Explicit nonlinear waves of fluid models on extended domains and unbounded growth with backscatter

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Rotating shallow water equations with backscatter

We consider here the rotating shallow water equations with backscatter (red terms) on the unbounded domain $\mathbb{R}^2$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -f\boldsymbol{v}^{\perp} - g\nabla\eta - \begin{pmatrix} d_1\Delta^2 + b_1\Delta & 0 \\ 0 & d_2\Delta^2 + b_2\Delta \end{pmatrix} \boldsymbol{v}$$

$$\frac{\partial \eta}{\partial t} + (\boldsymbol{v} \cdot \nabla)\eta = -(H_0 + \eta)\mathrm{div}(\boldsymbol{v}).$$

- Velocity field $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x}) \in \mathbb{R}^2$ and deviation $\eta = \eta(t, \boldsymbol{x}) \in \mathbb{R}$ with characteristic fluid depth $H_0 > 0$ and fluid thickness $H_0 + \eta$.
- Here simplified backscatter, which comes from the subgrid parametrisation and is intended to provide energy consistency in the simulations (e.g. [Jansen & Held, 2014]).
- We are looking for solutions of the full nonliear equations, which we call explicit solutions here.

Explicit solutions

We consider solutions of the form

$$\boldsymbol{v}(t,\boldsymbol{x}) = \psi(t,\boldsymbol{k}\cdot\boldsymbol{x})\boldsymbol{k}^{\perp}, \quad \eta(t,\boldsymbol{x}) = \phi(t,\boldsymbol{k}\cdot\boldsymbol{x}),$$

with wave-vector $\boldsymbol{k} \in \mathbb{R}^2$ and sufficiently smooth wave-shapes ψ and ϕ.

- With this approach the nonlinear terms of the material derivative vanish, since the gradient of ψ and ϕ are orthogonal to the velocity $\boldsymbol{v}$ (for more details also in other models see [Prugger & Rademacher]).
- A linear problem remains, which implies a linear behavior of the explicit solutions, as long as the orthogonality condition is satisfied.
- The usual shallow water equations (choosing $b_i, d_i = 0$) for example, have the large set of explicit stationary geostrophic solutions

$$\boldsymbol{v}(\boldsymbol{x}) = \psi'(\boldsymbol{k}\cdot\boldsymbol{x})\boldsymbol{k}^{\perp}, \quad \eta(\boldsymbol{x}) = \frac{f}{g}\psi(\boldsymbol{k}\cdot\boldsymbol{x}),$$

for any wave-vector $\boldsymbol{k} \in \mathbb{R}^2$ and wave-shape $\psi \in C^2(\mathbb{R})$.

Explicit solutions

The whole equations with backscatter have explicit solutions

$$\boldsymbol{v}(t,\boldsymbol{x}) = \alpha_1 e^{\beta t}\cos(\boldsymbol{k}\cdot\boldsymbol{x}+\delta)\boldsymbol{k}^{\perp}, \quad \eta(\boldsymbol{x}) = \alpha_2 \frac{f}{g}\sin(\boldsymbol{k}\cdot\boldsymbol{x}+\delta)$$

with $\beta = 0$ or $\alpha_2 = 0$, and satisfying the conditions

$$\beta = (b_1 - d_1||\boldsymbol{k}||^2)k_2^2 + (b_2 - d_2||\boldsymbol{k}||^2)k_1^2,$$

$$\frac{\alpha_2-\alpha_1}{\alpha_1} f = \big((d_1-d_2)||\boldsymbol{k}||^2 + b_2 - b_1\big)k_1 k_2.$$

- There are stationary and exponentially growing/decaying flow.
- These explicit solutions have arbitrary amplitudes.
- Superposition of these solutions with vector $s\boldsymbol{k}$ and arbitrary factor $s \in \mathbb{R}$ are also explicit solutions.

Example of the existence of explicit solutions

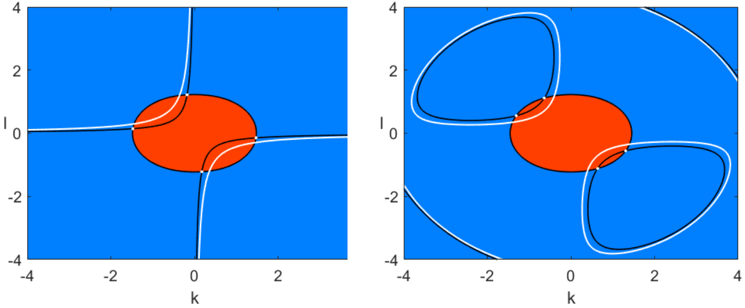
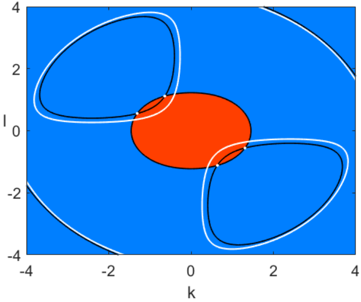


Figure: Left: $d_2 = 1, \alpha_2 = 0.5$, right: $d_2 = 1.04, \alpha_2 = -0.5$. Red: $\beta > 0$, blue $\beta < 0$. White solid: existence of the solutions for $\alpha_2 = 0$, white dots $\alpha_2 \neq 0$. Black curves: β for $\alpha_2 \neq 0$. Fixed parameters: $d_1 = 1, b_1 = 1.5, b_2 = 2.2, f = 0.3, g = 9.8, H_0 = 0.1, \alpha_1 = 1$.

Instability of some explicit solutions

An important question is the stability of stationary flow. For some of them we can show that they are unstable:

- We consider the explicit stationary solution, which is described by one of the white dots on the black ellipse.
- Superpose with an exp. increasing solution (white curve on the red region) with wave-vector in the same direction.
- This yields an explicit solution, which is a perturbation of the stationary flow and is exp. increasing in time.



Summary and conclusion

- The orthogonality between wave-vectors and velocity-directions can remove the nonlinear terms caused by the material derivative.
- The remaining linear problem implies a certain linear behavior of the explicit solutions.
- With this property we can show the instability of certain explicit stationary flow.
- This approach of finding explicit solutions of the full nonlinear problem can be used in different fluid models, but we focused on the backscatter model here.
- Here: energy accumulates in selected scales, causing exponentially and unboundedly growing ageostrophic nonlinear flow.

Thank you for your attention!

Artur Prugger & Jens D. M. Rademacher:
Explicit internal wave solutions in nonlinear fluid models on the whole space.
2020, in preparation, arXiv link: https://arxiv.org/abs/2003.07824

Malte F. Jansen & Isaac M. Held:
Parameterizing subgrid-scale eddy effects using energetically consistent backscatter.
Ocean Modelling, Vol. 80, 2014, pp. 36-48