# Adaptive 2D shallow water simulation based on a MultiWavelet Discontinous Galerkin approach


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# Basic problem

Shallow-water problems in the environment (floods, rivers, coasts) are large spatial domains that require local very high resolution to capture local features which may govern the flow

# Basic solution

Dynamic adaptive meshing, capturing the features of the terrain (strong source term in SWE) and the state variables (depth, velocity)

# Technical problems in existing strategies

- ad-hoc thresholding for coarsening & refinement
- ensuring conservation, well-balancing
- proper grading of neighbors (2 small neighbours per edge, not more)

# Proposed strategy: **MWDG**

- Discontinuous Galerkin (high-order, locality, HPC)
- MultiWavelets to decompose DG solution into multi-resolution data
- Multiresolution analysis to assess what is relevant at each spatial resolution
- Objective, general thresholding strategy
- Flexible, generic, scalable approach for hyperbolic PDEs

# Take home message

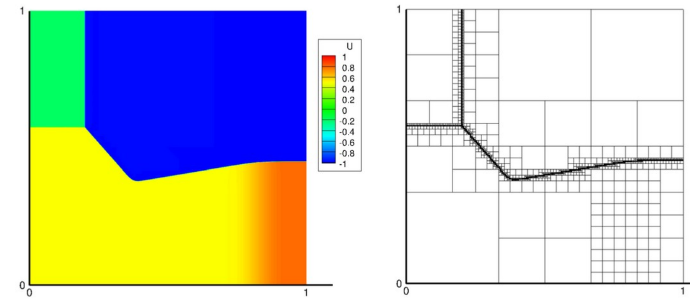
Dynamically-adaptive scheme which allows to significantly reduce the number of grid cells, while achieving extremely high spatial resolution.

Applicable to complex and realistic shallow water problems, spanning many spatial scales of interest.

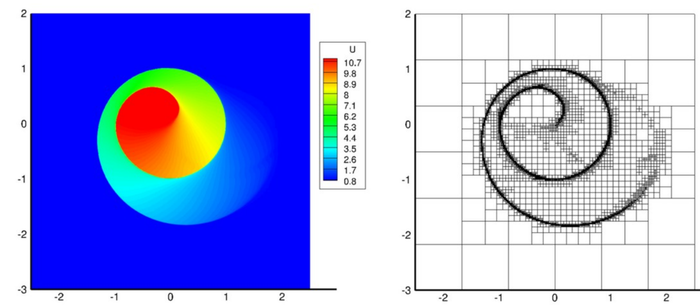
The strategy is robust and flexible, requires only a single control parameter, and keeps all the properties of the reference numerical scheme actually solving the equations.

# Some background on MWDG

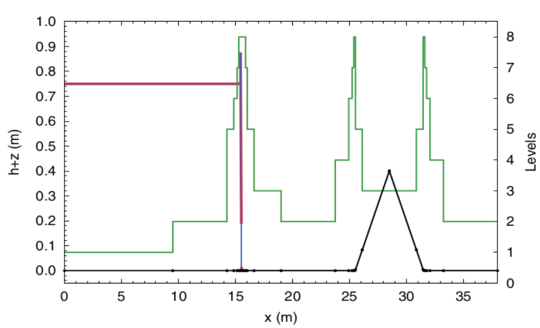
Developed for systems of hyperbolic equations –Euler eqs.–
(Hovhannisyan et al., Math. Comput. 2014)
(Gerhard et al., Journal of Scientific Computing. 2015)
(Gerhard & Müller. Comp. Appl. Math. 2016)



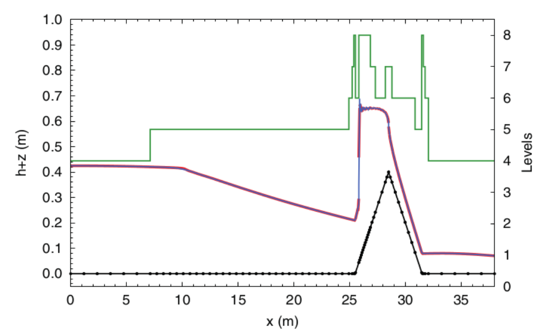
Mathematical formulation and proof for SWE
(Gerhard et al., J. Computational Physics 2015)



Application and assessment in 1D for SWE
(Caviedes-Voullième & Kesserwani. Advances in Water Resources. 2015)




(a) $t = 0.0\,s$




(d) $t = 8.88\,s$

In this work, we show the applicability of MWDG to complex and realistic experimental benchmark cases in two-dimensions.

# How does it work?

Shallow water equations $$\mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{x}, \mathbf{u})$$

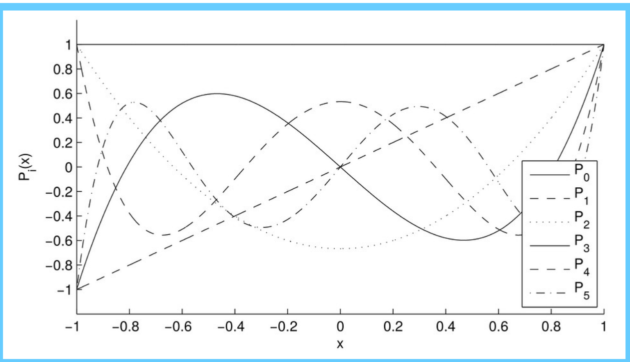
$$\mathbf{u} = (\mathrm{h},\ \mathrm{h}\ \mathbf{v})$$
$$\mathbf{f} = (\mathrm{h}\ \mathbf{v}, \mathrm{h}\ \mathbf{v} \otimes \mathbf{v} + 0.5 g h^2 \mathbf{I})$$
$$\mathbf{s} = (0,\ \text{-g h}\ (\nabla z + \sigma))$$

Projection into DG space (SWE solution → DG modes)

DG-SWE (single scale, "classical" modal DG)

$$\int_{V_\lambda} \frac{\partial \mathbf{u}_h}{\partial t} \cdot \mathbf{w}_h \, d\mathbf{x} - \int_{V_\lambda} \mathbf{f}(\mathbf{u}_h) : \nabla \mathbf{w}_h + \mathbf{s}(\mathbf{x}, \mathbf{u}_h) \cdot \mathbf{w}_h \, d\mathbf{x} + \int_{\partial V_\lambda} \hat{\mathbf{f}} \cdot \mathbf{w}_h \, dS = \mathbf{0},$$

Explicit time-integration with Strong-stability preserving Runge-Kutta scheme

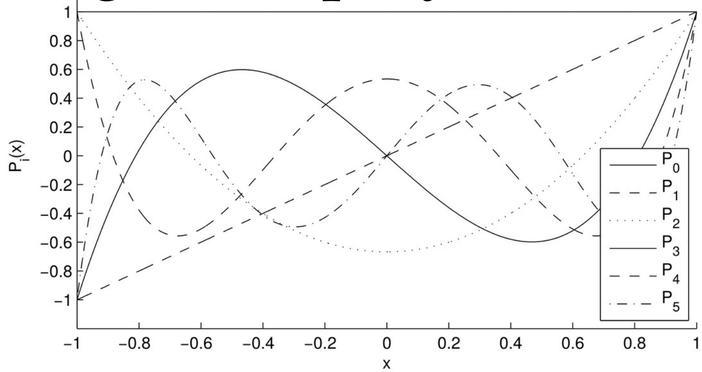



Legendre polynomials

# How does it work?

Shallow Water Equations

Legendre polynomials

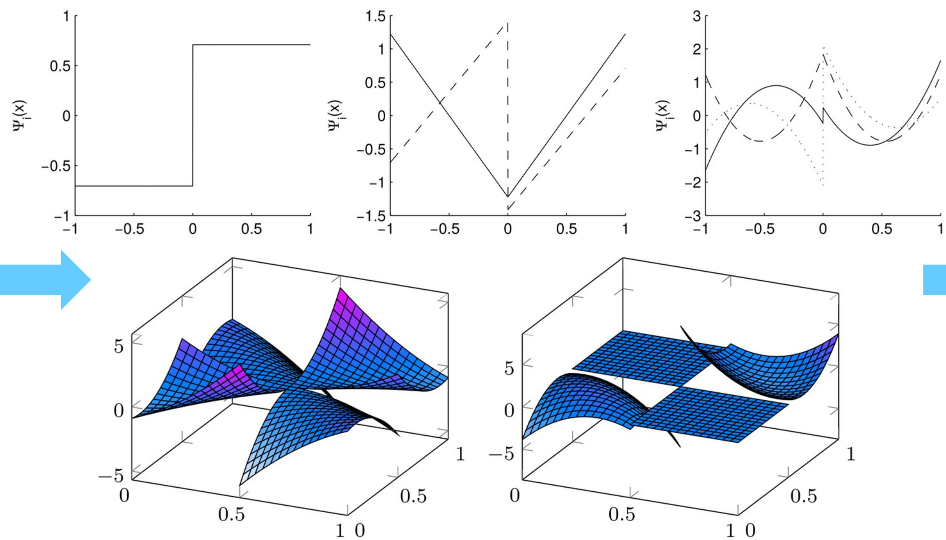



Projection into DG space
(SWE solution → DG modes)

DG-SWE
(single scale)

Multiwavelets (a wavelet for every DG mode)

Single scale DG-SWE




Multiresolution spaces

MWDG-SWE
(multiscale)

# How does it work?

## Two scale transformation

A change and decomposition of basis functions, which exist in nested subspaces (**multiwavelets**)

Single-scale basis function

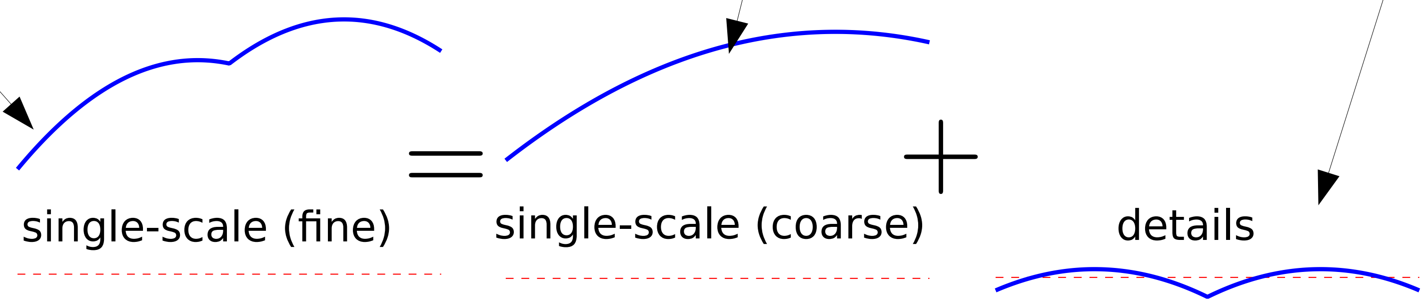
Basis function in complement space

$$\mathbf{u}\Big|_{V_\lambda} = \sum_{\mu \in \mathcal{M}_\lambda} \sum_{i \in \mathcal{P}} \mathbf{u}_{\mu,i}\phi_{\mu,i} = \sum_{i \in \mathcal{P}} \mathbf{u}_{\lambda,i}\phi_{\lambda,i} + \sum_{i \in \mathcal{P}^*} \mathbf{d}_{\lambda,i}\psi_{\lambda,i}$$

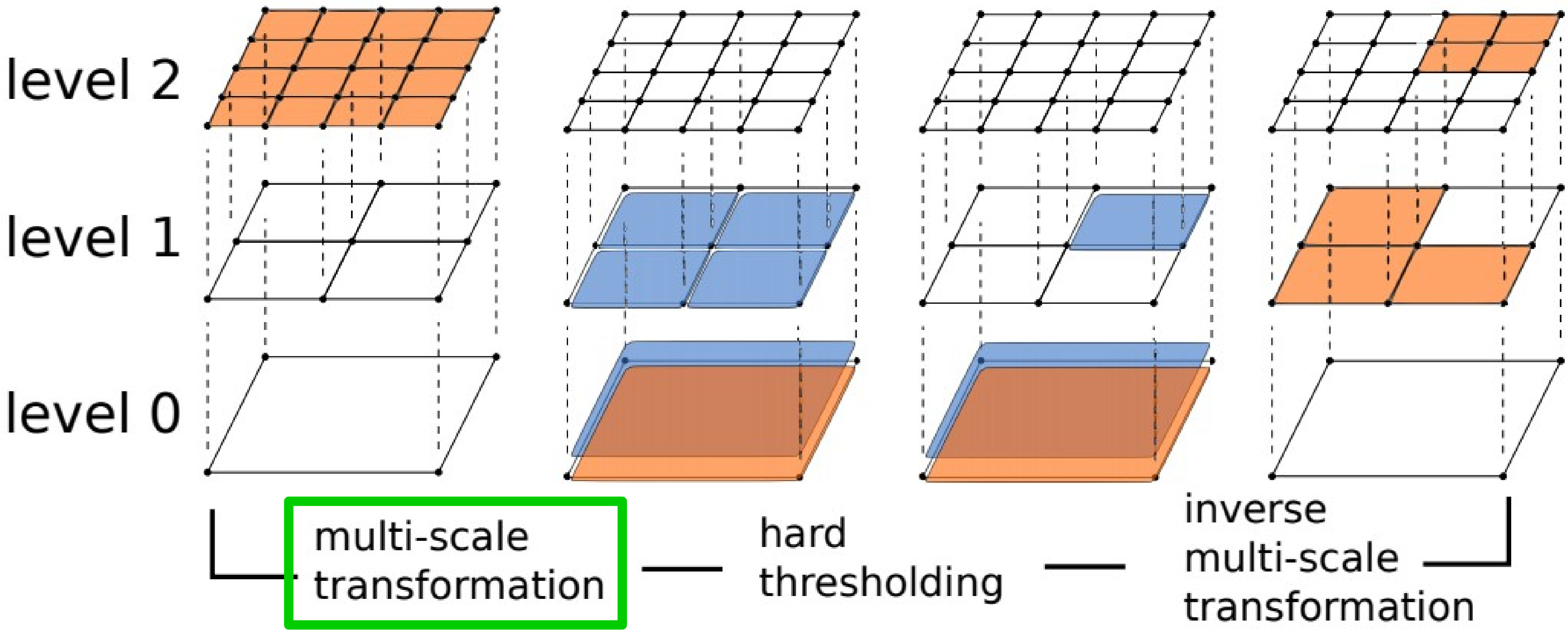
DG approximate solution

Coarse representation

Detail coefficients



# How does it work?

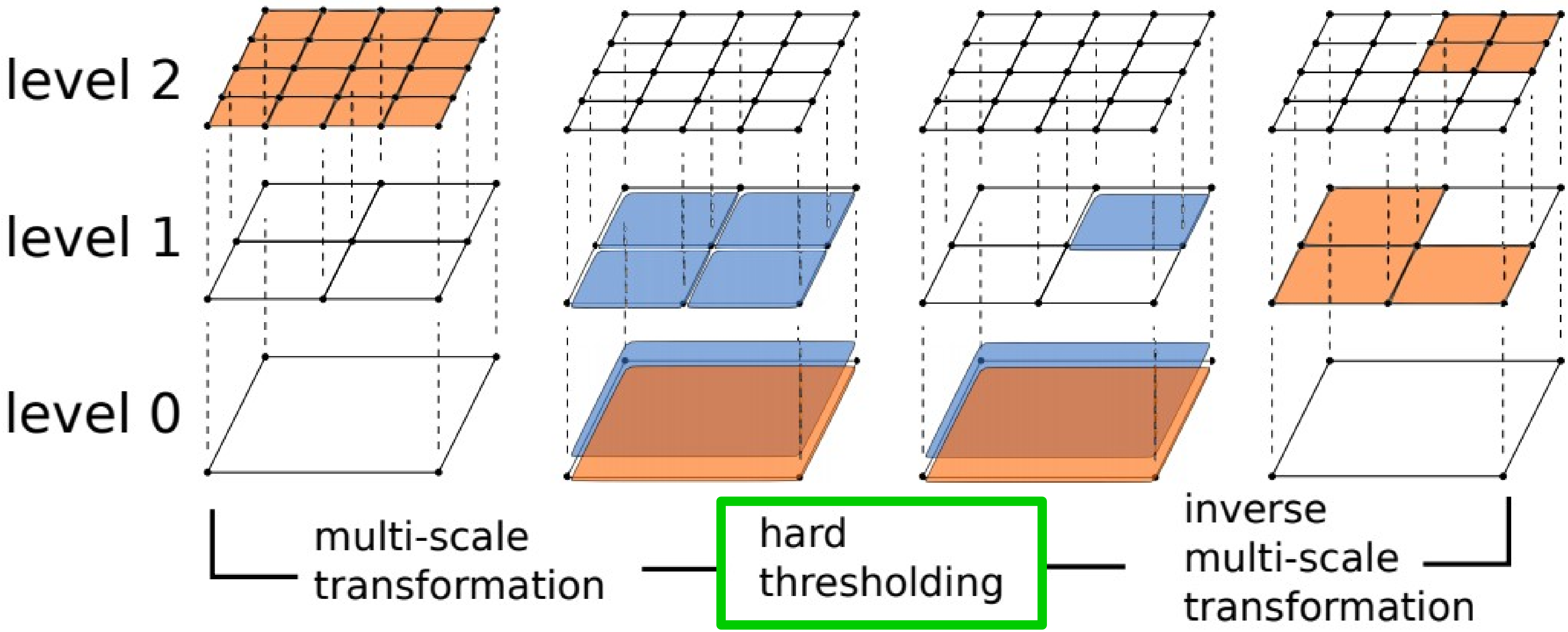



Multiscale transformation is a generalisation of the 2-scale transform.

Decompose fine resolution into coarse data and details across scales

$$\sum_{\mathrm{i}\in\mathcal{I}_L^S} \mathbf{u}_\mathrm{i}\phi_\mathrm{i} = \sum_{\mathrm{i}\in\mathcal{I}_0^S} \mathbf{u}_\mathrm{i}\phi_\mathrm{i} + \sum_{\ell=0}^{L-1} \sum_{\mathrm{i}\in\mathcal{I}_\ell^W} \mathbf{d}_\mathrm{i}\psi_\mathrm{i},$$

# How does it work?

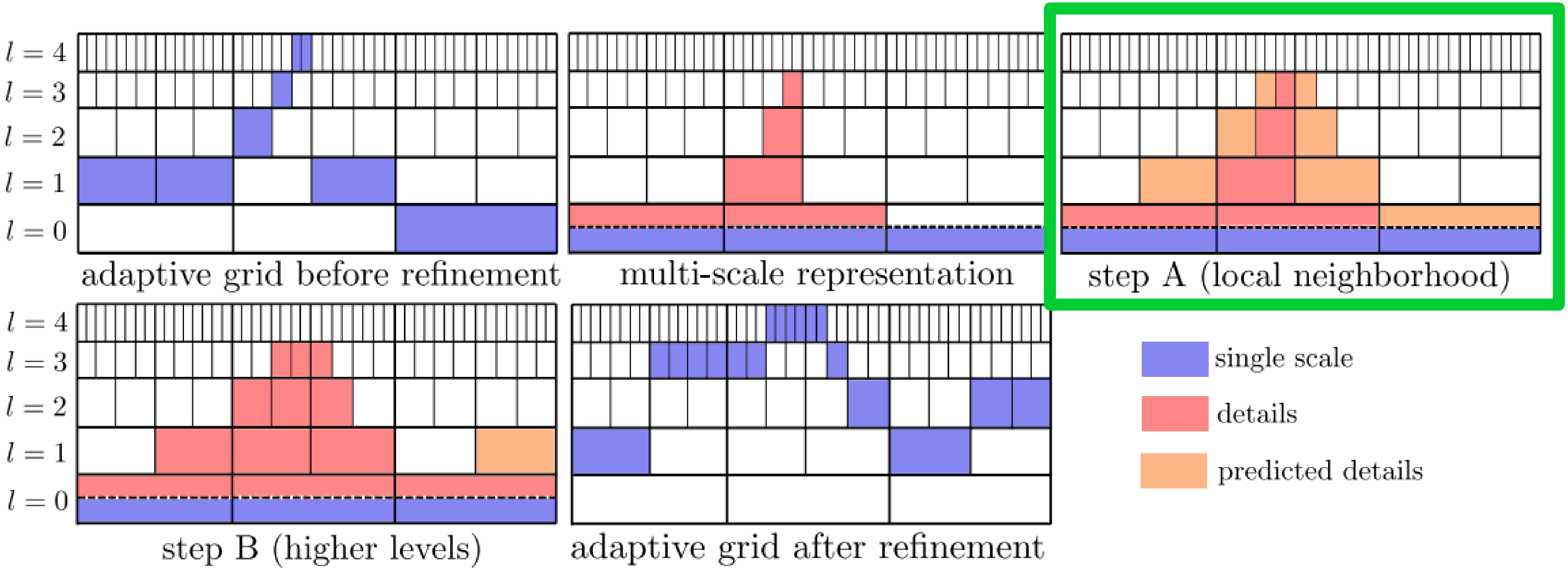



## Hard thresholding

Identify, in each resolution which details are **significant,** and keep the resolution that ensures that those details are kept. If details are not significant, allow coarsening.

Magnitude of details is normalised across scales and state (conserved) variables, which allows for a **single threshold** to compare against.

# How does it work?




Also necessary to predict details in the neighboring cells, as they may become relevant in the next time step.

Shocks in the flow field are always refined to highest level of refinement.

Bed discontinuities are treated as shocks of the z(x,y) field.

Wet/dry fronts are also refined to highest level.

# Some features of the method

MWDG strategy preserves all features of the underlying DG solvers, namely:

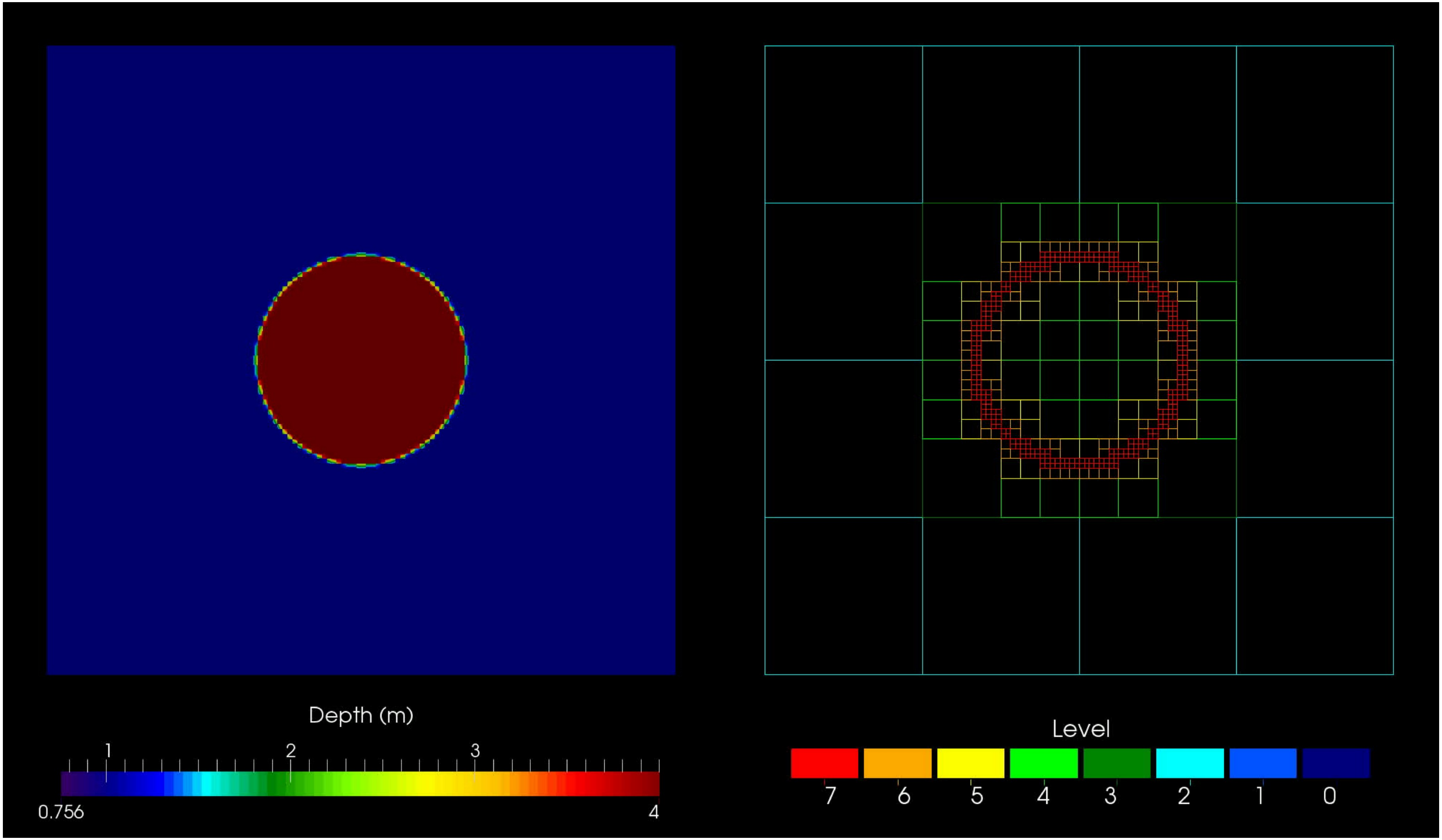
- DG-SWE is flexible in terms of order-of-accuracy. In practice, we don't take it beyond $3^{rd}$ order.
- The DG-SWE scheme is locally mass conservative, well-balanced, can deal with wet/dry interfaces.

The adaptive scheme has some interesting properties:

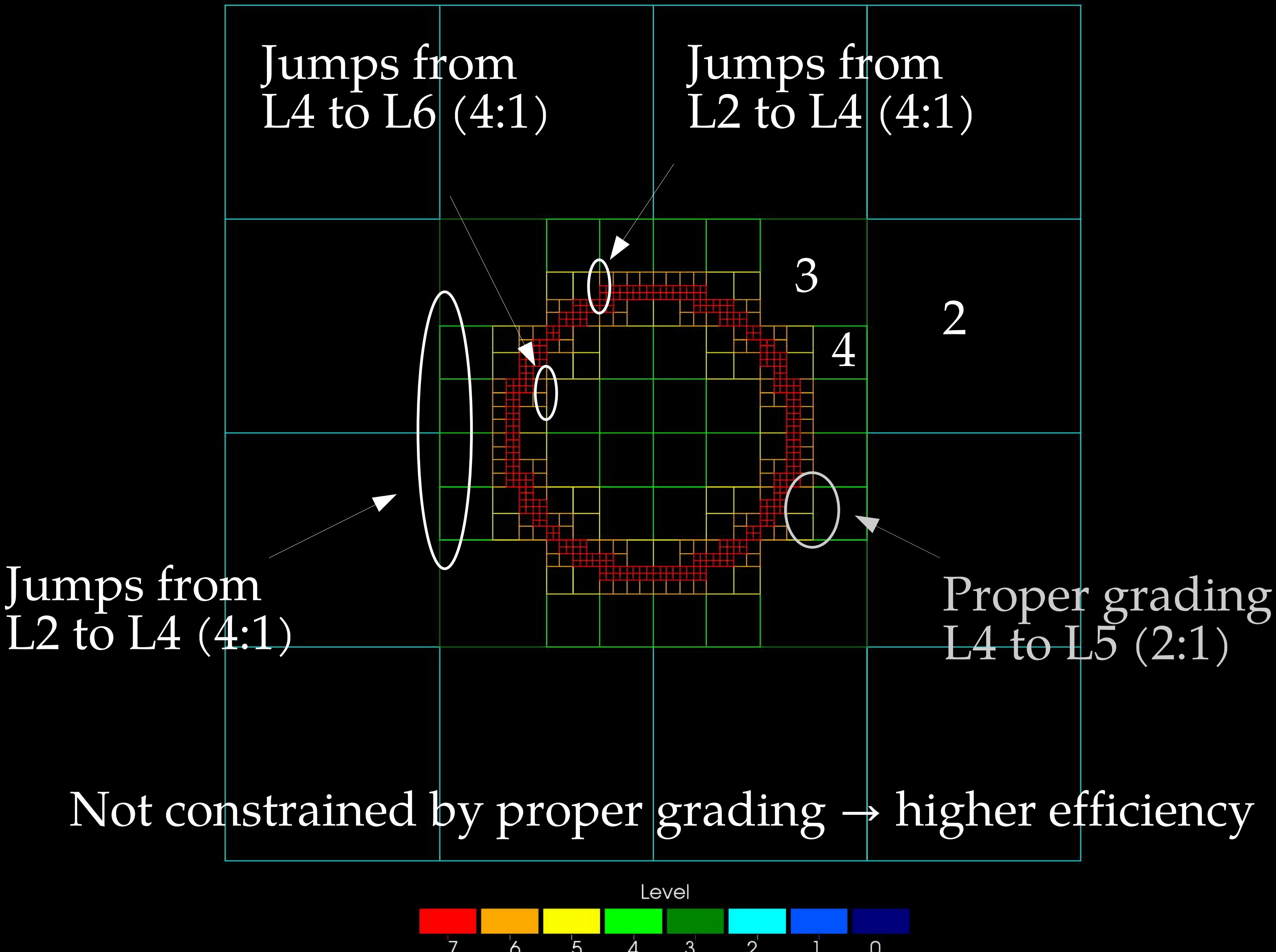
- Refinement/coarsening threshold is controlled by a **single parameter**, regardless of number of state variables and levels
- Not constrained by proper grading (only two finer neighbors to a coarser cell, 2:1 ratio)
- Does not require a *fine enough* coarsest mesh, as other strategies.

# How does the adaptive grid look?

Circular dam break (frictionless & flat bed → homogeneous PDE)



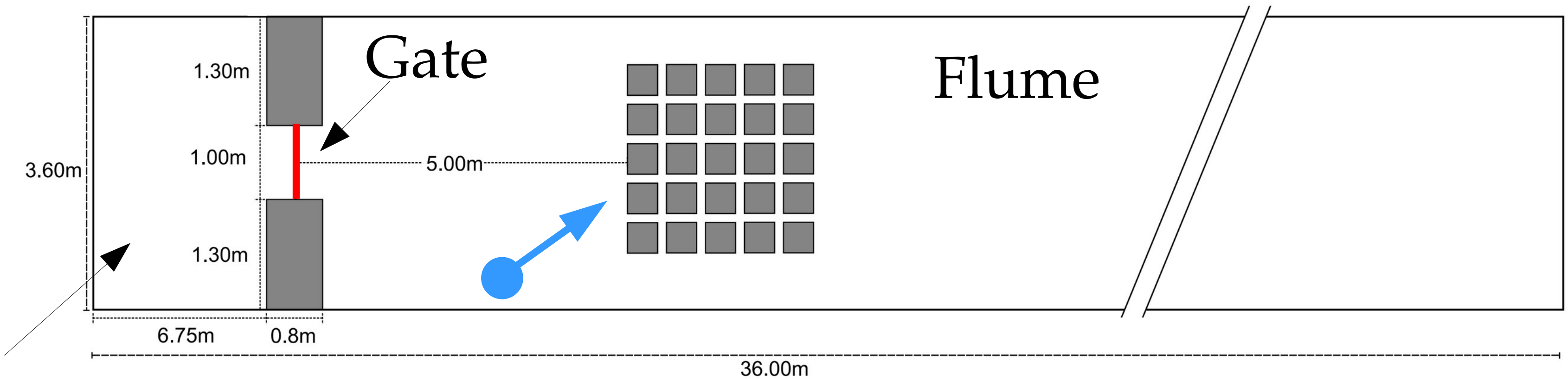
# Some features of the adaptive grid




Not constrained by proper grading → higher efficiency

# Some results: experimental dam-break

Well-known experimental benchmark
(Soares-Frazao & Zech, Journal of Hydraulic Research, 2008)



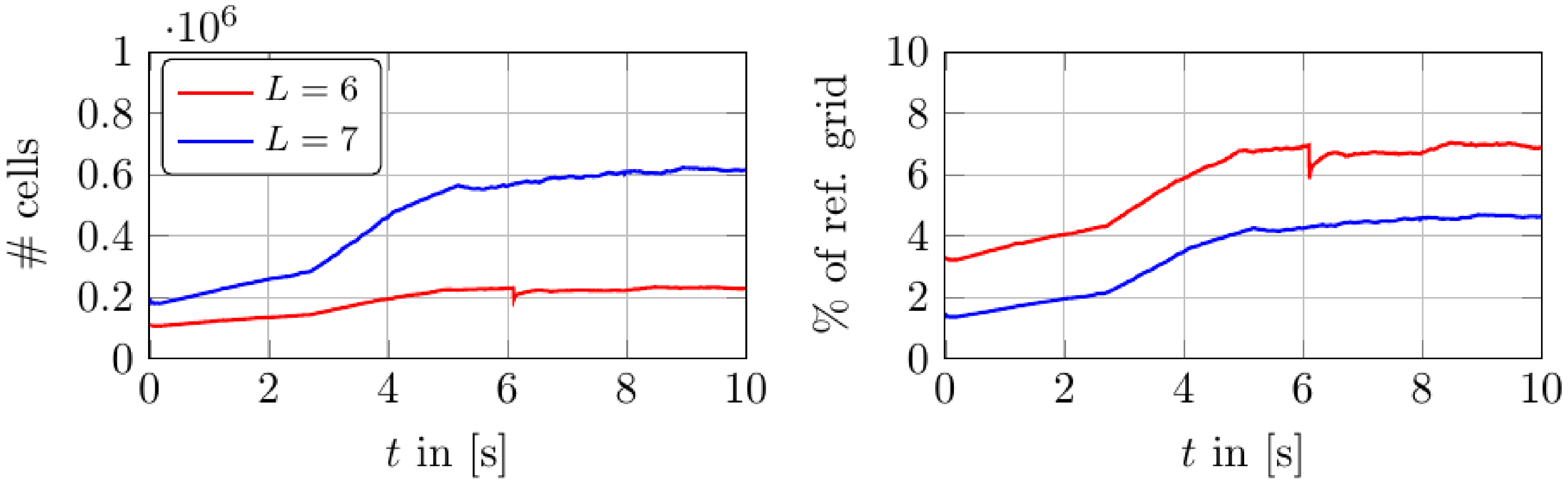

# Some results: experimental dam-break

Solved with MWDG3 (quadratic test functions)

Coarsest mesh (level 0): 90 x 9 = 810 cells

Finest mesh (L=6): 5760 x 576 = 3.3 x $10^6$ cells    dx = 6.25 mm

Finest mesh (L=7): 11520 x 1152 = 13.3 x $10^6$ cells    dx = 3.125 mm




L = 6 → number of computational cells is 7% of equivalent highest resolution mesh (93% reduction, achieving same resolution)

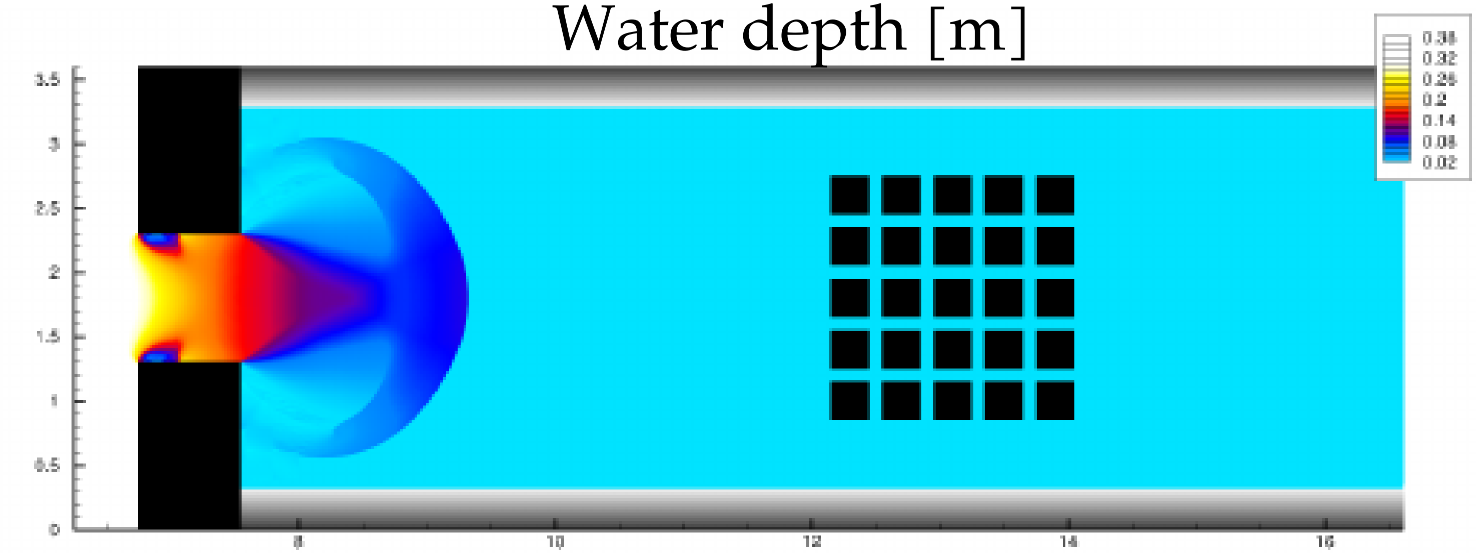
L = 7 → number of computational cells is 4.5% (95.5% reduction, achieving same resolution)

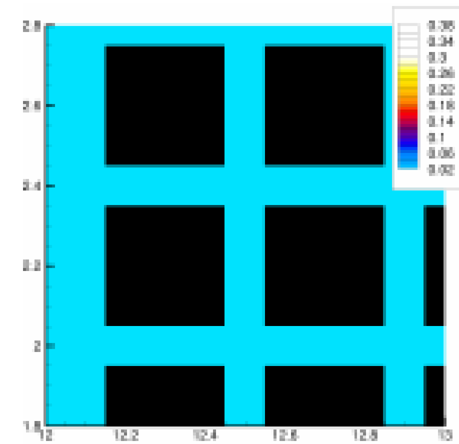
# Some results: experimental dam-break

Time: 1.0 s

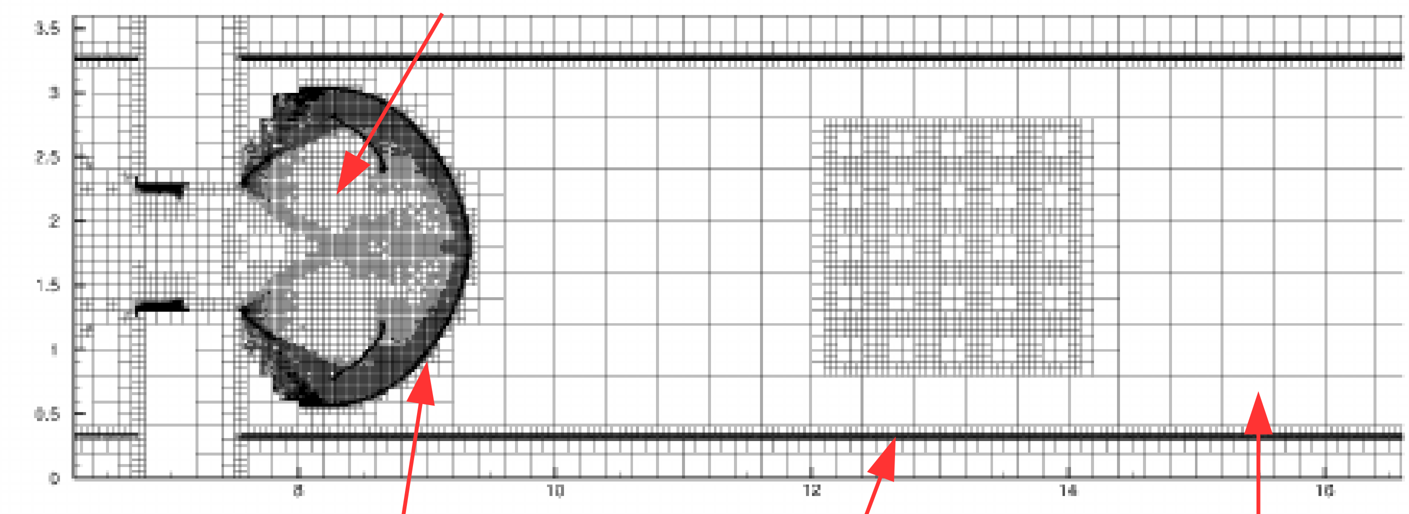
L = 7 solution

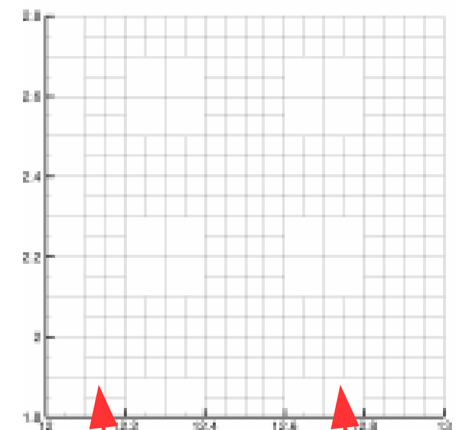
Water depth [m]



Intermediate refinement behind the shock





L7 shock refinement

High resolution bed kink

L0 cells (still water)

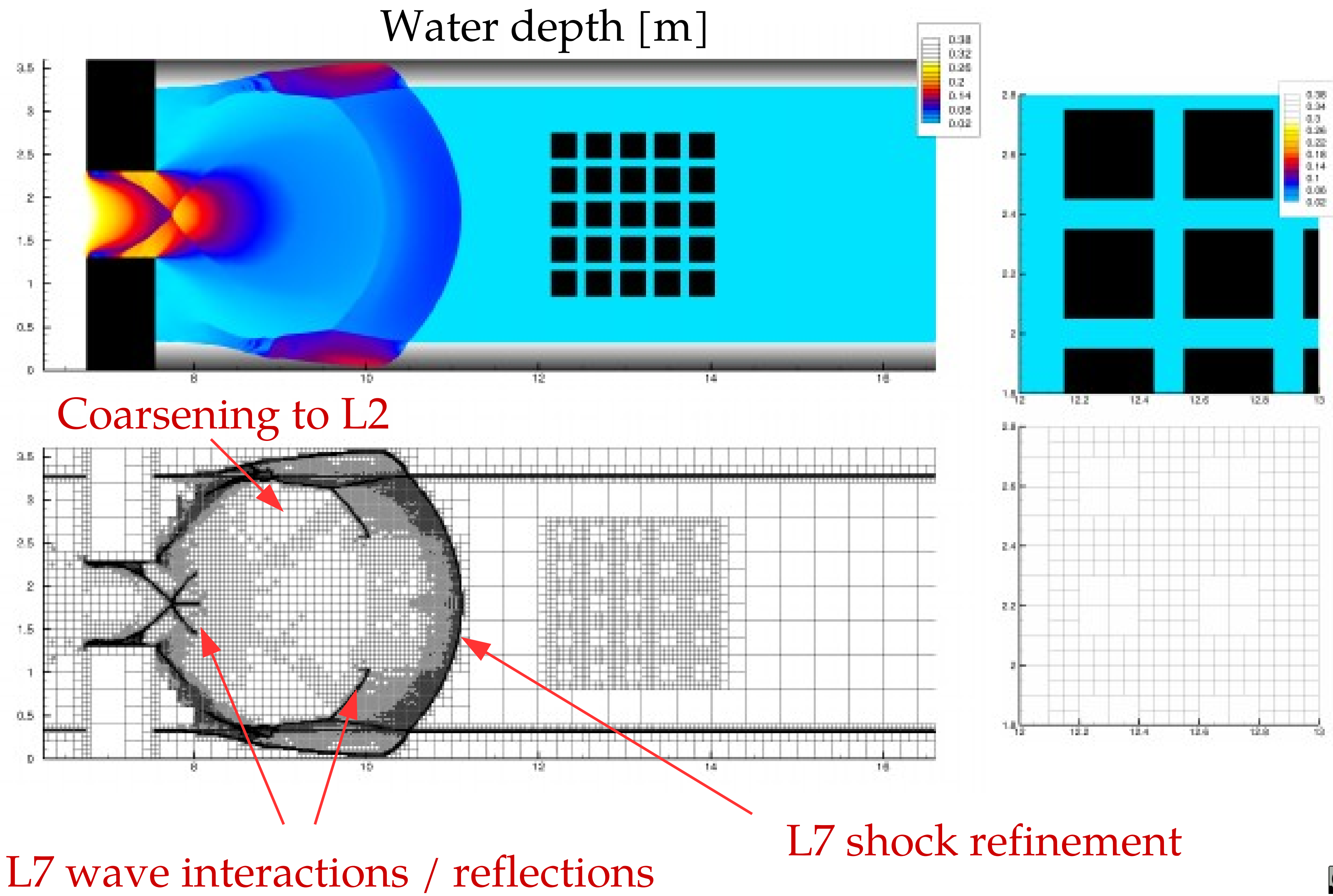
L3 cells

L2 cells

# Some results: experimental dam-break

Time: 2.0 s

L = 7 solution

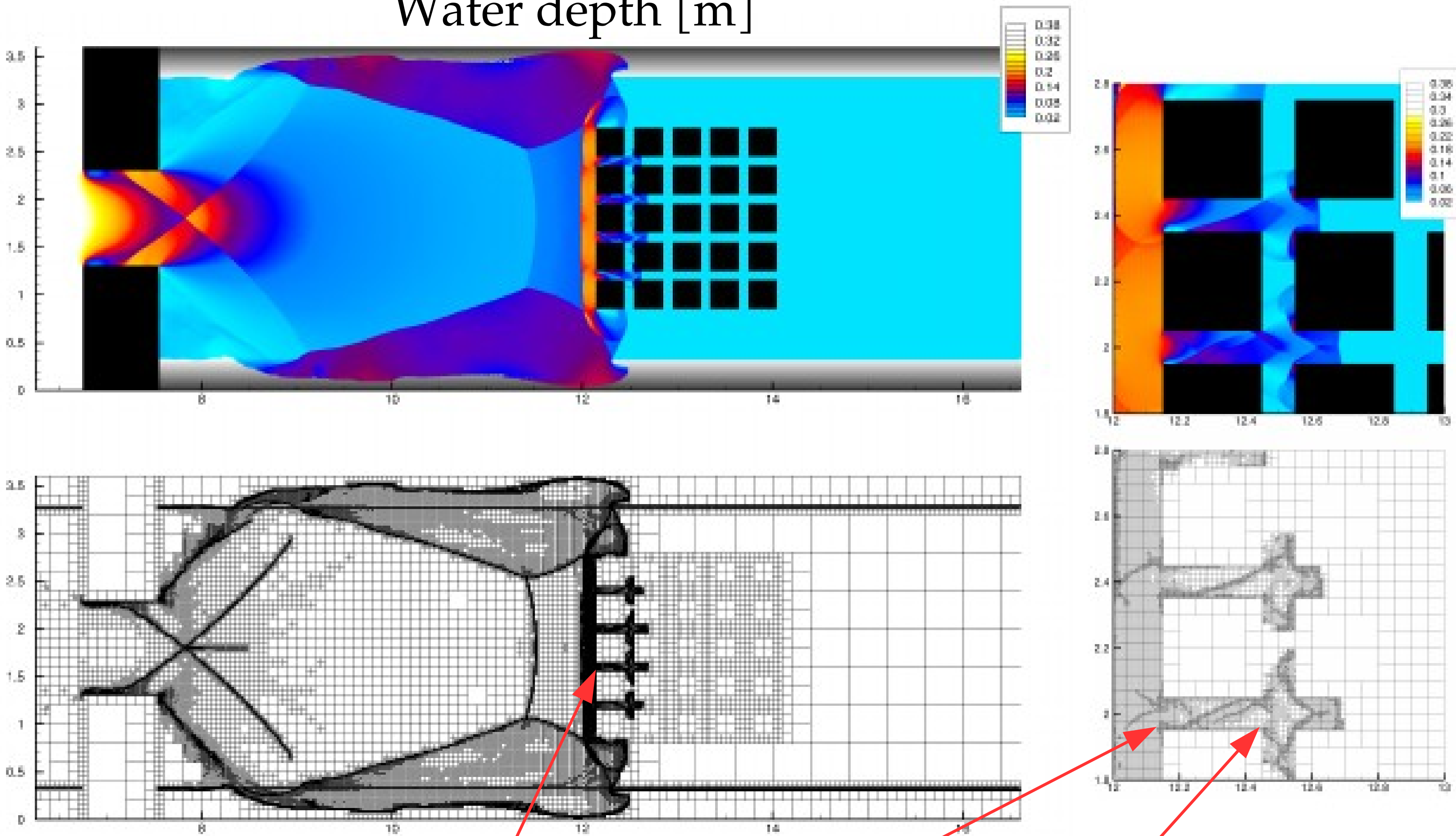


# Some results: experimental dam-break

Time: 3.0 s

L = 7 solution

Water depth [m]



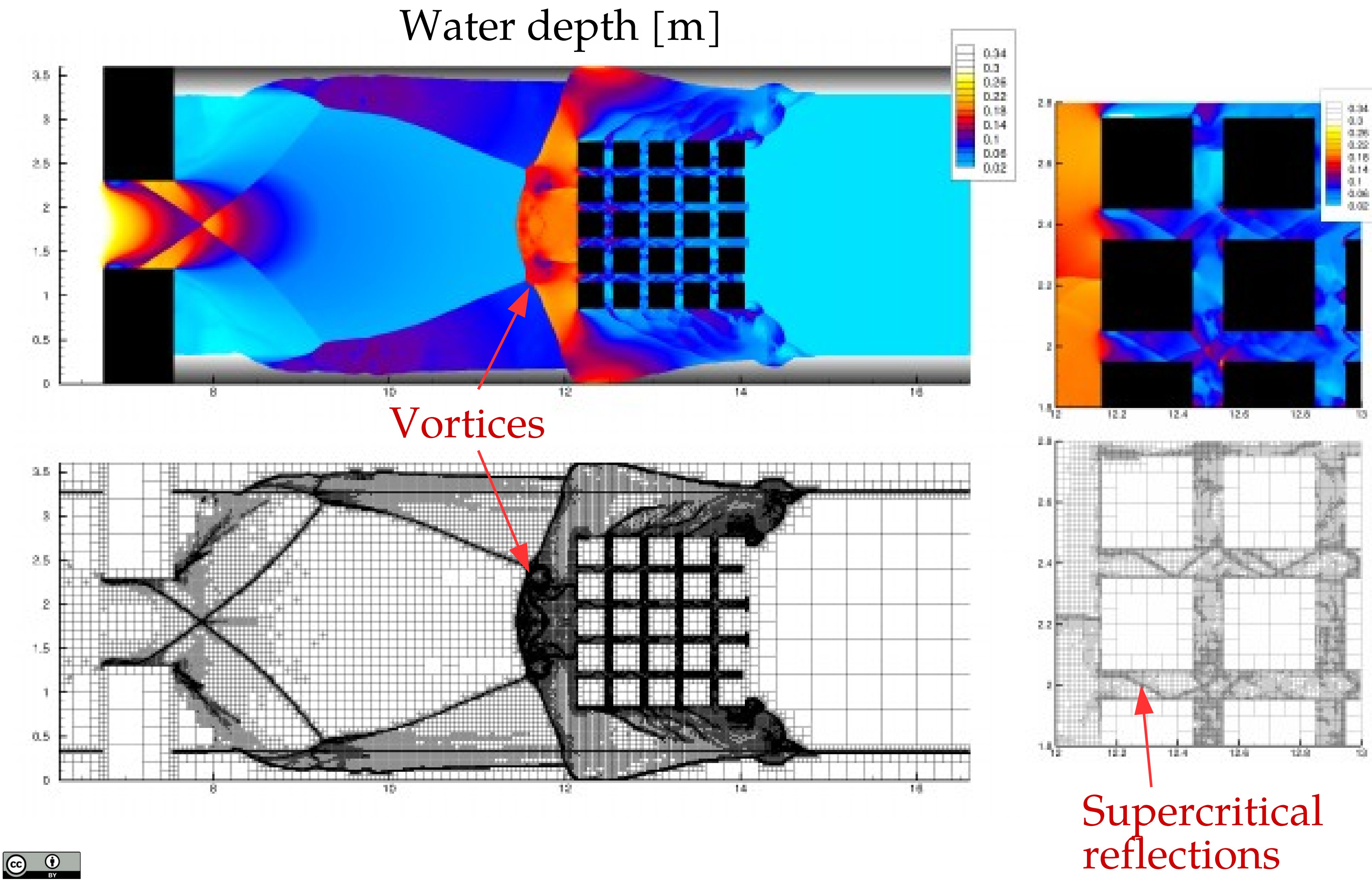
Highly variable, high resolution

Very sharp transition from L4 to L7 cells. No *proper grading*.

# Some results: experimental dam-break

Time: 4.0 s

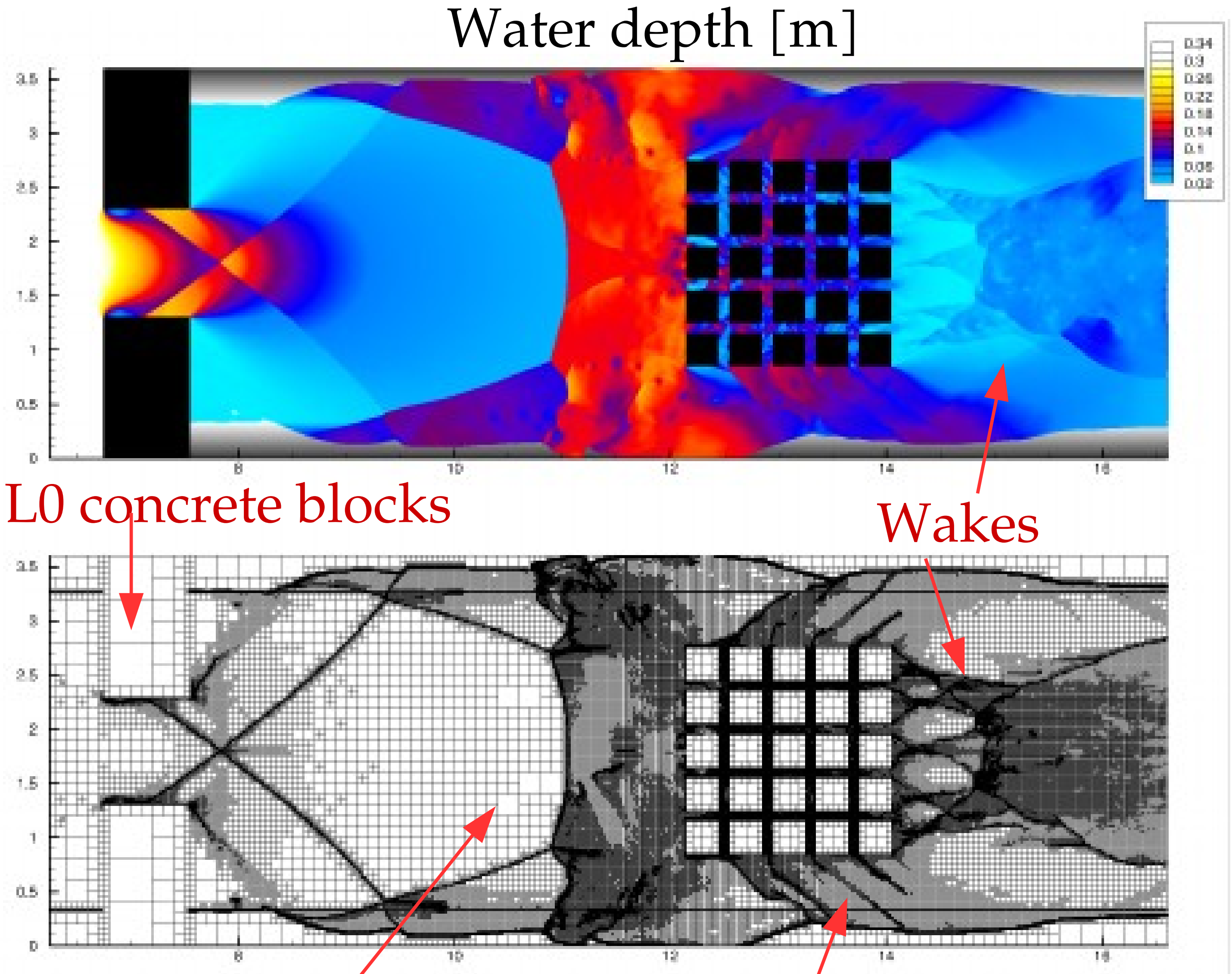
L = 7 solution

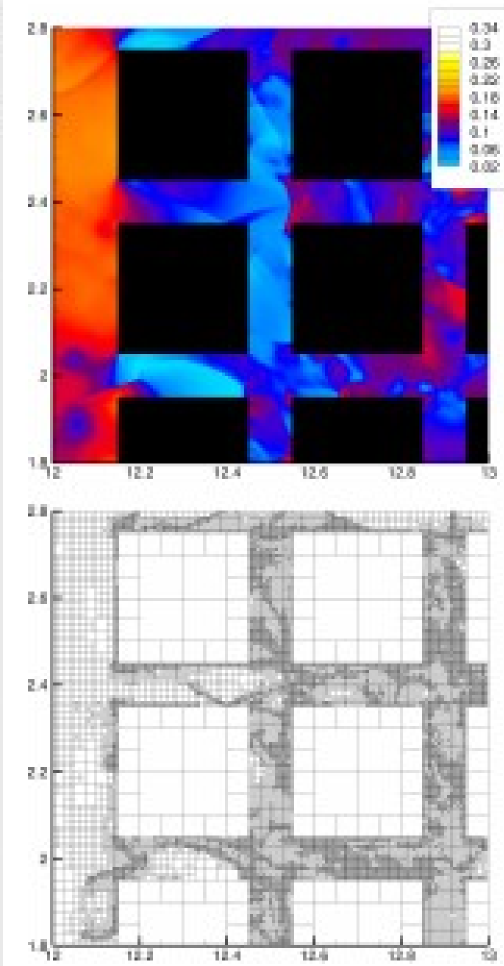


# Some results: experimental dam-break

Time: 7.0 s

L = 7 solution

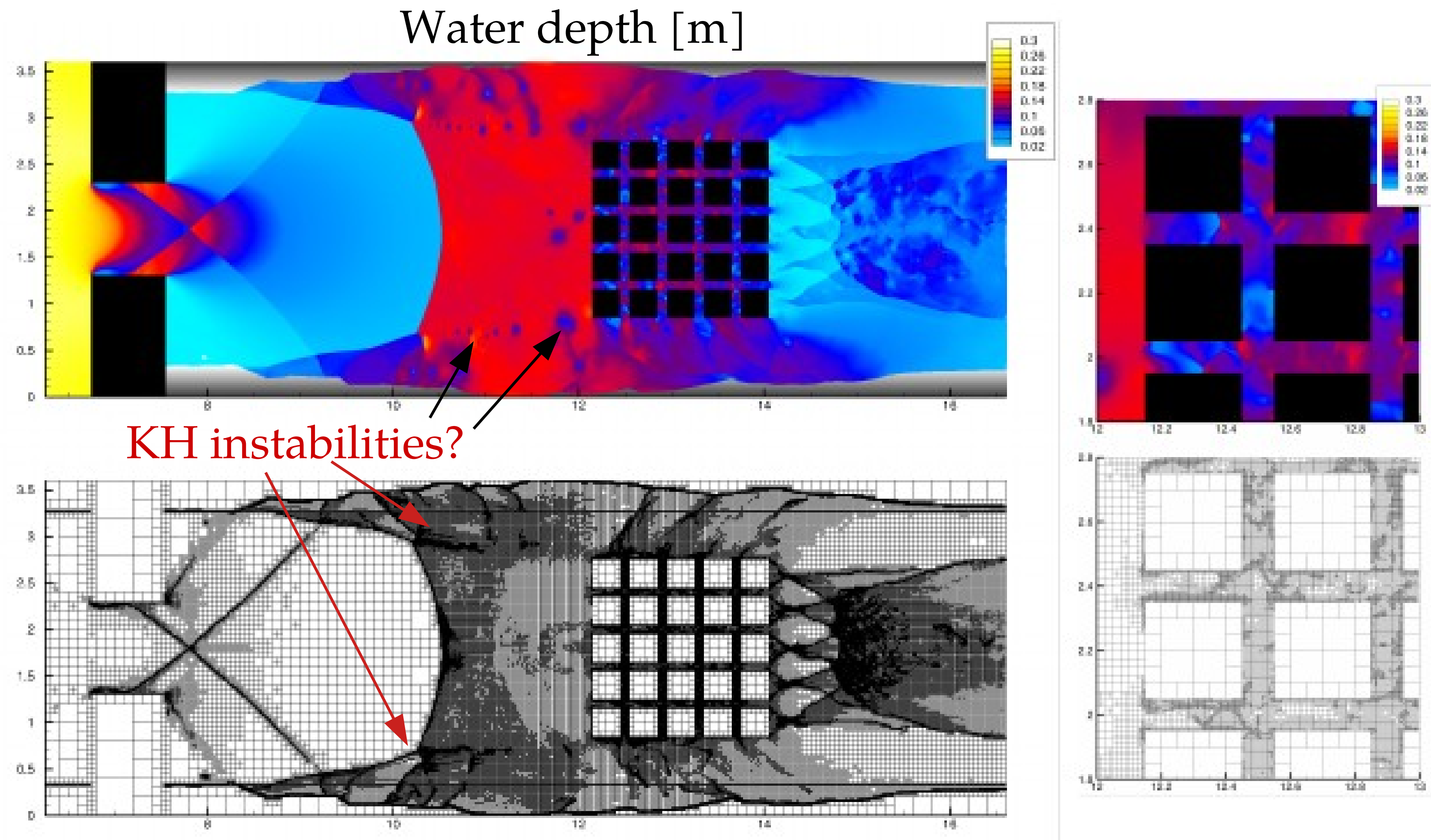


# Some results: experimental dam-break

Time: 10.0 s

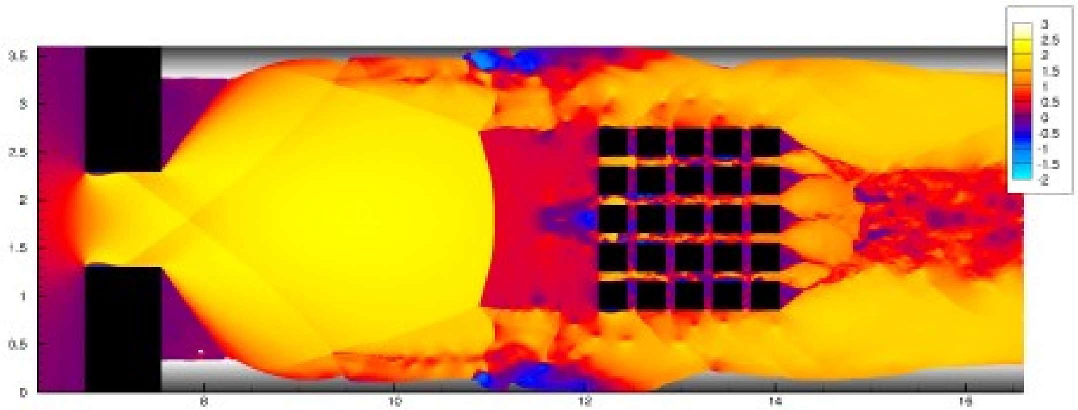
L = 7 solution



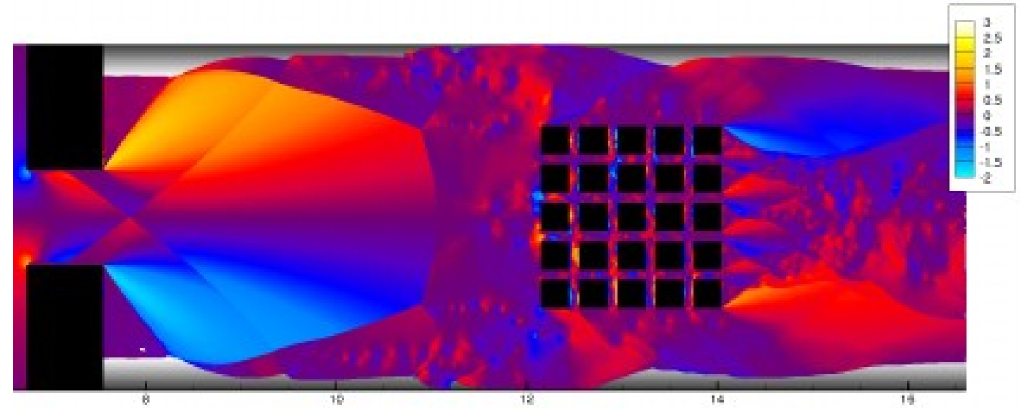

High resolution + High order → very little numerical dissipation
We can observe complex structures

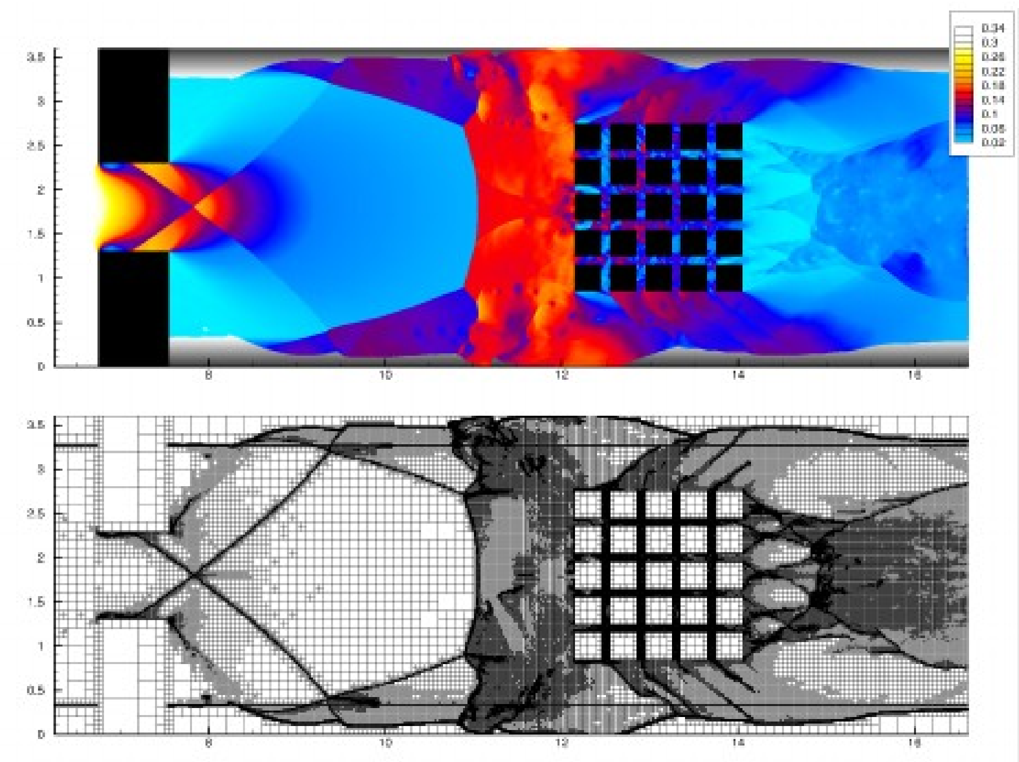
# Some results: experimental dam-break

x-velocity

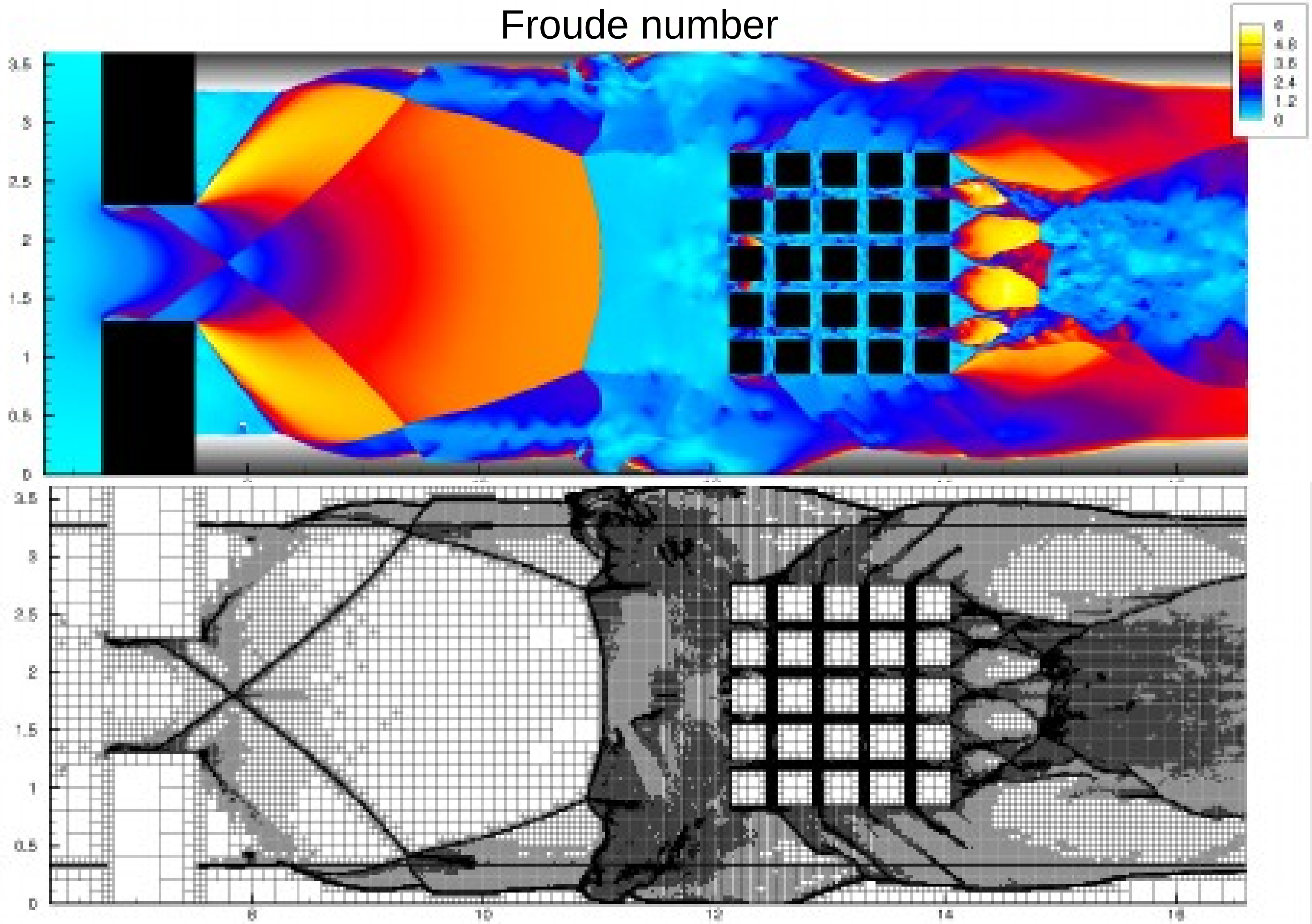


y-velocity



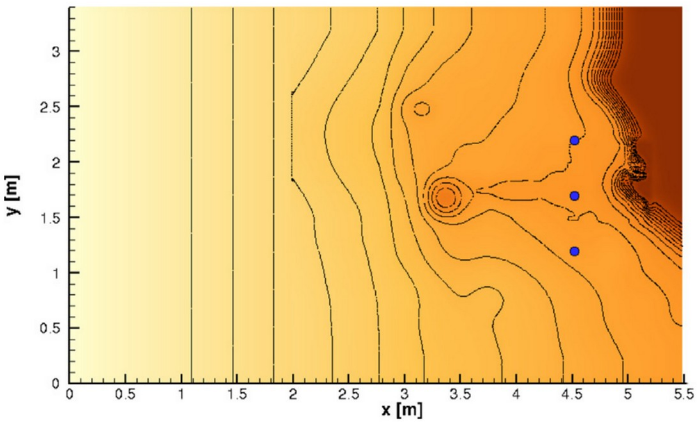


# Some results: experimental dam-break




Froude number provides good insight into the flow regimes and the flow's space of characteristics.
Although the grid is not adapted by the Froude number, it certainly captures its features. Good emergent model property.

# Some results: Tsunami case




Experimental case of tsunami in Monai valley

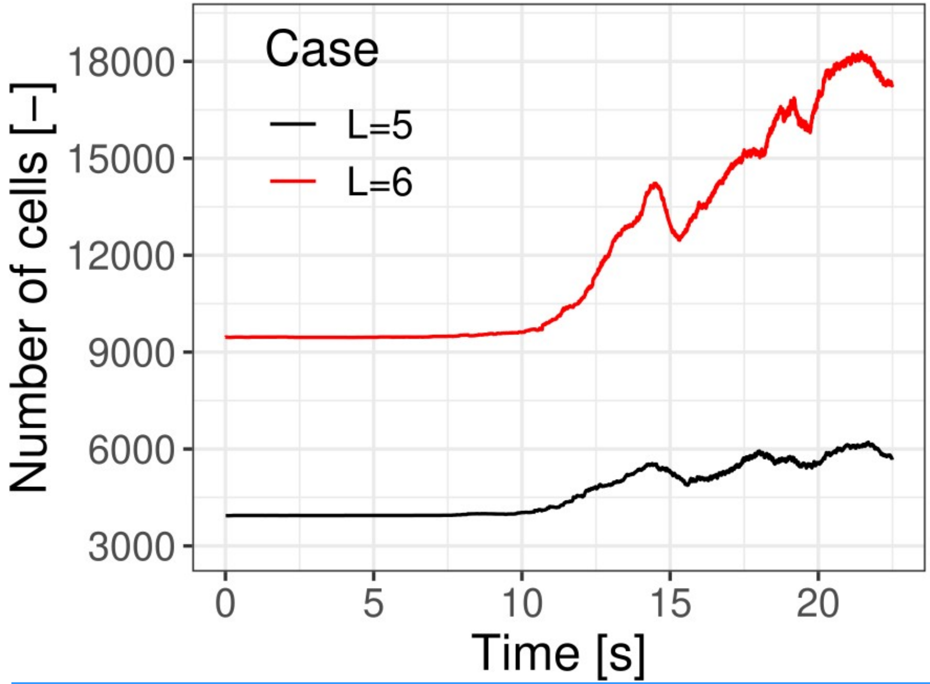
http://isec.nacse.org/workshop/2004_cornell/bmark2.html

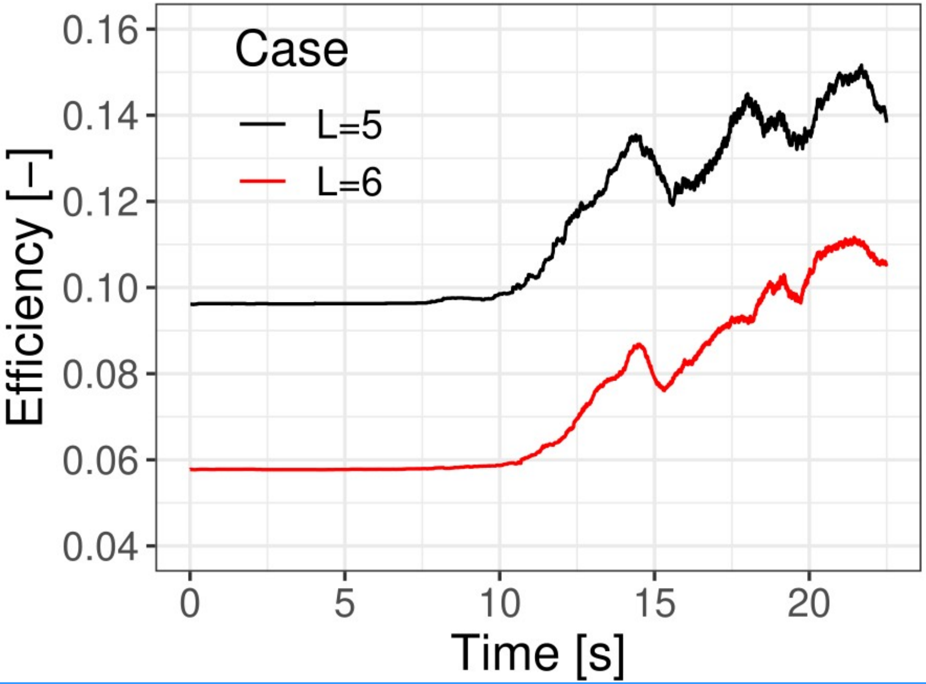
Solved with MWDG2 (linear test functions)

Coarsest mesh (level 0): 8 x 5 = 40 cells

Finest mesh (L=5): 256 x 160 = 40960 cells    dx = 2.14 cm

Finest mesh (L=6): 512 x 320 = 163840 cells    dx = 1.07 cm

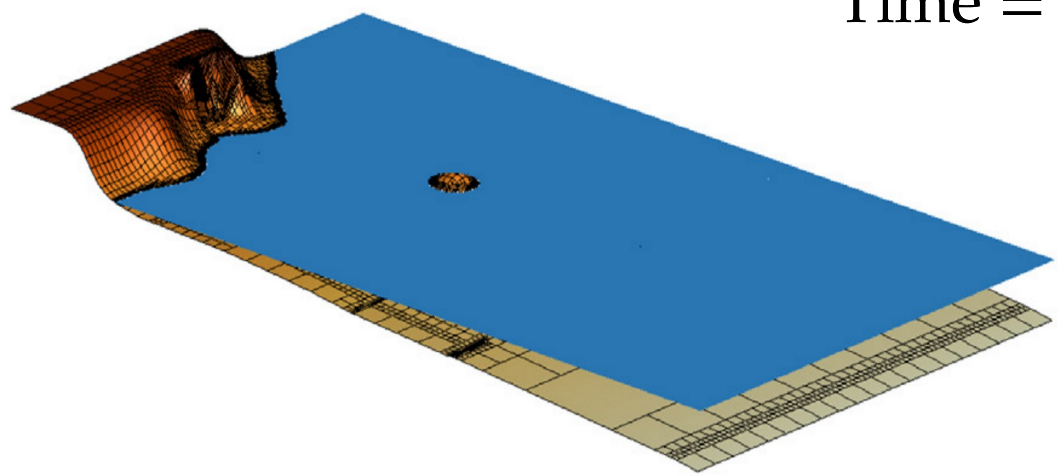


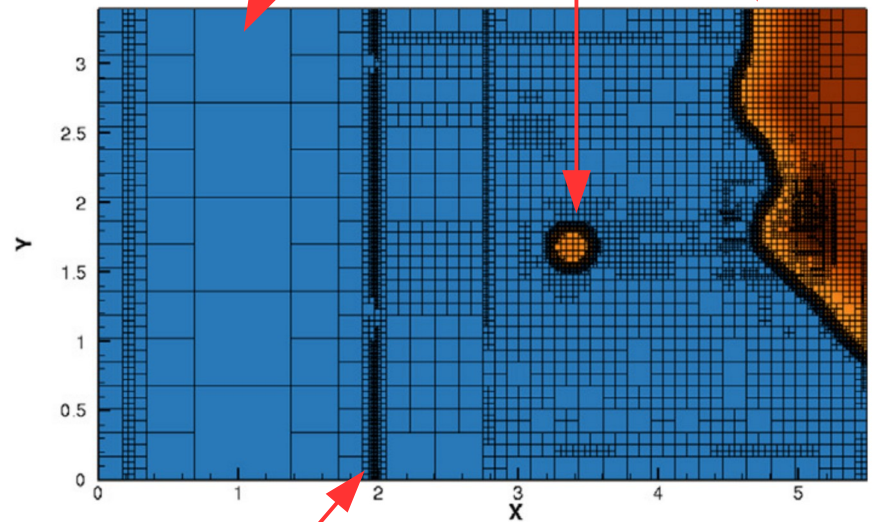




For L=6 case, the most expensive instantaneous grid uses only 11% of the size of the reference mesh. For L=5, it is 15%, although L6 results in roughly three times more cells.

# Some results: Tsunami case

Time = 0s

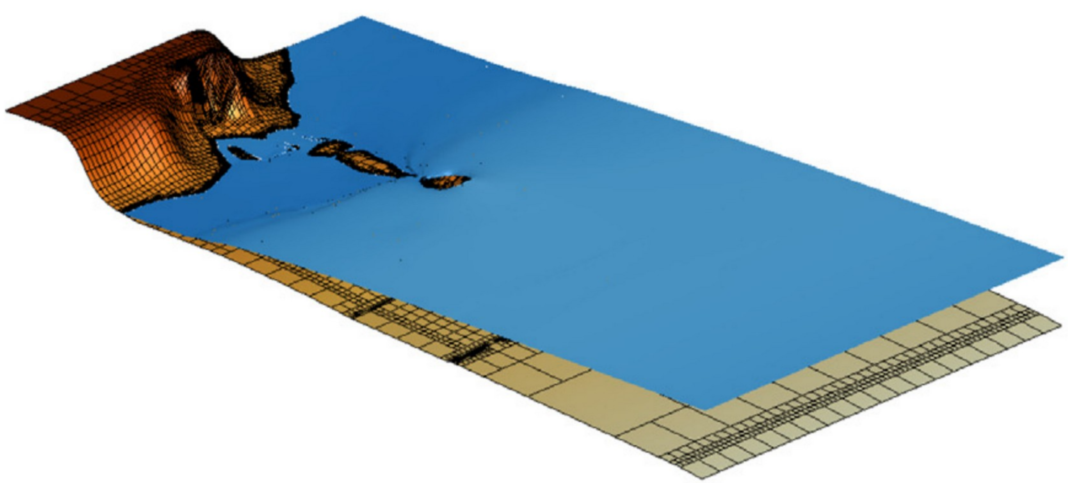


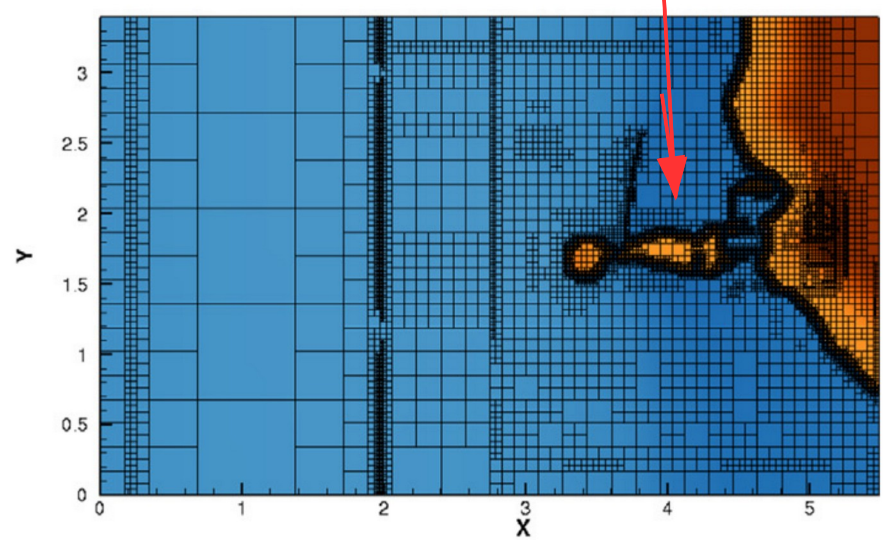



Bed kink

Drying fronts

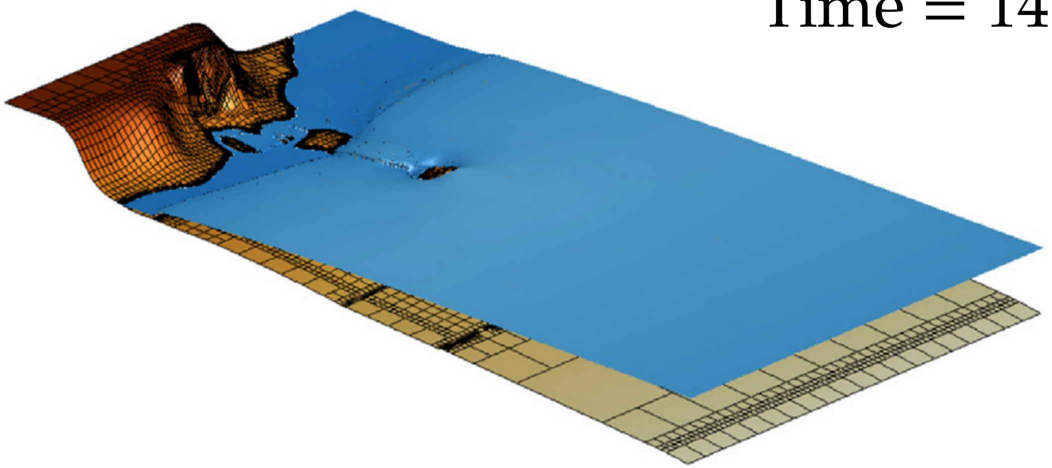
Time = 13.5s



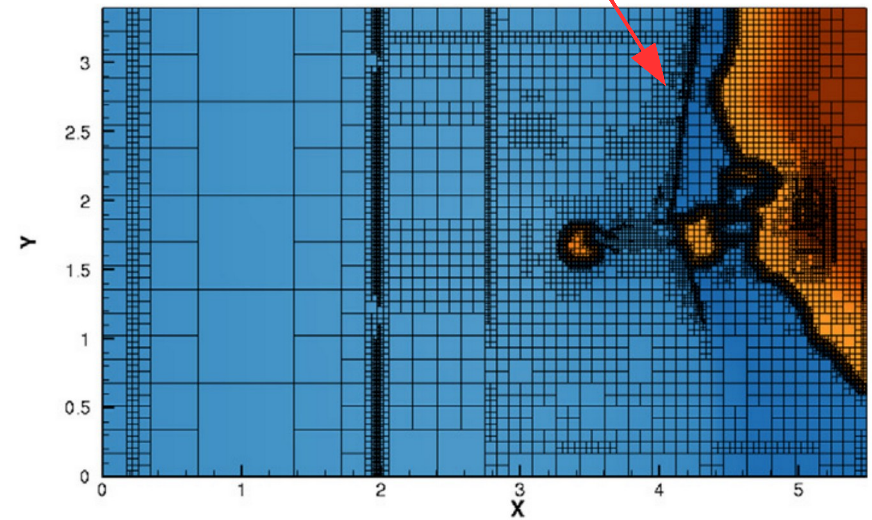


# Some results: Tsunami case

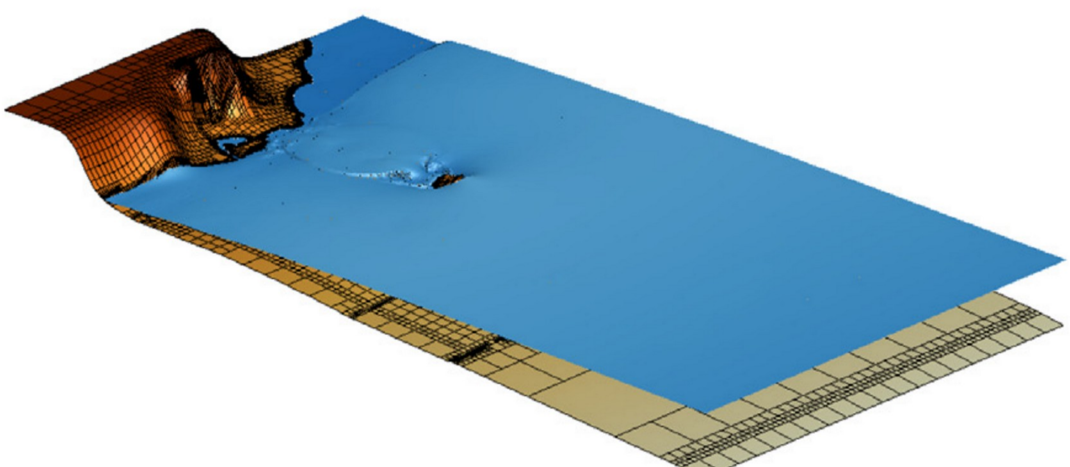
Time = 14.3s



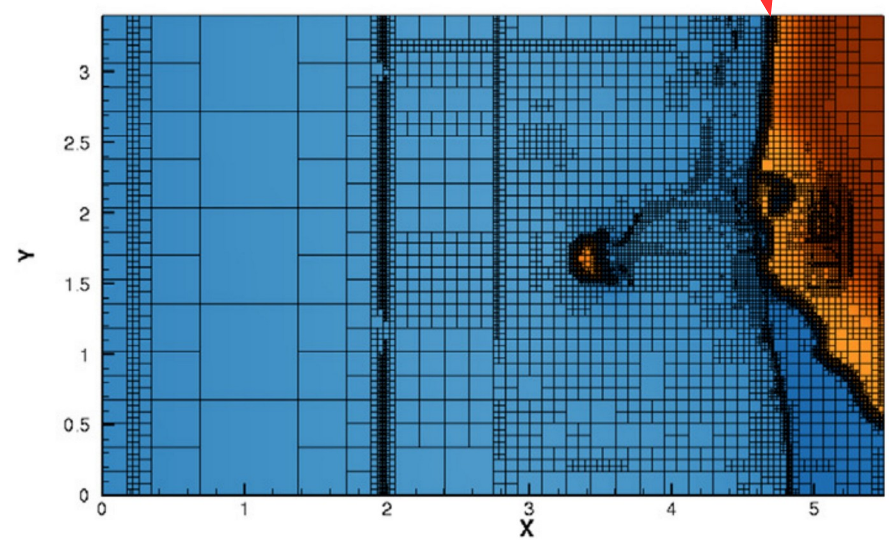
Incoming wave




Time = 15.0s

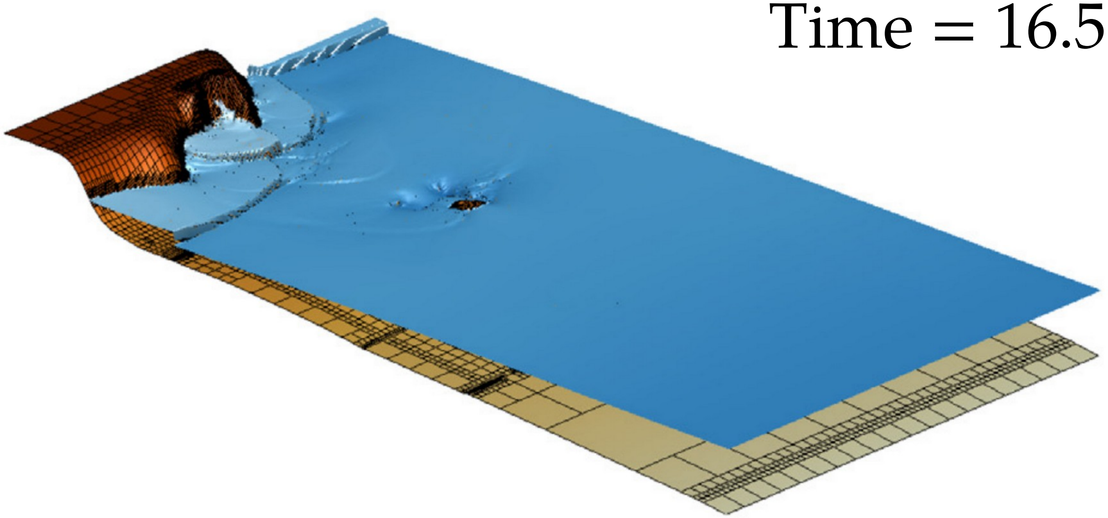


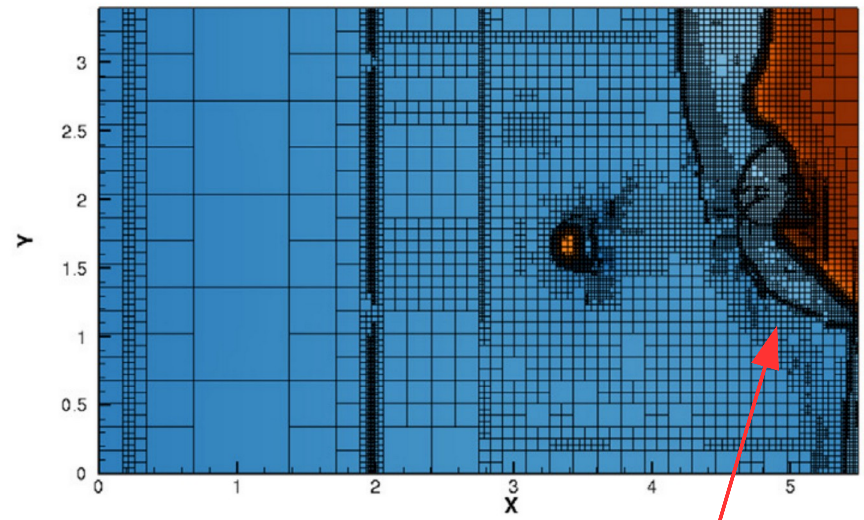
Moving wetting front



# Some results: Tsunami case

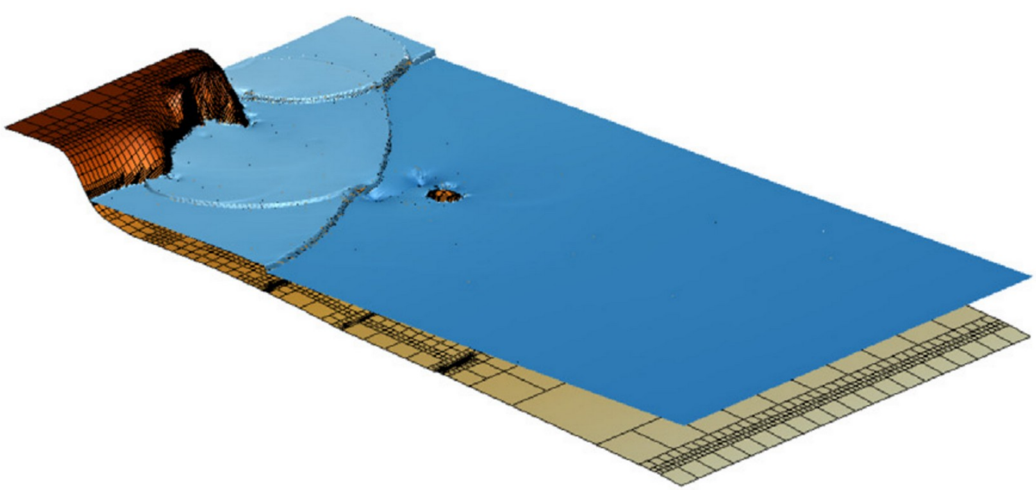
Time = 16.5s

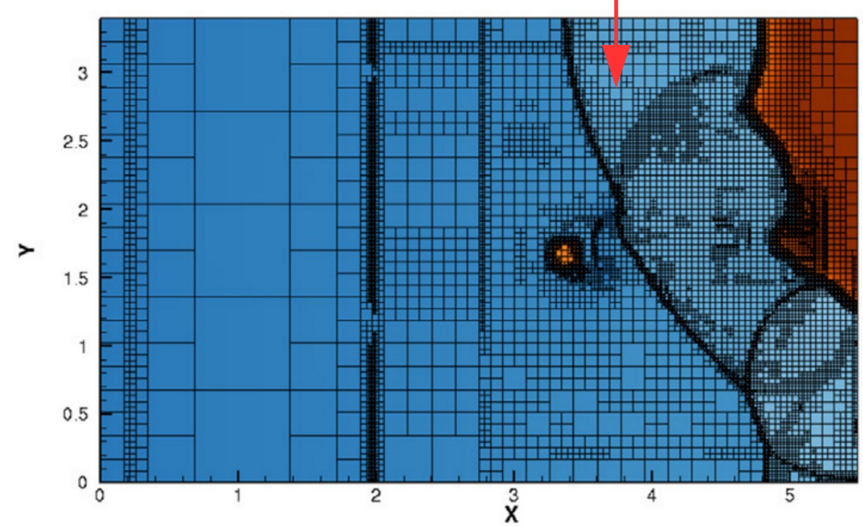





Wave reflections

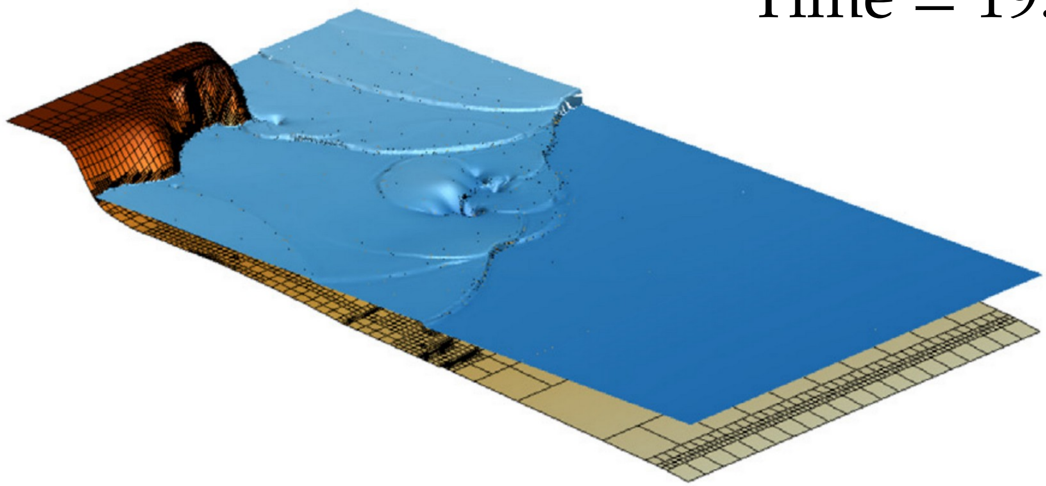
Time = 18.0s

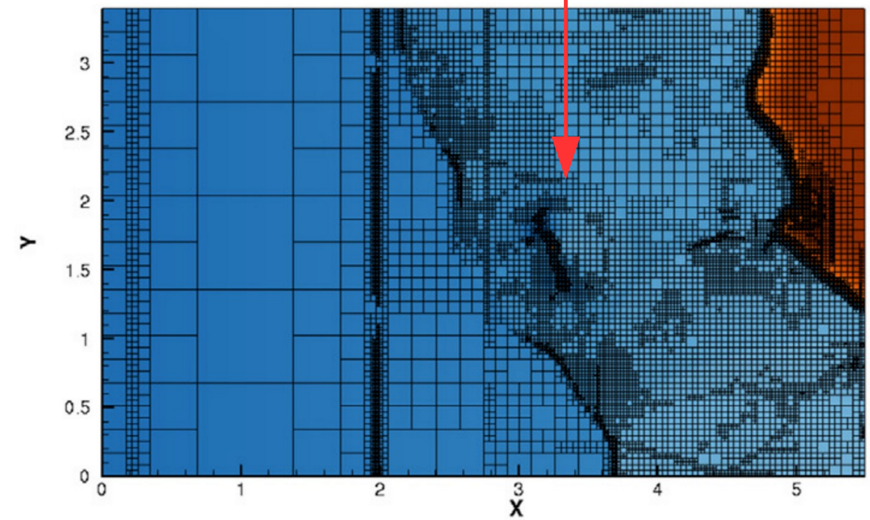




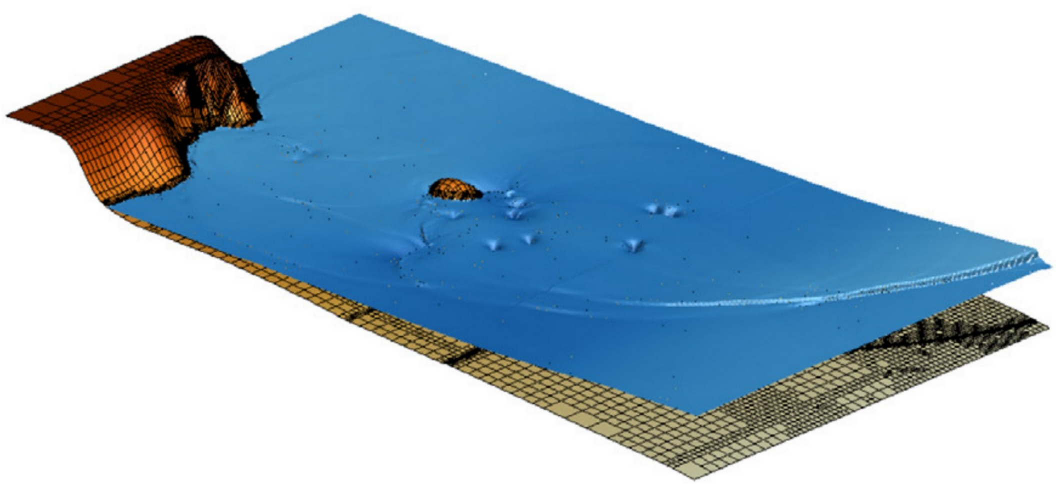
# Some results: Tsunami case

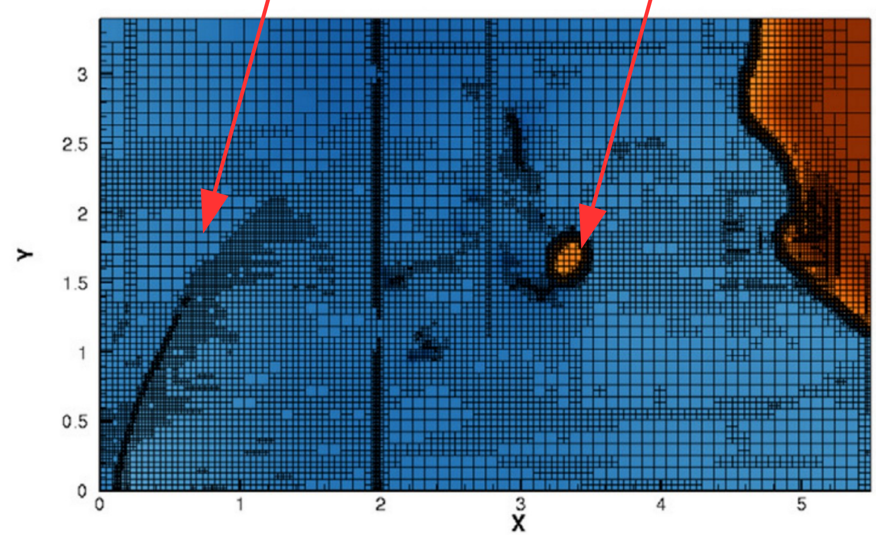
Time = 19.5s






Time = 22.5s





# Some results: Tsunami case

At the same resolution, MWDG-SWE strategy is more efficient in reducing cell numbers than all other strategies

Cell count normalised by number of cells of L6 reference mesh. Smaller is better for same resolution.

Comparison of mesh properties for tsunami case simulations.

| Simulation | Scheme | $N_a$ | $\delta x$ [$m$] | $\eta$ | $\hat{\eta}$ | $\tilde{\eta}$ | Notes |
|---|---|---|---|---|---|---|---|
| MWDG $L = 5$ | DG3 | 6211 | 0.0214 | 0.85 | 0.04 | 0.08 | DAMR |
| MWDG $L = 6$ | DG3 | 18139 | 0.0107 | 0.89 | 0.11 | 0.11 | DAMR |
| Kesserwani and Liang (2012a) | DG2 | 41843 | 0.014 | 0.56 | 0.26 | 0.33 | DAMR |
| Hou et al. (2018) | MUSCL | 30130 | 0.014 | 0.56 | 0.26 | 0.33 | FAMR cartesian. $N_a$ is estimated |
| Arpaia and Ricchiuto (2018) | ALE | 7000 | 0.025 | – | 0.23 | 0.21 | moving mesh |
| Arpaia and Ricchiuto (2018) | ALE | 36911 | 0.01 | – | 0.08 | 0.19 | moving mesh |
| Murillo and García-Navarro (2012) | FV | 47432 | 0.028 | – | | | Uniform triangular |
| Murillo and García-Navarro (2012) | FV | 3035648 | 0.0035 | – | 18.5 | 6.05 | Uniform triangular |
| Gonzàlez et al. (2011) | FV | 95256 | 0.014 | – | 0.58 | 0.75 | Uniform cartesian |
| Murillo et al. (2009) | MUSCL | 750 | 0.224 | — | 0.005 | 0.096 | Uniform triangular |
| Murillo et al. (2009) | MUSCL | 762048 | 0.007 | – | 4.65 | 3.04 | Uniform triangular |
| Morales-Hernández et al. (2014) | FV | 23716 | 0.028 | – | 0.15 | 0.38 | Uniform cartesian |
| Morales-Hernández et al. (2014) | FV | 94864 | 0.014 | – | 0.58 | 0.76 | Uniform cartesian |
| Vater et al. (2019) | DG2 | 393216 | 0.007 | – | 2.4 | 1.6 | Uniform triangular |

Compared to Kesserwani and Liang (2012) and to Hou et al (2018): MWDG L6 is 3 times less cells to achieve even higher resolution

Compared to Vater et al (2019), MWDG L6 results in an order of magnitude reduction in maximum cell numbers, achieving similar resolution

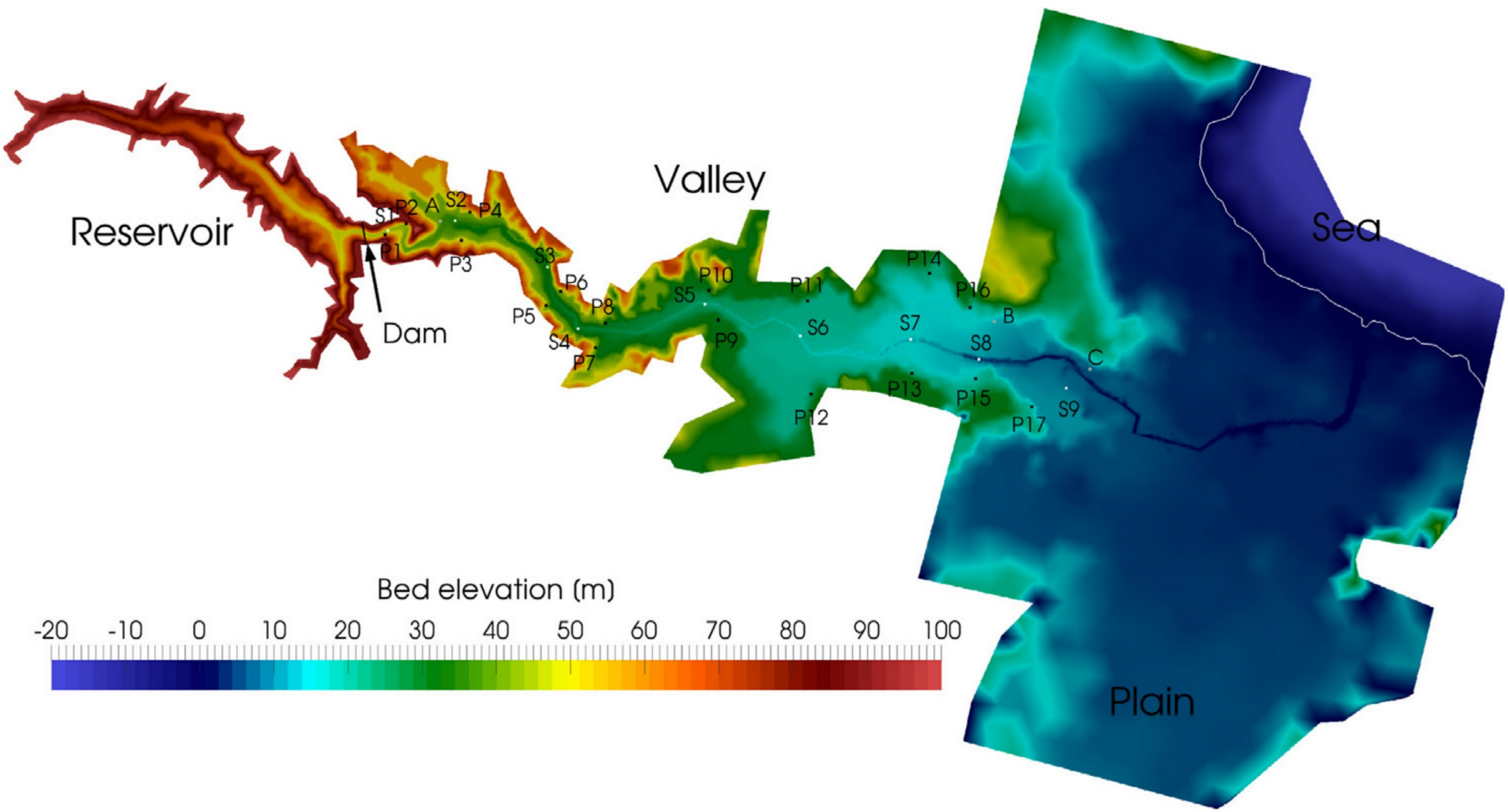
# Some results: Malpasset dam-break

Well-known benchmark test of real dam-break, with field and experimental data

Solved with MWDG2 (linear test functions)

Coarsest mesh (level 0): 27x15 = 405 cells dx = 640 x 424 m

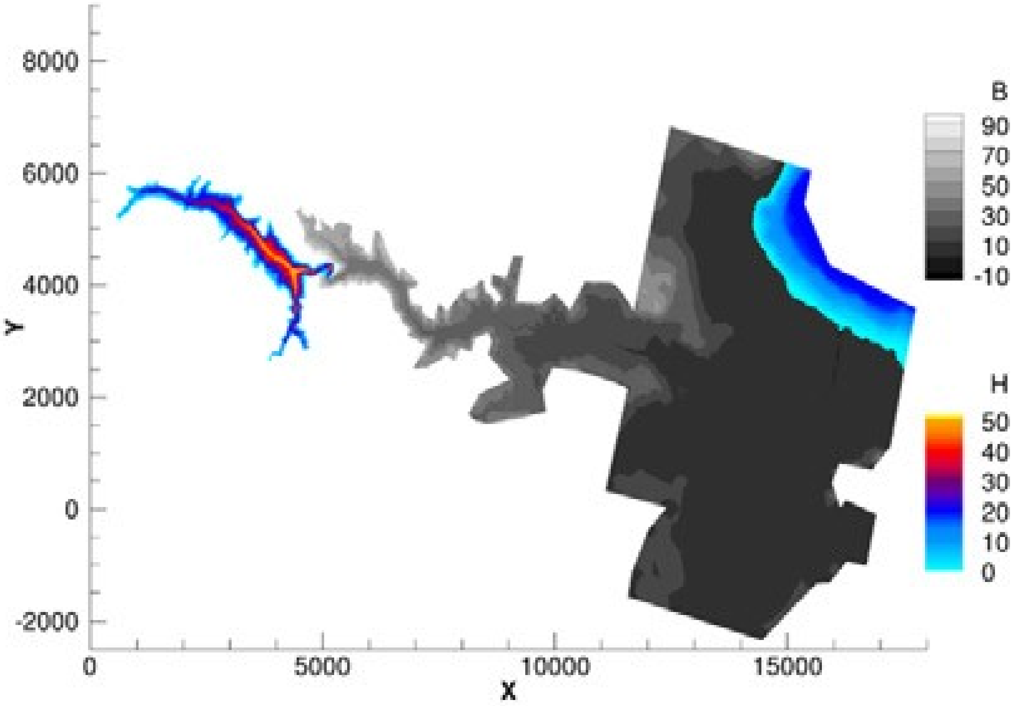
Finest mesh (L=7): 3456 x 1920 = $6.63 \times 10^6$ cells dx = 15 x 3 m



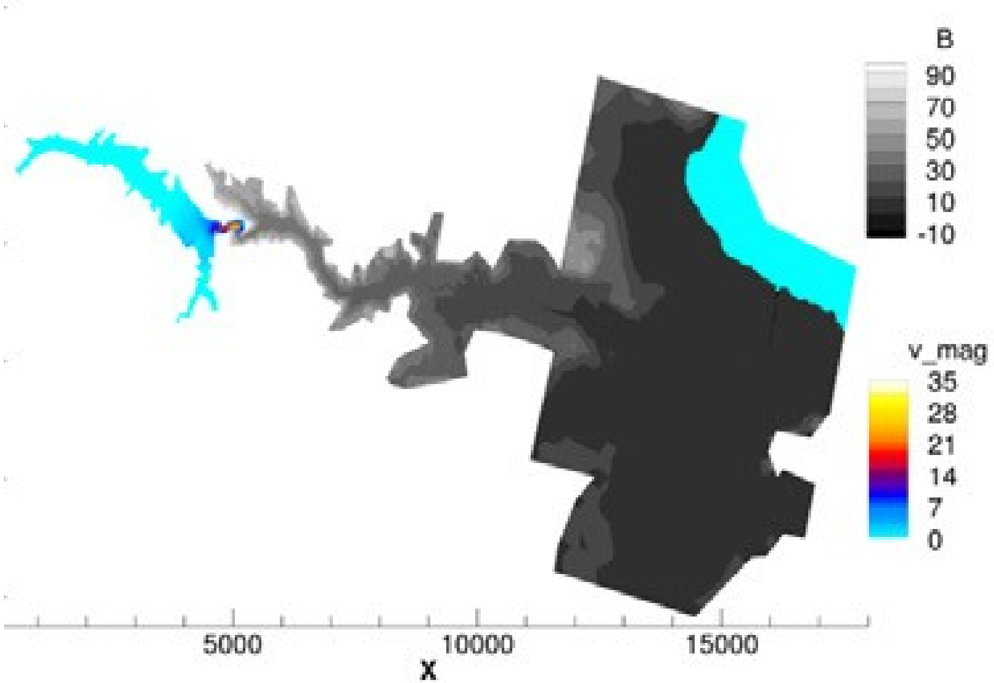
# Some results: Malpasset dam-break

Time: 50s

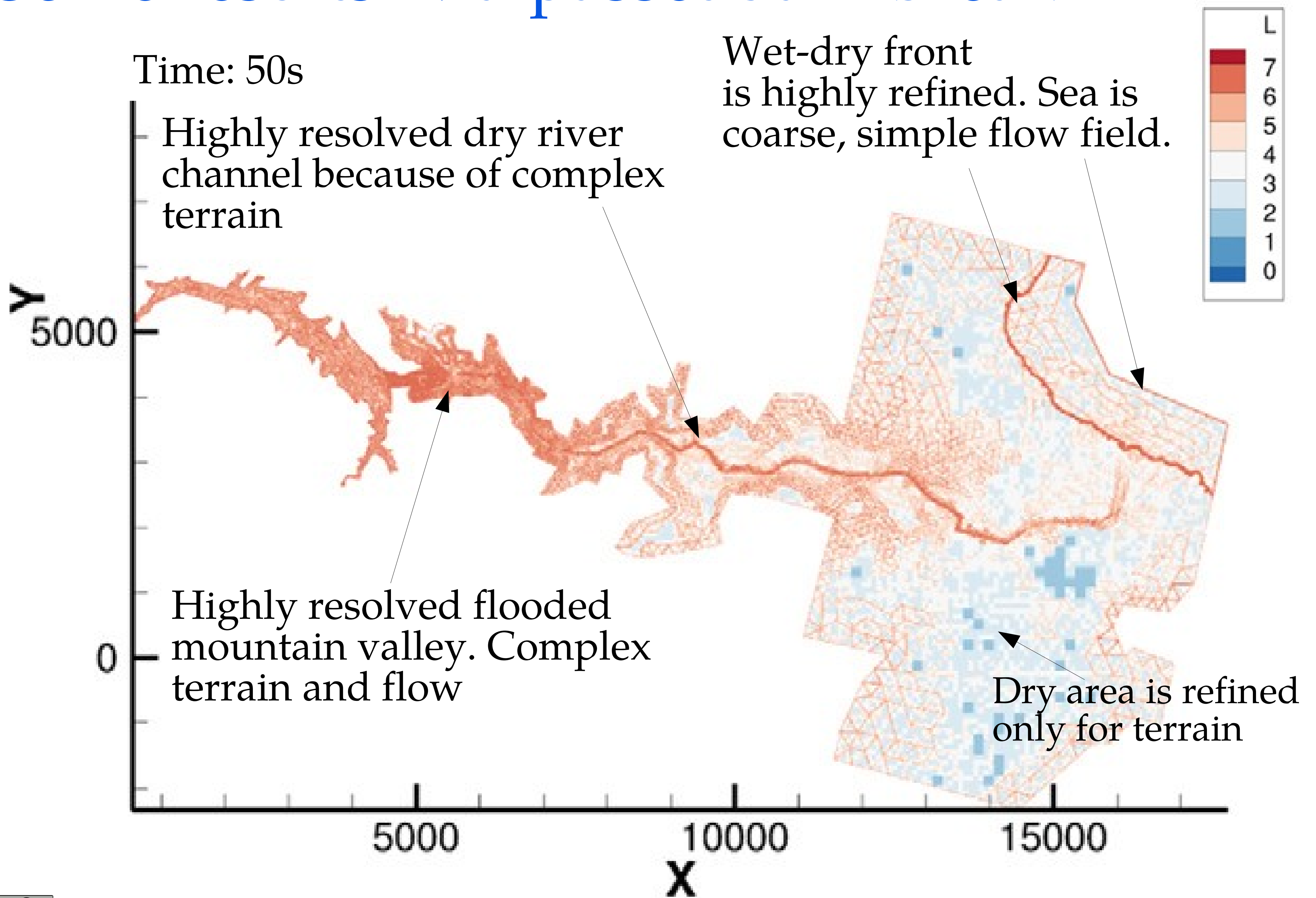
Depth [m]




Velocity [m/s]



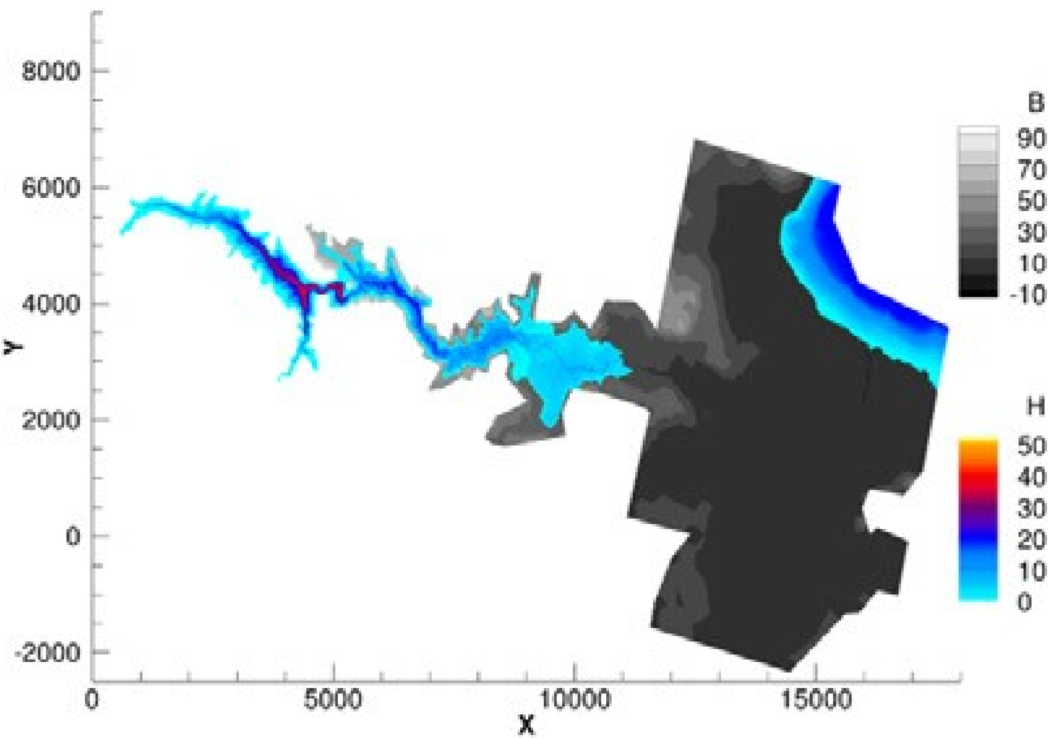
# Some results: Malpasset dam-break



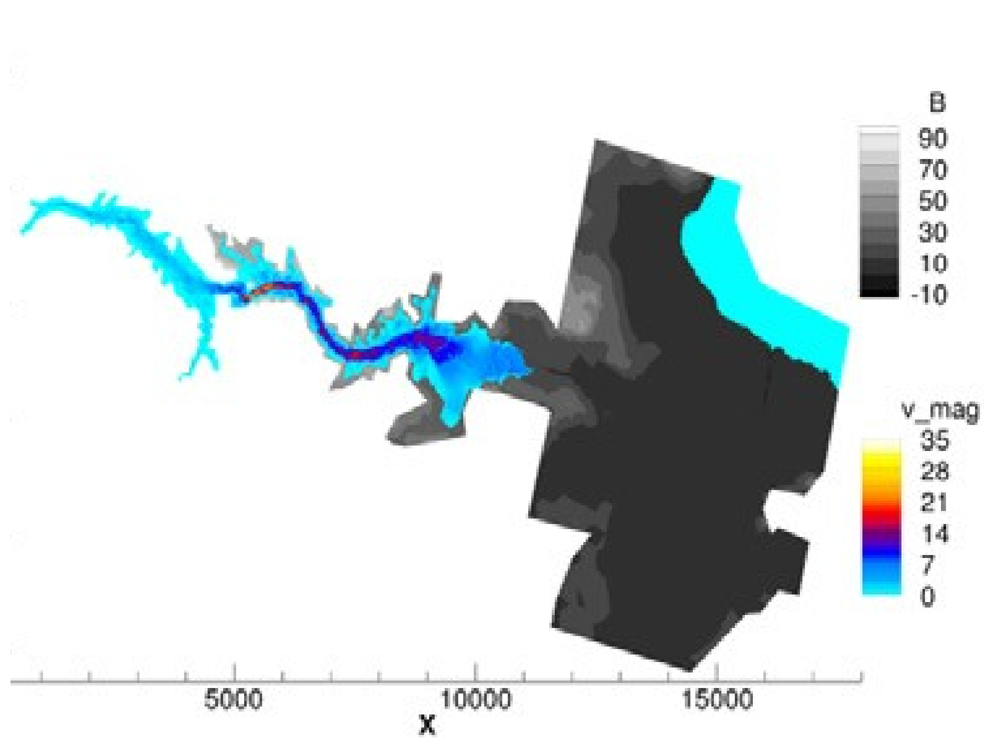
# Some results: Malpasset dam-break

Time: 1000s

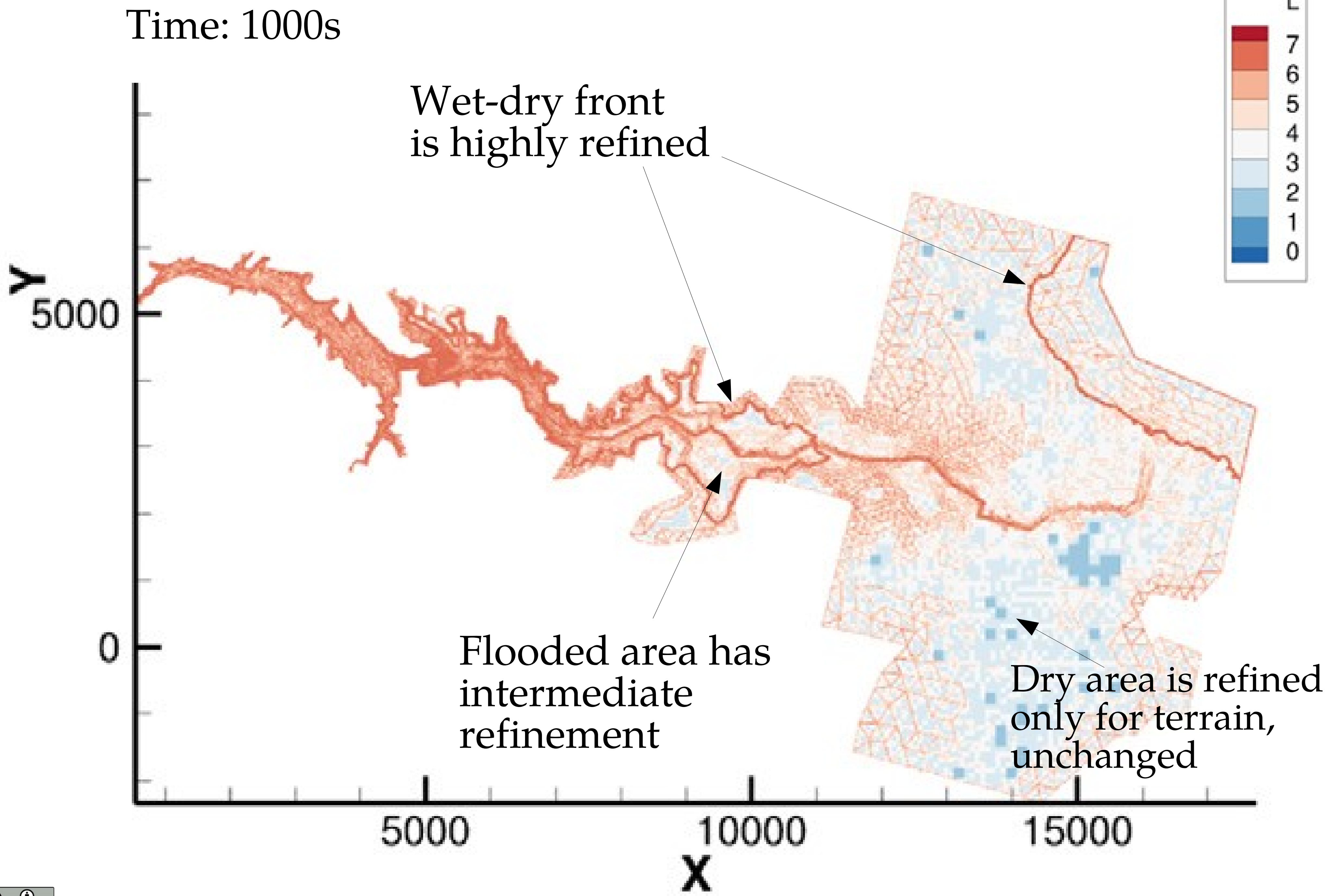
Depth [m]




Velocity [m/s]



# Some results: Malpasset dam-break



# Conclusions

- MWDG is applicable to complex and realistic 2D shallow-water problems.
- It is possible to achieve very high resolution and accuracy, otherwise very difficult to achieve with non-adaptive solvers.
- Since the method is not constrained by proper gradient, it can achieve significantly higher efficiency than previous strategies.
- MWDG captures all the good (conservation, well-balancing, wet/dry treatment) features of the underlying reference SWE solver, but also highlights deficiencies (wet/dry limiters, spurious momentum waves).
- Locality of DG allows for high-order solutions without compromising parallelisation.
- Strong reduction in number of cells, together with locality & parallelisation make MWDG competitive with high-order FV solvers.
- Further exploit *hp*-adaptivity also based on multiresolution analysis. Small cells are required at shocks and wet/dry interfaces, but no need for high-order there. This also allows for larger time steps.

# More results and details in our paper...

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## Multiwavelet-based mesh adaptivity with Discontinuous Galerkin schemes: Exploring 2D shallow water problems

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ABSTRACT

In Gerhard et al. (2015a) a new class of adaptive Discontinuous Galerkin schemes has been introduced for shallow water equations, including the particular necessary properties, such as well-balancing and wetting-drying treatments. The adaptivity strategy is based on multiresolution analysis using multiwavelets in order to encode information across different mesh resolution levels. In this work, we follow-up on the previous proof-of-concept to thoroughly explore the performance, capabilities and weaknesses of the adaptive numerical scheme in the two-dimensional shallow water setting, under complex and realistic problems. To do so, we simulate three well-known and frequently used experimental benchmark tests in the context of flood modelling, ranging from laboratory to field scale. The real and complex topographies result in complex flow fields which pose a greater challenge to the adaptive numerical scheme and are computationally more ambitious, thus requiring a parallelised version of the aforementioned scheme. The benchmark tests allow to examine in depth the resulting adaptive meshes and the hydrodynamic performance of the scheme. We show that the scheme presented by Gerhard et al. (2015a) is accurate, i.e., allows to capture simultaneously large and very small flow structures, is robust, i.e., local grid refinement is controlled by just one parmeter that is auotmatically chosen and is more efficient in terms of the adaptive meshes than other shallow-water adaptive schemes achieving higher resolution with less cells.

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