Extremes for high dimensional dynamical systems

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Main questions

Can one link statistical and dynamical properties

- that is extremal index and local dimension of the attractor?
- Is this link generic?
- Is this link stable?

Extreme Value theory for i.i.d.

Given time series $X_1, X_2, X_3, \ldots$, consider

$$M_n := \max_{k=1,\ldots,n} X_k.$$

Note the analogy with $S_n := \sum_{k=1}^n S_k$.
One has an extreme value theory, if an analogue of a Central Limit Theorem holds for $(M_n)_{n\geq 1}$ for a proper resacling

$$\frac{M_n - b_n}{a_n} \to S.$$

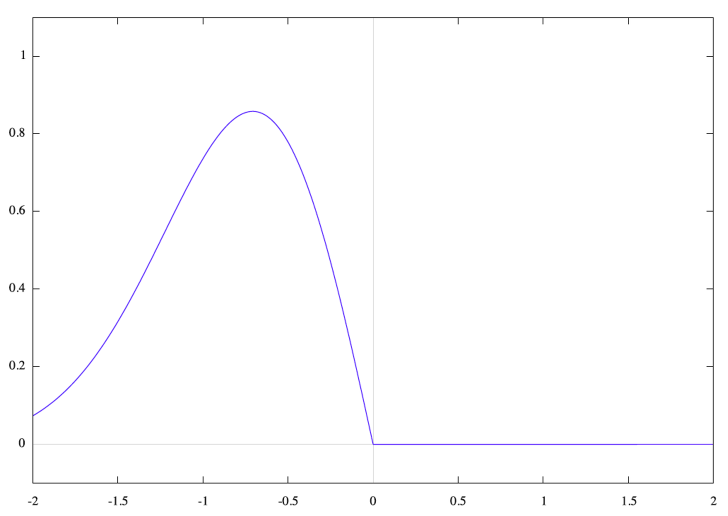
Fischer-Tippet-Gnedenko-Theorem: Potential limiting distribution $P_{\lim}$ of S are parametrized

- by two scaling factors $x \mapsto ax + b$,
- and the so-called *extreme value index* ξ.

The limit has to be a *Generalized extreme value distribution*

$$P_{\lim}(z) = (1+\xi z)^{-1-1/\xi} e^{-(1+\xi z)^{-1/\xi}} \qquad \text{for } 1+\xi x > 0.$$

Here we consider mostly $\xi < 0$, that is Weibull distribution.



Pickands-Balkema-deHaan-Theorem:
A distribution with cumulative distribution function F has as limit law extreme value index $\xi < 0$ iff $x^* = \inf\{x : P(X > x) = 0\} < \infty$ right endpoint of the distribution and for $x \downarrow 0$

$$\frac{P(X_1 > x^* - tx)}{P(X_1 > x^* - x)} \to \ell(x) t^{-1/\xi}, \qquad (1)$$

where ℓ log-factors (slowly varying).
Note that (1) is a conditional probability over threshold (compare (3) below)

$$P(X \geq x^* - tx | X \geq x^* - x) \qquad (2)$$

Leadbetter-Theorem:
Same properties holds also for *non* i.i.d. whenever maxima in blocks are mixing with distance between blocks and absence of clustering (one over threshold event in a block)

Main Advantage of Extreme value theory

Extreme value techniques will be robust to deviations from power laws as non power law parts of the distribution are automatically suppressed. (1) means that F is of form

$$P(X_1 > x^* - x) = \ell(x) x^{-1/\xi}$$

where ℓ are logarithmic factors which will be washed out by the method.
What the CLT is for the bulk, the extreme value theory is for the extremes

Extreme value theory for dynamical systems

Traditionally, extreme value theory has been consider for random system, that is distributions with a density. Why dynamical systems?

- Features of non-equilibrium
  - Highly complex system, that is, d very large
  - Attractor lower dimensional
- physical measure is singular (in contrast to stochastic systems)
- Multi-scale in this case fractal structure.

Recently intensively studied: e.g. P. Collet, A. Freitas, J. Freitas, M. Todd, C. Gupta, M. Holland, M. Nicol, G. Turchetti, and S. Vaienti and others. See monograph in references.

Perfectly chaotic systems

Let M be a closed bounded subset of $\mathbb{R}^d$ (d very large). Consider discrete time and assume that attractor $\Omega \subset M$ is perfectly chaotic, that is f restricted to Ω splits the space uniformly in expanding and contracting direction for the dynamics.
ν denotes the *physical measure*, that is the statistics of frequency of points along a typical path.
Denote by d_s the local dimension of the attractor in the stable (contracting direction) and by

Extremal Index vs. dimension of attractor: distance observable

For an observable A for which its maximum on the attractor is a local maximum on $\mathbb{R}^d$. Holland et al. suggest that the extremal index is in one to one correspondence with the local scaling of the volume

$$\lim_{r\downarrow 0} \frac{\ln(\nu(d(|x-x_0| \leq r))}{\ln r} = d_s + d_u.$$

Around the extremum

$$A(x) = A(x_0) + \nabla^{\otimes 2} A(x_0) x^{\otimes 2} + \cdot$$

Hence

$$\frac{\ln(\nu(A(x) \geq A(x_0) - ta))}{\ln(\nu(A(x) \geq A(x_0) - a))} \sim (d_s + d_u)(\ln(\sqrt{ta}) - \ln(\sqrt{a})) = t^{d_s + d_u} \qquad (3)$$

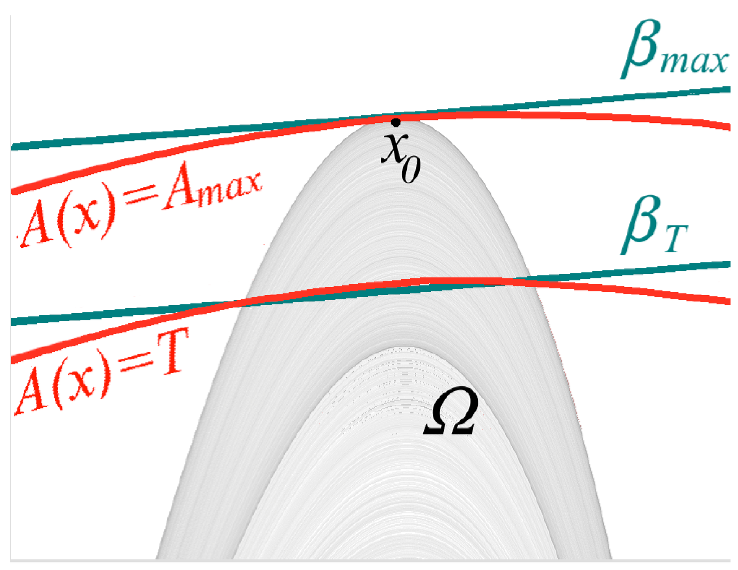
This is equivalent to (2) for example when ν is non-singular. Easy counter example shows that this is not true for general invariant measures.

Extremal Index vs. dimension of attractor: generic observable

For a generic observable A, as the attractor is lower dimensional, the maximal value is attained at the surface of the attractor. Expand around this point x_0 then

$$A(x) = A(x_0) + \nabla A(x_0) \cdot x + \nabla^{\otimes 2} A(x_0) x^{\otimes 2} + \cdot$$

The level sets are in first approximation hyperplanes β orthogonal to $\nabla A(x_0)$ and in second order paraboloids.



Everything in the hyperplane β scales parabolic all other directions linear.
As maximum, all unstable direction are in β.

$$\frac{d_u}{2} + d_s \geq -\frac{1}{\xi}$$

Claim: Generically all stable direction are not in β. Then

$$\frac{d_u}{2} + d_s = -\frac{1}{\xi}.$$

This can be used as a looking glass into the surface of the attractor,

Generic?

Is the level set of A tangential to the surface of the attractor in the unstable directions only. This depends on the challenging question whether the surface structure of the attractor is non-smooth. at a generic surface points.

Stable? – Response theory

An easier question is whether the extreme value index is stable under perturbation of the law of the dynamics. In Baladi et al. we employed the most recent results in the theory of dynamical systems. We were able shows either *Hölder continuity of the response* under very restrictive conditions or the following general result:
Let $\alpha \mapsto f_\alpha$ be a C^3 maps of C^4-diffeomorphism with a compact hyperbolic attractor. Let

$$\varphi(x) = h(x)\theta(g(x) - a)$$

with $h, g : M \to \mathbb{R}$ in C^4 and $a \in \mathbb{R}$ not a critical value of g and assume that

$$\{x \in M : g(x) = a\} \cap \mathrm{supp}(h)$$

admits a C^4-foliation of admissible stable pseudo leaves. Note the level sets are non-tangential to the expanding directions.
Then the map $\alpha \mapsto \rho_\alpha$ is differentiable in the weak sense.

References

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