The finite element method for solving the oblique derivative boundary value problems in geodesy

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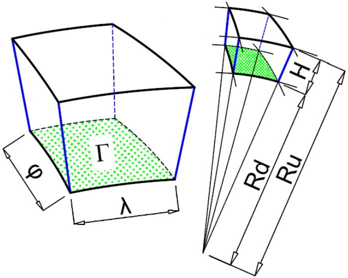
Formulation of the FGBVPs

- Fixed gravimetric boundary value problem (FGBVP) in the bounded domain Ω

$$\begin{aligned} \Delta T(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3, & (1) \\ \nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}) &= -\delta g(\mathbf{x}), \quad \mathbf{x} \in \Gamma \subset \partial\Omega, & (2) \\ T(\mathbf{x}) &= T_{SAT}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega - \Gamma, & (3) \end{aligned}$$



Solution of the GBVP by the FEM on non uniform mesh

We multiply the differential equation (1) by $w \in V$ and using Green's identity (we omit $(\mathbf{x})$ to simplify the notation in the following equations) we get

$$\int_{\Omega} \nabla T \cdot \nabla w \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\partial\Omega} \nabla T \cdot \mathbf{n} \, w \, \mathrm{d}\sigma, \quad w \in V. \tag{4}$$

Now we split the oblique vector $\mathbf{s}$ into one normal and two tangential components

$$\mathbf{s} = c_1 \mathbf{n} + c_2 \mathbf{t}_1 + c_3 \mathbf{t}_2, \tag{5}$$

where $\mathbf{n}$ is the normal vector and $\mathbf{t}_1$, $\mathbf{t}_2$ are tangent vectors to $\Gamma \subset \partial\Omega \subset R^3$. These three vectors together form an orthonormal basis.

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Then we put (5) into (2) to obtain

$$\nabla T \cdot \mathbf{s} = c_1 \nabla T \cdot \mathbf{n} + c_2 \nabla T \cdot \mathbf{t}_1 + c_3 \nabla T \cdot \mathbf{t}_2 = -\delta g. \tag{6}$$

From (6) we express the normal derivative

$$\nabla T \cdot \mathbf{n} = \frac{-\delta g}{c_1} - \frac{c_2}{c_1}\frac{\partial T}{\partial \mathbf{t}_1} - \frac{c_3}{c_1}\frac{\partial T}{\partial \mathbf{t}_2} \tag{7}$$

and we insert it to (4) to get

$$\int\limits_{\Omega} \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\limits_{\partial\Omega} \left(\frac{-\delta g}{c_1} - \frac{c_2}{c_1}\frac{\partial T}{\partial \mathbf{t}_1} - \frac{c_3}{c_1}\frac{\partial T}{\partial \mathbf{t}_2} \right) w \,\mathrm{d}\sigma. \tag{8}$$

Solution of the GBVP by the FEM on non uniform mesh

Let the extension of the Dirichlet BC (6) given by T_{SAT} into the domain Ω be in $W_2^{(1)}(\Omega)$ and let $\delta g \in L^2(\Gamma)$. Then we define the weak formulation of BVP as follows: we look for a function T, such that $T - T_{SAT} \in V$ and

$$\int_\Omega \nabla T \cdot \nabla w \, \mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{c_2}{c_1} \int_\Gamma \frac{\partial T}{\partial \mathbf{t}_1} w \, \mathrm{d}\sigma + \frac{c_3}{c_1} \int_\Gamma \frac{\partial T}{\partial \mathbf{t}_2} w \, \mathrm{d}\sigma = \int_\Gamma \frac{-\delta g}{c_1} w \, \mathrm{d}\sigma. \tag{9}$$

The FEM is a numerical method that assumes discretization of the whole computational domain by a union of a collection of elements. For a three-dimensional problem, we use hexahedral elements with eight nodes.

Solution of the GBVP by the FEM on non uniform mesh

To calculate two integrals over a boundary Γ in Eq. (9) which include a tangential derivative, we approximate derivatives in tangential direction using values of basis functions at nodes N_i of element e

$$\frac{\partial \psi_j^{(e)}}{\partial \mathbf{t}_1} = \frac{\psi_j^{(e)}(N_3) - \psi_j^{(e)}(N_1)}{d(N_1, N_3)}, \tag{10}$$

$$\frac{\partial \psi_j^{(e)}}{\partial \mathbf{t}_2} = \frac{\psi_j^{(e)}(N_4) - \psi_j^{(e)}(N_2)}{d(N_2, N_4)}, \tag{11}$$

where d denotes the distance between nodes, i.e., length of diagonal of side of element e that lies on boundary Γ.

Local gravity field modelling in Slovakia

- Input data
  - gravity disturbances: generated from the detailed map of the Complete Bouguer Anomalies (Pašteka et al. 2014) using the CBA2G software (Marušiak et al. 2015)
  - disturbing potential on top and 4 side boundaries: EIGEN-6C4 geopotential model up to d/o 2160 (Förste et al. 2014)
  - terrestrial data: Earth's topography with the horizontal resolution 0.002 x 0.002 deg
- Computational domain
  - area: $\varphi \in \langle 47.5^\circ, 49.7^\circ \rangle$ and $\lambda \in \langle 16.5^\circ, 23.0^\circ \rangle$
  - number of elements: $3250 \times 1100 \times 640$, i.e. resolution, $d\varphi \times d\lambda \times dR = 200 \times 200 \times 360m$
- Computational costs
  - processing on 64 processors with 250 GB of distributed memory
  - total CPU time per procs: cca. 14 hours

Local quasigeoid model in Slovakia

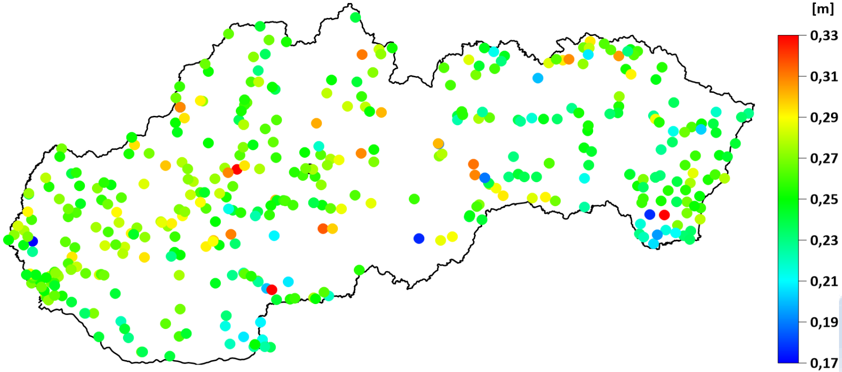

Figure: GNSS/levelling benchmarks with differences between the obtained local quasigeoid model and DVRM05.

Local quasigeoid model in Slovakia

Characteristic	For all points	Without outliers
Min. val.	0.124 m	0.157 m
Max. val.	0.366 m	0.366 m
Range	0.242 m	0.210 m
Mean val.	0.238 m	0.238 m
Median	0.239 m	0.239 m
St. dev.	0.027 m	0.024 m

Table: Statistics of the GNSS/levelling test in area of Slovakia.

Conclusions

- We have presented an numerical scheme to approximate the solution of the Laplace equation with an oblique derivative boundary condition by the finite element method.
- For more detail about numerical scheme see www.sciendo.com
- Our approach based on the local gravity field modelling in spatial domain using FEM on the unstructured 3D mesh about the real Earth's topography has resulted in the quasigeoid model whose accuracy is about 2.4 cm.