FVM approach for solving the oblique derivative BVP on unstructured meshes above the real Earth’s topography

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Outline

• We present a finite volume method (FVM) for the general Poisson problem with the Dirichlet and oblique derivative boundary condition

• We present local gravity field modelling in Slovakia based on the FVM approach considered on unstructured meshes above the real Earth’s topography
1. Mathematical formulation

• nonlinear satellite fixed geodetic boundary value problem
  \[-\Delta T(x) = 0, \quad x \in \Omega,\]
  \[\nabla T(x) \cdot V(x) = g(x), \quad x \in \Gamma,\]
  \[T_{SAT}(x) = 0, \quad x \in \partial\Omega\setminus\Gamma,\]
  where \(V(x) = n(x) + W(x)\)

• \(T\) - unknown disturbed potential

• \(V(x) = \frac{\nabla U(x)}{|\nabla U(x)|}\), where \(U\) is normal potential

• \(g(x)\) - gravity disturbances

Fig. 1: 2D illustration of a 3D computational domain \(\Omega\)
2. Generic Finite Volume method (FVM)

- Divide the computational domain $\Omega$ into the set of finite volumes $p$

\[ 0 = \iiint_{\Omega} -\Delta T \]

\[ = - \sum_{\sigma \in \mathcal{G}(p) \setminus \mathcal{G}_\Gamma} \iint_{\sigma} \nabla T \cdot n_{p,\sigma} - \sum_{\sigma \in \mathcal{G}(p) \cap \mathcal{G}_\Gamma} \iint_{\sigma} \nabla T \cdot (n_{p,\sigma} + W - W) = (*) \]

- Where $\nabla T \cdot (n_{p,\sigma} + W) = \nabla T \cdot V = g$

- Where inner fluxes are approximated by some FV scheme $\mathcal{F}_{p,\sigma}(T) \approx \int_{\sigma} \nabla T \cdot n_{p,\sigma}$

\[ (*) = - \sum_{\sigma \in \mathcal{G}(p) \setminus \mathcal{G}_\Gamma} \mathcal{F}_{p,\sigma}(T) - \sum_{\sigma \in \mathcal{G}(p) \cap \mathcal{G}_\Gamma} \iint_{\sigma} g - \nabla T \cdot W \]

Fig. 2: 2D illustration of a 3D FVM discretization of $\Omega$
2. Generic Finite Volume method (FVM)

\[ 0 = \int_{\sigma} \nabla T \cdot W = \int_{\sigma} \nabla \Gamma \cdot (TW) - T \nabla \Gamma \cdot W \]

\[ = \sum_{e \in \mathcal{E}(\sigma)} \int_{e} T \ W \cdot n_{\sigma,e} - \int_{\sigma} T \nabla \Gamma \cdot W = (*) \]

- Choice of central scheme
  - Approximate \( T \) on the edge \( e \) by constant \( T_e \)
  - Approximate \( T \) on the face \( \sigma \) by constant \( T_\sigma \)

\[ (*) = \sum_{e \in \mathcal{E}(\sigma)} T_e \int_{e} W \cdot n_{\sigma,e} - T_\sigma \int_{\sigma} \nabla \Gamma \cdot W \]

Fig. 2: 2D illustration of a 3D FVM discretization of \( \Omega \)
2. Generic Finite Volume method (FVM)

• From a numerical analysis [1] we add a small amount of boundary diffusion for a stability purposes

• Resulting scheme

\[
\sum_{\sigma \in \mathcal{G}(p) \setminus \Gamma} F_{p,\sigma}(T) + \sum_{\sigma \in \mathcal{G}(p) \setminus \Gamma} \sum_{e \in \mathcal{E}(\sigma)} T_e \int_e W \cdot n_{\sigma,e} - T_{\sigma} \iiint_{\sigma} \nabla T \cdot W
\]

\[
+ R h_{\Gamma} \sum_{\sigma \in \mathcal{G}(p) \setminus \Gamma} \sum_{e \in \mathcal{E}(\sigma)} F_{p,\sigma}(T) = \sum_{\sigma \in \mathcal{G}(p) \setminus \Gamma} \iiint_{\sigma} g
\]
3. Choice of fluxes discretization

- Chose some finite volume approximation of inner volume fluxes $F_{p,\sigma}^\Omega (T)$
- Chose some finite volume approximation of boundary fluxes $F_{p,\sigma}^\Omega (T)$
- For our choices see [1]

$$
\sum_{\sigma \in \Xi(p) \setminus \bar{\Gamma}} F_{p,\sigma}^\Omega (T) + \sum_{\sigma \in \Xi(p) \setminus \bar{\Gamma}} \sum_{e \in \Xi(\sigma)} T_e \int_e W \cdot n_{\sigma,e} - T_\sigma \int_\sigma \nabla \cdot W + R_{\bar{\Gamma}} \sum_{\sigma \in \Xi(p) \setminus \bar{\Gamma}} \sum_{e \in \Xi(\sigma)} F_{p,\sigma}^\Omega (T) = \sum_{\sigma \in \Xi(p) \setminus \bar{\Gamma}} \int_\sigma g
$$

Fig. 2: 2D illustration of a 3D FVM discretization of $\Omega$
4. Numerical results for local gravity field modelling in the area of Slovakia

![Fig. 3: Topography in the area of Slovakia](image)

<table>
<thead>
<tr>
<th>Boundaries</th>
<th>Resolution</th>
<th>#points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude direction</td>
<td>16.5° - 23°</td>
<td>0.002° (200 m)</td>
</tr>
<tr>
<td>Longitude direction</td>
<td>47.5° - 49.7°</td>
<td>0.002° (200 m)</td>
</tr>
<tr>
<td>Radial direction</td>
<td>Topography – 230 km</td>
<td>250 m – 1 km</td>
</tr>
</tbody>
</table>
4. Numerical results for local gravity field modelling in the area of Slovakia

**Boundary conditions:**

- Bottom boundary condition (the gravity disturbances) generated
  - inside Slovakia using the CBA2G software [2]
  - Outside Slovakia interpolated from the GGMPlus database [3]
- Upper boundary condition (disturbing potential) generated from the GO_CONS_GCF_2_DIR_R5 geopotential model up to d/o 300 [4]
- Side boundaries condition (disturbing potential) generated from the EIGEN-6C4 geopotential model up to d/o 2160 [5]
4. Numerical results for local gravity field modelling in the area of Slovakia

Fig. 5: Local quasigeoid model in the area of Slovakia obtained from the FVM solution
4. Numerical results for local gravity field modelling in the area of Slovakia

Statistics of the GNSS/Levelling test:

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>For all points</th>
<th>Without outliers</th>
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<tbody>
<tr>
<td>Number of points</td>
<td>404</td>
<td>395</td>
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<tr>
<td>Minimum</td>
<td>0.131 m</td>
<td>0.147 m</td>
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<tr>
<td>Maximum</td>
<td>0.352 m</td>
<td>0.352 m</td>
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<tr>
<td>Range</td>
<td>0.221 m</td>
<td>0.205 m</td>
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<tr>
<td>Mean</td>
<td>0.231 m</td>
<td>0.231 m</td>
</tr>
<tr>
<td>Median</td>
<td>0.230 m</td>
<td>0.230 m</td>
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<tr>
<td>Standard deviation</td>
<td>0.028 m</td>
<td>0.026 m</td>
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</tbody>
</table>

Fig. 6: GNSS/levelling test of the local quasigeoid model in Slovakia at 404 benchmarks
References


