) N(x, y) \, d_x S \\
KT(y) &= -\frac{1}{4\pi} \int_{\Omega_{\text{ell}}} \delta \left[ T(x), h(x) \right] N(x, y) \, d_x V,
\end{align*}

where \( F(y) \) is a harmonic function and \( KT(y) \) is an integro-differential operator applied on \( T \), such that

\[ \Delta_{\text{ell}} KT = \delta(T, h) \quad \text{in} \quad \Omega_{\text{ell}} \quad \text{and} \quad \frac{\partial KT}{\partial n} = 0 \quad \text{on} \quad \partial \Omega_{\text{ell}}. \]

Under this notation the problem is to find \( T \) from

\[ T = F + KT. \]

We apply the method of successive approximations, i.e.

\[ T = \lim_{n} T_n, \quad T_n = F + KT_{n-1}, \]

where \( T_0 \) is the starting approximation, e.g. \( T_0 = F \).
8. Function Spaces and Estimates of a Particular Solution of Poisson’s Equation

Our aim is to examine whether the iteration process converges. The investigation will be based on Banach’s fixed point theorem interpreted for functions from a Sobolev weight space.

Let \( W_2^{(1)}(\Omega_{ell}) \) be a Sobolev weight space equipped by the norm

\[
\|u\|_1 = \left[ \int_{\Omega_{ell}} \frac{1}{2} u^2 \, dV + \int_{\Omega_{ell}} \|\text{grad } u\|^2 \, dV \right]^{1/2},
\]

Similarly, let \( W_2^{(2)}(\Omega_{ell}) \) be a Sobolev weight space with the norm

\[
\|u\|_2 = \left[ \|u\|_1 + \sum_{i=1}^{3} \int_{\Omega_{ell}} \|\text{grad } (\text{grad}_i u)\|^2 \, dV \right]^{1/2}. \tag{1}
\]

It is obvious that \( W_2^{(2)}(\Omega_{ell}) \subset W_2^{(1)}(\Omega_{ell}) \).
The crucial point for the use of Banach’s theorem is to show that the operator \( K \) is a contraction mapping. In our case it means to show that there is a constant \( \alpha < 1 \) such that the inequality

\[
\| Ku - Kv \|_2 \leq \alpha \| u - v \|_2
\]

holds for arbitrary functions \( u \) and \( v \) from \( W_2^{(2)}(\Omega_{ell}) \), provided that also \( F \) belongs to \( W_2^{(2)}(\Omega_{ell}) \). Considering the linearity of the operator \( K \), we can see that it is enough to show that

\[
\| KT' \|_2 \leq \alpha \| T' \|_2
\]

with \( \alpha < 1 \) holds for any \( T' \) from \( W_2^{(2)}(\Omega_{ell}) \), still provided that \( F \) belongs to \( W_2^{(2)}(\Omega_{ell}) \).
Recall that in our investigation we particularly have
\[ u(y) = KT'(y) = -\frac{1}{4\pi} \int_{\Omega_{ell}} \delta[T'(x), h(x)] N(x, y) \, dx \, V \] (2)

We will study first how the estimate \( u \) by means of \( \delta \).

From the fundamental properties of Neumann’s function it can be deduced that \( u \) is a solution of the following boundary-value problem
\[ \Delta_{ell} u = \delta(T', h) \quad \text{for} \quad z > b, \ i.e. \ y \in \Omega_{ell} \] (3)
and
\[ \frac{\partial u}{\partial n} = 0 \quad \text{for} \quad z = b, \ i.e. \ y \in \partial \Omega_{ell} . \] (4)

It is thus clear that \( u \) can be estimated as the solution of the above problem in \( W_{2}^{(2)}(\Omega_{ell}) \).
In the 1st step we will estimate $u$ as a function from $W_2(\Omega_{ell})$, i.e. as the so-called \textit{weak solution of the problem} mentioned above. We multiply Eq. (3) by an arbitrary function $v$ from $W_2(\Omega_{ell})$ and integrate this product over $\Omega_{ell}$. We obtain

$$\int_{\Omega_{ell}} v \Delta_S u \, dV = \int_{\Omega_{ell}} v \delta(T', h) \, dV.$$ 

Moreover, the \textit{integration by parts (Green’s first identity)} on the left hand side together with the boundary condition (4) results in

$$A(v, u) = - \int_{\Omega_{ell}} v \delta(T', h) \, dV, \quad \text{where}$$

$$A(v, u) = \int_{\Omega_{ell}} \langle \text{grad } v, \text{grad } u \rangle \, dV$$

Similarly, we denote the right hand side of Eq. (5) by $Fv$, i.e.

$$Fv = - \int_{\Omega_{ell}} v \delta(T', h) \, dV.$$
Following now the Hölder inequality, we have

\[
|Fv| \leq \left( \int_{\Omega_{ell}} z^2 \delta^2 (T', h) \, dV \right)^{1/2} \left( \int_{\Omega_{ell}} \frac{1}{z^2} v^2 \, dV \right)^{1/2}, \text{ i.e.}
\]

\[
|Fv| \leq \left\| z \delta (T', h) \right\|_{L^2(\Omega_{ell})} \left\| v \right\|_1
\]

Thus \( Fv \) is a continuous functional and for its norm the following estimate holds

\[
\left\| F \right\| \leq \left\| z \delta (T', h) \right\|_{L^2(\Omega_{ell})}.
\]

To estimate the norm of \( u \) we still have to examine whether \( A(v,u) \) is a coercive bilinear form. We have to show that

\[
A(v,v) \geq \alpha \left\| v \right\|_1^2, \quad \alpha = \text{const.} > 0,
\]

is valid for all \( v \) from \( W^{(1)}_2(\Omega_{ell}) \). According to (Holota, 1991, 1997) we can put

\[
\alpha = \frac{1}{5}.
\]
Our further considerations are based on the simple \textit{Lax-Milgram generalization of the famous Riesz representation theorem.}

The assumptions of the Lax-Milgram theorem are met. Indeed, the bilinear form $A(v,u)$ is coercive, as we have shown above. Moreover, it is also continuous, i.e., there exists a constant $(\text{const.})$ such that

$$|A(v,u)| \leq (\text{const.}) \|v\|_1 \|u\|_1$$

holds for all $v$ and $u$ from $W_2^1(\Omega_{\text{ell}})$. A simple proof is left for the reader. Recalling our notation, we have

$$A(v,u) = Fv$$

in view of Eq. (5) and we can apply the \textit{Lax-Milgram theorem} immediately. We arrive at

$$\|u\|_1 \leq 5 \|z \delta(T', h)\|_{L_2(\Omega_{\text{ell}})}$$

which is our desired estimate.
9. Calderón-Zygmund Inequality

Now we will estimate $u$ as a function from $W_2^{(2)}(\Omega_{ell})$, i.e. in the norm

$$\|u\|_2 = \left[ \left\| u \right\|_1 + \sum_{i=1}^{3} \int_{\Omega_{ell}} \left| \text{grad}(\text{grad}_i u) \right|^2 dV \right]^{1/2}.$$

As to $\|u\|_1$, we can use the result of the preceding section. Hence it remains to estimate the term

$$D = \sum_{i=1}^{3} \int_{\Omega_{ell}} \left| \text{grad}(\text{grad}_i u) \right|^2 dV = \sum_{i=1}^{3} \int_{b < z} \left| \text{grad}(\text{grad}_i u) \right|^2 dV.$$

Let us approach first

$$D' = \sum_{i=1}^{3} \int_{b < z < b_e} \left| \text{grad}(\text{grad}_i u) \right|^2 dV.$$

Putting

$$D_e = \sum_{i=1}^{3} \int_{z < b_e} \left| \text{grad}(\text{grad}_i u) \right|^2 dV,$$

we have $D' \leq D_e$. 

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Applying now the Green identity twice, we obtain

\[ D_e = \sum_{i=1}^{3} \int_{z=b_e} (\nabla_i u) \frac{\partial}{\partial n} (\nabla_i u) dS - \sum_{i=1}^{3} \int_{z<b_e} (\nabla_i u) \Delta_{ell} (\nabla_i u) dV = \]

\[ = \sum_{i=1}^{3} \int_{z=b_e} (\nabla_i u) \frac{\partial}{\partial n} (\nabla_i u) dS - \int_{z<b_e} \left\langle \nabla u, \nabla \left( \Delta_{ell} u \right) \right\rangle dV = \]

\[ = \sum_{i=1}^{3} \int_{z=b_e} (\nabla_i u) \frac{\partial}{\partial n} (\nabla_i u) dS - \int_{z=b_e} \left( \Delta_{ell} u \right) \frac{\partial u}{\partial n} dS + \int_{z<b_e} \left( \Delta_{ell} u \right)^2 dV \]

and recalling the integral representation \( u \), Eq. (2), we easily deduce that asymptotically

\[ \nabla_i u = O \left( z^{-2} \right), \quad \frac{\partial}{\partial n} \nabla_i u = O \left( z^{-3} \right) \quad \text{and} \quad \delta(T, h) = O \left( z^{-4} \right) \]

as \( z \to \infty \) (Note. \( O \) means the Landau symbol).
Hence \( D = D' \leq \int_{0<z} \left( \Delta_S u \right)^2 dV \) for \( b_e \to \infty \),

which yields the desired estimate

\[
D \leq \int_{\Omega_{\text{ell}}} \delta^2(T, h) dV = \left\| \delta(T, h) \right\|_{L_2(\Omega_{\text{ell}})}^2
\]

in view of Eq. (3). Combining now

\[
\left\| u \right\|_1 \leq 5 \left\| z \delta(T', h) \right\|_{L_2(\Omega_{\text{ell}})}
\]

with our last result, we obtain

\[
\left\| u \right\|_2^2 \leq 25 \left\| z \delta(T', h) \right\|_{L_2(\Omega_{\text{ell}})}^2 + \frac{1}{b^2} \left\| z \delta(T', h) \right\|_{L_2(\Omega_{\text{ell}})}^2,
\]

which enables us to write conclusively that

\[
\left\| KT' \right\|_2 = \left\| u \right\|_2 \leq \left( 25 + \frac{1}{b^2} \right)^{1/2} \left\| z \delta(T', h) \right\|_{L_2(\Omega_{\text{ell}})}
\]
10. Contraction Mapping

Recall that our original aim is to prove the contractivity of the operator $K$. Therefore, in view of

$$\left\| KT' \right\|_2 \leq \left(25 + \frac{1}{b^2}\right)^{1/2} \left\| z \delta(T', h) \right\|_{L_2(\Omega_{ell})}$$

it remains to estimate the right hand side by means of $\left\| T' \right\|_2$, i.e. to get

$$\left\| z \delta(T', h) \right\|_{L_2(\Omega_{ell})} \leq (\text{const.}) \left\| T' \right\|_2,$$

where

$$\delta(T, h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{w(z + \omega h, \beta)}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial^2 T}{\partial z \partial \lambda}$$

and $A_i$, $i = 1, 2, 3, 4$, are topography dependent coefficients.
In consequence the factor \( \alpha \) in
\[
\| KT' \|_2 \leq \alpha \| T' \|_2
\]
depends on essential supreme values
\[
\text{supess } | A_i | \text{ of } A_i, \ i = 1, 2, 3, 4 \text{ in the domain } \Omega_{\text{ell}}
\]

Note. Since \( \alpha \) depends on the essential supreme values of \( A_i \), \( i = 1, 2, 3, 4 \), we can expect that the contraction nature of \( K \) will be kept also for larger slopes, provided that they do not occur too frequently.

The inequality \( \alpha < 1 \) (sufficient for \( K \) to be a contraction mapping) has been proved for the topography of a realistic range of heights and relatively gentle slopes.
11. Operator with Reduced Degree of Derivatives

It is convenient to modify the operator $K$ in order to reduce the degree of derivatives involved in $\delta(T,h)$ and to display the mutual interplay of individual terms in $\delta(T,h)$ more explicitly. Integrating by parts, we get

$$(KT)_p = -\frac{1}{4\pi} \int_{z=b} NA_2 \delta g \, dS - \frac{1}{4\pi} \int_{b < z < b_{\text{ext}}} NA_5 \frac{\partial T}{\partial z} \, dV +$$

$$+ \frac{1}{4\pi} \int_{b < z < b_{\text{ext}}} \left( A_2 \frac{\partial N}{\partial z} + A_3 \frac{1}{z} \frac{\partial N}{\partial \beta} + A_4 \frac{1}{z \cos \beta} \frac{\partial N}{\partial \lambda} \right) \frac{\partial T}{\partial z} \, dV ,$$

where

$$A_5 = A_1 - \frac{\partial A_2}{\partial z} - \frac{2}{z} A_2 - \frac{1}{z} \frac{\partial A_3}{\partial \beta} + \frac{\sin \beta}{z \cos \beta} A_3 - \frac{1}{z \cos \beta} \frac{\partial A_4}{\partial \lambda} .$$

Note. the quantities with and without the subscript $P$ are referred to the computation and the variable point of the integration.
13. Experiment France - Auvergne

*Shuttle Radar Topography Mission*  

*Gravity Data $\delta g$*

1 arcsec resolution (*SRTM1*)
Starting Approximation

\[
(T_0)_P = \tilde{F}_P = \frac{1}{4\pi} \int_{z=b} N(1 - A_2) \delta g \, dS
\]

\[
h = 0 \text{ m} \quad h = 10000 \text{ m} \quad h = 25000 \text{ m} \quad h = 50000 \text{ m}
\]
**Computation of** $T_1$

**Topography Dependent Coefficients for** $z = b$

\[ A_1 = \frac{1}{b^2 + E^2 \sin^2 \beta} \left[ -2h - \frac{\sin \beta}{\cos \beta} \frac{\partial h}{\partial \beta} + \frac{\partial^2 h}{\partial \beta^2} + \frac{b^2 + E^2 \sin^2 \beta}{(b^2 + E^2) \cos^2 \beta} \frac{\partial^2 h}{\partial \lambda^2} \right] \]

\[ A_2 = -\frac{2bh + h^2}{b^2 + E^2 \sin^2 \beta} - \frac{1}{b^2 + E^2 \sin^2 \beta} \left[ \left( \frac{\partial h}{\partial \beta} \right)^2 + \frac{b^2 + E^2 \sin^2 \beta}{(b^2 + E^2) \cos^2 \beta} \left( \frac{\partial h}{\partial \lambda} \right)^2 \right] \]

\[ A_3 = \frac{2}{\sqrt{b^2 + E^2 \sin^2 \beta}} \frac{\partial h}{\partial \beta}, \quad A_4 = \frac{2}{\sqrt{b^2 + E^2 \cos \beta}} \frac{\partial h}{\partial \lambda} \]
Computation of $T_1$

Topography Dependent Coefficients for France - Auvergne and $z = b$ (SRTM - 1 arcsec)

\[ 1 - A_2 \quad A_3 \quad A_4 \quad A_5 \]

\[ A_5 = A_1 - \frac{\partial A_2}{\partial z} - \frac{2}{z} A_2 - \frac{1}{z} \frac{\partial A_3}{\partial \beta} + \frac{\sin \beta}{z \cos \beta} A_3 - \frac{1}{z \cos \beta} \frac{\partial A_4}{\partial \lambda} \]
The approach thus rests: on the solution of the transformed Laplace differential equation for the input gravity disturbances given on the surface of the oblate ellipsoid of revolution.

Green’s function representation of the solution and the method of successive approximations are used:

\[ T = \lim_{n} T_n , \quad T_n = F + KT_{n-1} \]

where

\[ F(y) = \frac{1}{4\pi} \int_{\partial \Omega_{ell}} \delta g(x) N(x, y) d_x S \]

is a harmonic function and

\[ KT_{n-1}(y) = -\frac{1}{4\pi} \int_{\Omega_{ell}} \delta [T_{n-1}(x) h(x)] N(x, y) d_x V \]

is an integro-differential operator applied on \( T_{n-1} \).
Thank you for your attention!

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