Fundamental Notions in Relativistic Geodesy - physics of a timelike Killing vector field

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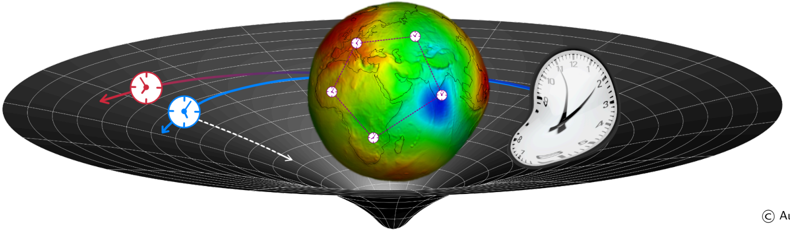
with Eva Hackmann, Claus Lämmerzahl, Volker Perlick, Dirk Pützfeld, Jürgen Müller



EGU2020, Vienna

geo

RESEARCH TRAINING GROUP Models of Gravity

ZENTRUM FÜR ANGEWANDTE RAUMFAHRTTECHNOLOGIE UND MIKROGRAVITATION

ZARM

Contents

Mostly based on:

- DP, Volker Perlick, Dirk Puetzfeld, Eva Hackmann, and Claus Lämmerzahl, *Definition of the relativistic geoid in terms of isochronometric surfaces*, Phys. Rev. D 95, 104037 (2017)
- DP, PhD thesis *Theoretical Aspects of Relativistic Geodesy*, (2019)
- DP, Eva Hackmann, Claus Lämmerzahl, and Jürgen Müller, *Relativistic geoid: Gravity potential and relativistic effects*, Phys. Rev. D 101, 064032 (2020)

Geodesy and its Objectives

"Geodesy is what geodesists do for their living."

Helmut Moritz

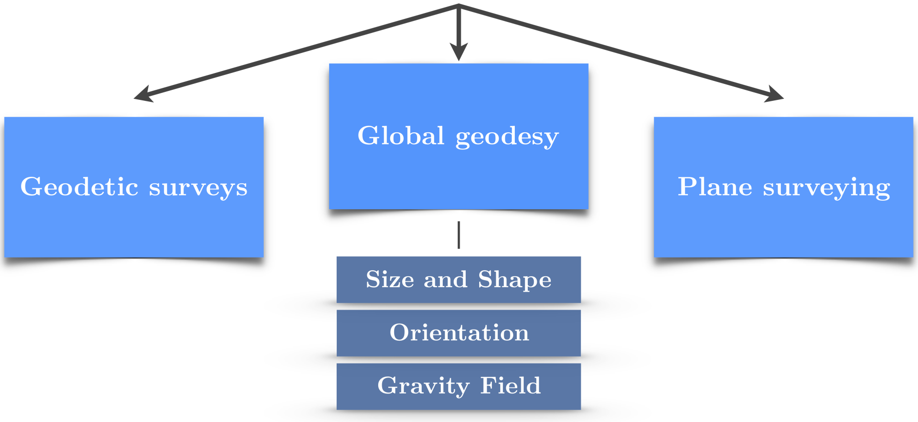
"Geodesy is the science of measurement and mapping of the Earth's surface."

Friedrich R. Helmert

"The objective of geodesy is to determine the figure and external gravity field of the Earth, as well as its orientation in space, as a function of time, from measurements on and exterior to the Earth's surface."

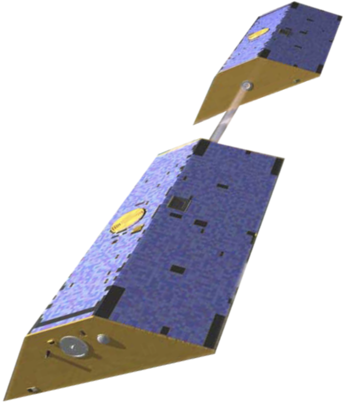
[Torge, Müller 2012]

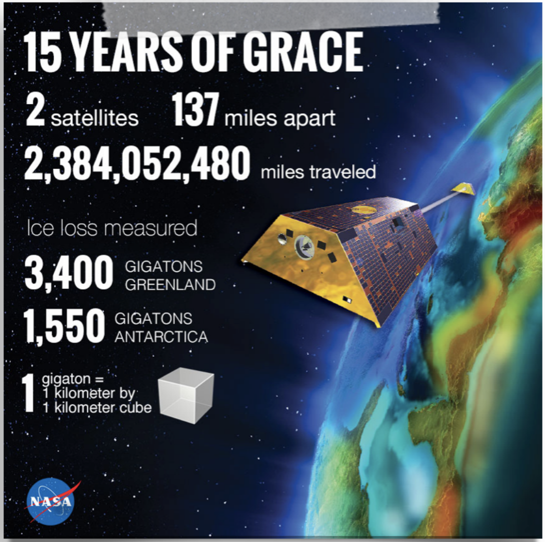
Geodesy and its Objectives



- Geodesy allows us to learn more about our planet, its properties and dynamics.
- Results have an impact on important related fields such as navigation and climate research.

GRACE Mission

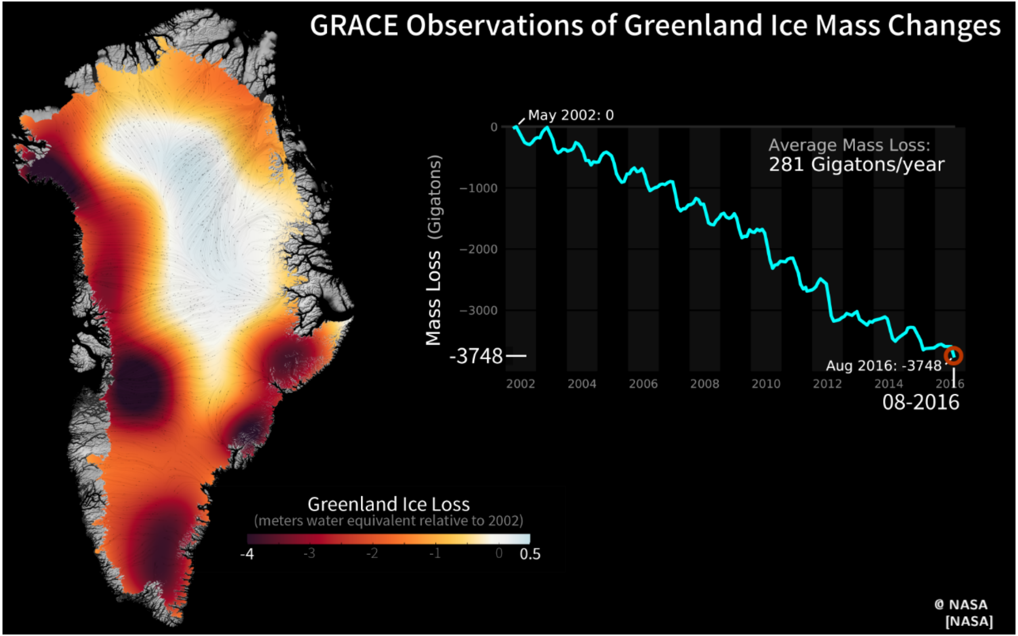




Why?

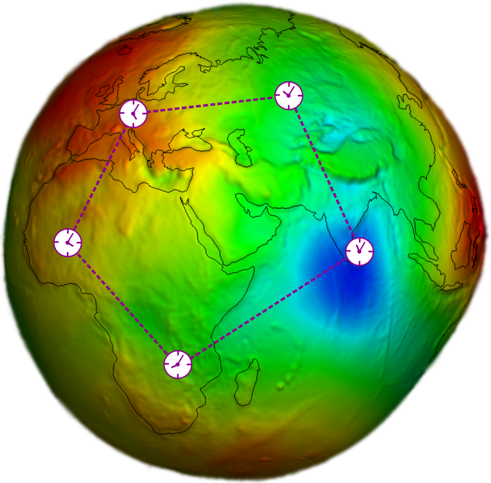

CLIMATE
climate change
warming
changes
Americans skeptical
YOU CAN'T IGNORE Climate Change
DER SPIEGEL
Der verheizte Planet
Wie die Gier nach Wachstum unser Klima zerstört
DHAKA THURSDAY SEPTEMBER 3, 2009
UN chief 'alarmed' Arctic glacier melt
DER SPIEGEL
Ozon-Loch, Pol-Schmelze, Treibhaus-Effekt: Forscher warnen
DIE KLIMA-KATASTROPHE
2050 sea level
2030 sea level
I DON'T BELIEVE IN GLOBAL WARMING

GRACE Mission - Impact

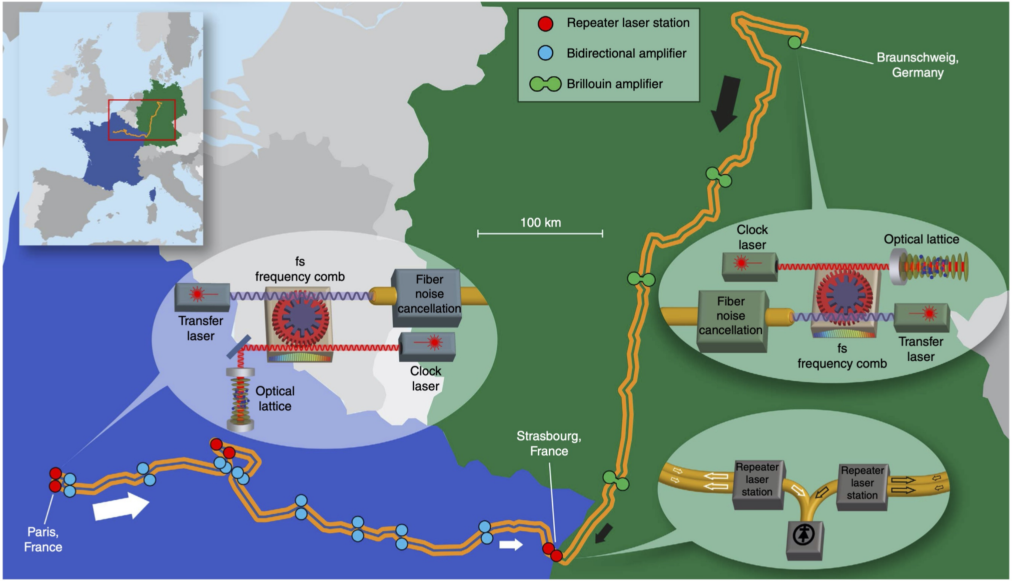


Chronometric Geodesy

Relativity offers yet another measurement device to probe grav. fields: clocks



Clocks and Fiber Networks

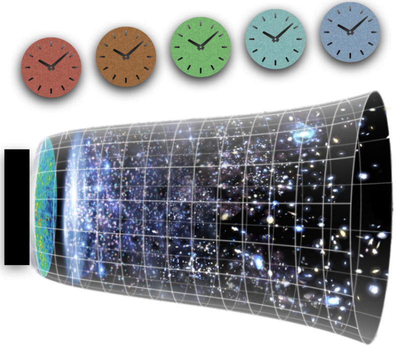


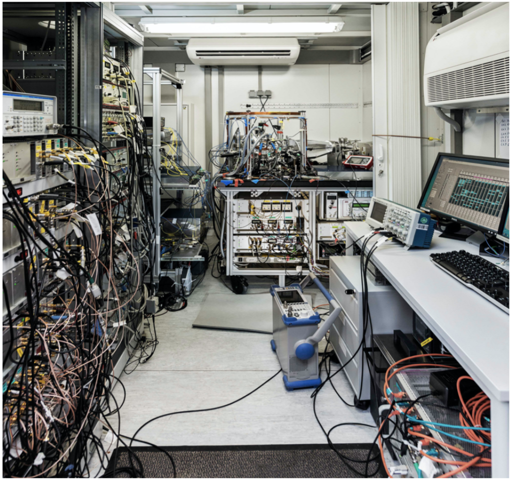
[Lisdat et al. 2016]

Optical atomic clocks

Modern clocks at the 10^{-18}-level go off one second in the age of the universe

$$10^{-18} = 0.000000000000000001$$





[PTB]

Fundamental Notions

- According to Newton, the gravitational force is

$$\vec{F}_{12} = -G\,\frac{m_1 m_2}{r^2}\,\vec{e}_r \tag{1}$$

- For the conservative force, we introduce the gravitational potential

$$\vec{a} = -\vec{\nabla} U\,, \quad \Delta U\big(\vec{X}\big) = 4\pi G \rho\big(\vec{X}\big) \tag{2}$$

- Including centrifugal effects, we define the gravity potential

$$W\big(\vec{X}\big) = U\big(\vec{X}\big) + V\big(\vec{X}\big)\,, \quad V\big(\vec{X}\big) = -\frac{1}{2}\omega^2 R^2 \sin^2\Theta \tag{3}$$

- Rigidly co-rotating observers measure gravity

$$\vec{g} = -\vec{\nabla} W\,, \quad g = ||\vec{g}|| \tag{4}$$

- In our (physical) sign convention, the grav. potential is negative since it refers to an attractive force.

Fundamental Notions II

- U can be expanded into a series of spherical harmonics,

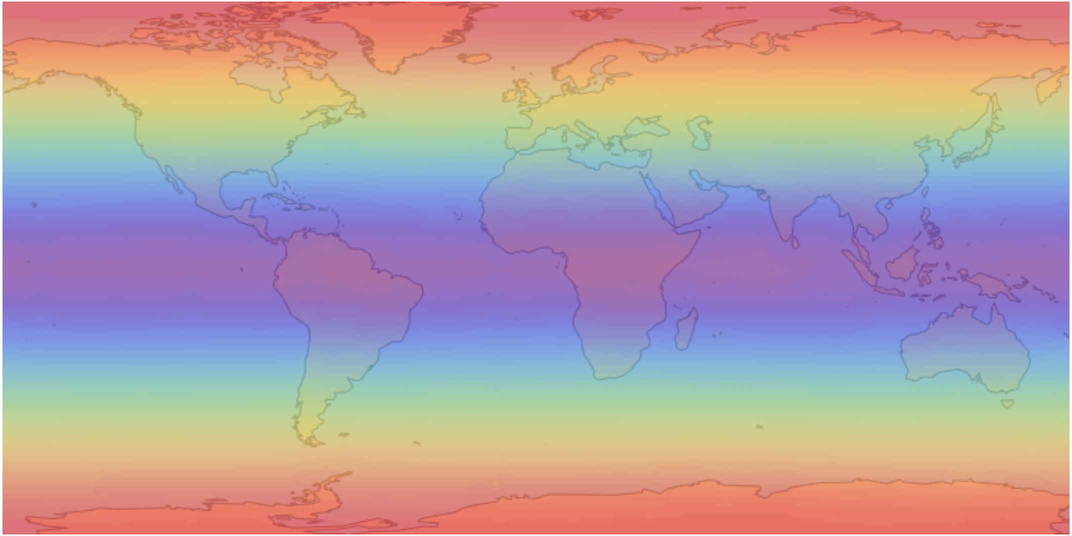
$$U = -\frac{GM}{R} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\mathsf{ref}}}{R} \right)^{l} P_l(\cos\Theta) \left[C_{lm} \cos(m\Phi) + S_{lm} \sin(m\Phi) \right] . \tag{5}$$

- Under the assumption of axisymmetry the expansion reduces to

$$U(R,\Theta) = -\frac{GM}{R} \sum_{l=0}^{\infty} \left(\frac{R_{\mathsf{ref}}}{R} \right)^{l} J_l \, P_l(\cos\Theta) \, . \tag{6}$$

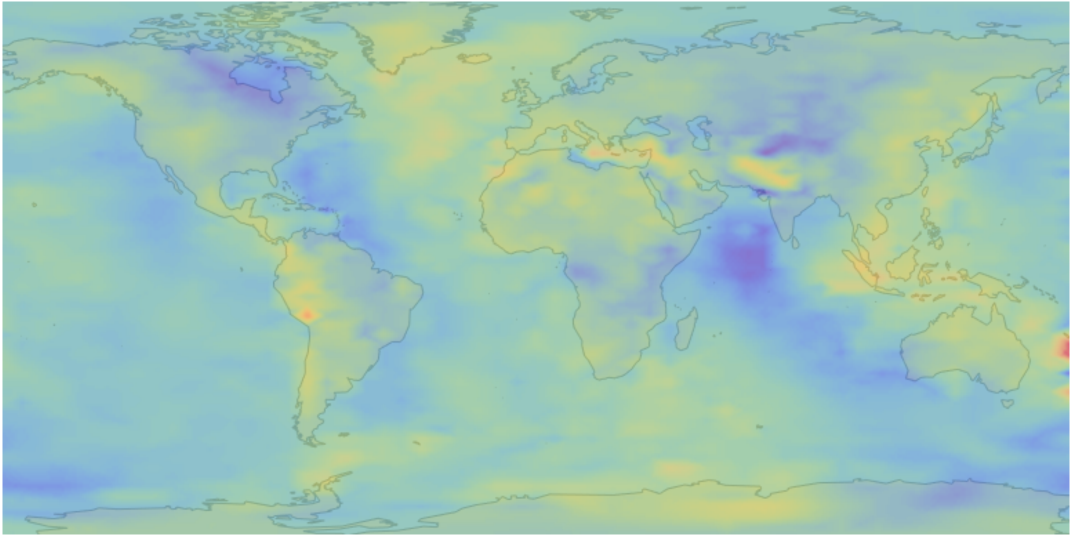
- The expansion parameters (multipole moments) C_{lm}, S_{lm}, J_l are dimensionless.

Earth Gravity Model EGM96 - Global and Local Scales



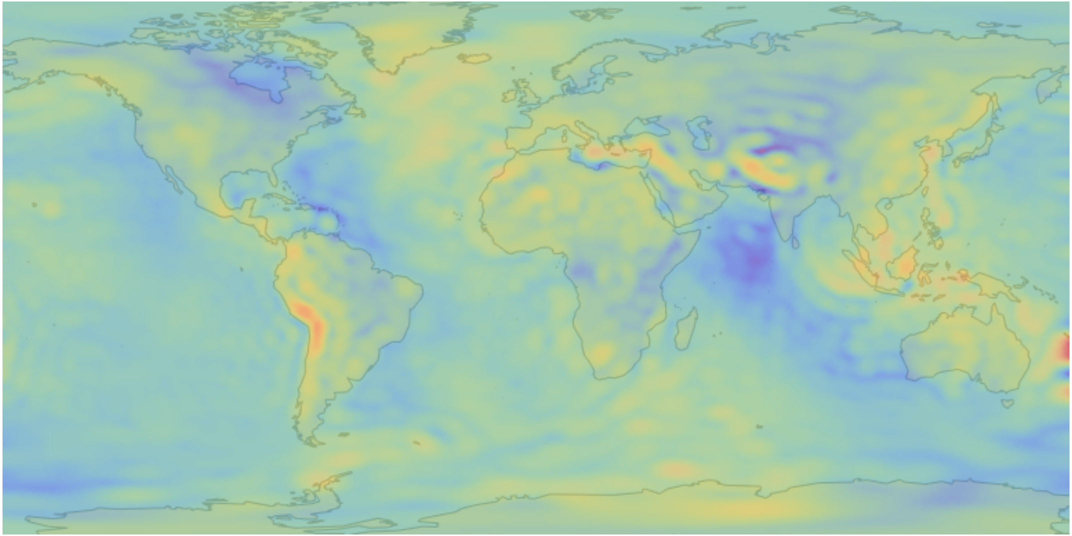
red: higher gravity, purple: lower gravity

Earth Gravity Model EGM96 - Global and Local Scales



red: higher gravity, purple: lower gravity

Earth Gravity Model EGM96 - Global and Local Scales



red: higher gravity, purple: lower gravity

Reference Surfaces

The Earth's geoid is defined by the level surface of the gravity potential W which coincides best with mean sea level, such that

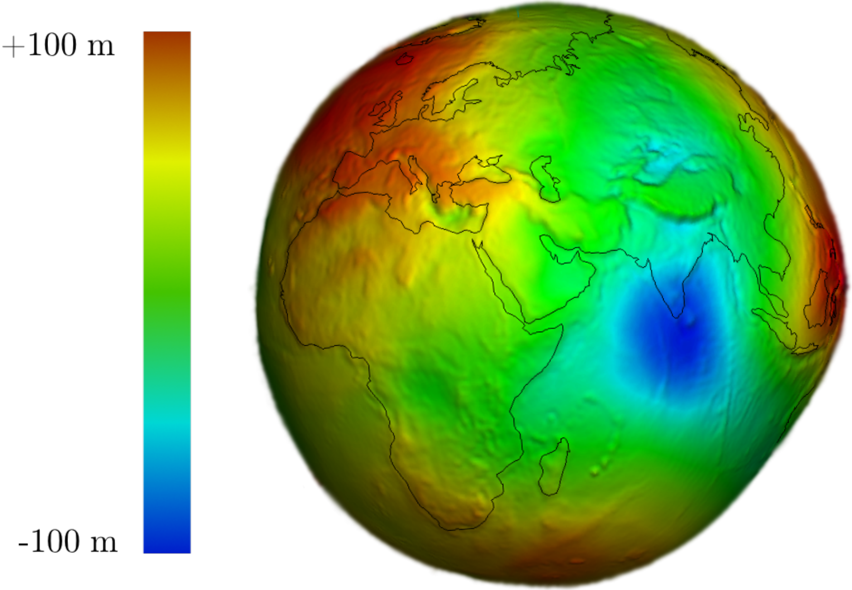
$$-W(\vec{X})\big|_{\text{geoid}} = W_0 = \text{constant}\,,$$

with a constant $W_0 = 6.263\,685\,60 \times 10^7\,\text{m}^2\,\text{s}^{-2}$, the numerical value that complies with modern conventions; see [Sanchez *et al.* 2016].

The Earth's reference ellipsoid is a bi-axial ellipsoid of revolution that is a best fit to the Earth's geoid. It is a geometrical concept and defined by any two parameters of the set $\{a, b, f, e\}$.

- The (height) differences between the geoid and reference ellipsoid are called geoid undulations.

Geoid Undulations



Time-independent Geoid in Newtonian Gravity

- For the existence of a time-independent geoid, we make the following three idealizing assumptions: [DP *et al.* 2017]

(A1): The Earth is in rigid motion.

(A2): The Earth rotates with constant angular velocity about a fixed rotation axis.

(A3): There are no external forces acting on the Earth.

- For $\vec{x} = \vec{x}_0(t) + \boldsymbol{R}(t)\,\vec{x}\,'$ with $\boldsymbol{\omega}(t) = \dot{\boldsymbol{R}}(t)\,\boldsymbol{R}(t)^{-1}$ this translates into

(A1') The velocity gradient $\nabla \otimes \vec{v}$ is antisymmetric.

(A2') The time derivative of the matrix $\boldsymbol{\omega}$ vanishes, $\dot{\boldsymbol{\omega}} = 0$.

(A3') The change of the acceleration is $\dot{\vec{a}} = \boldsymbol{\omega}\,\vec{a}$.

The Normal Gravity Field

- Let the reference ellipsoid be also an equipotential surface of an artificial gravity potential $W_N(\vec{X})$. Then, we call it a level ellipsoid [Torge, Müller 2012].

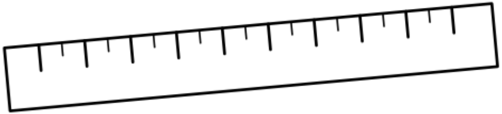
The Earth's normal gravity potential W_N is uniquely defined by (i) postulating that the reference ellipsoid is indeed a level ellipsoid of W_N and (ii) the values for the mass constant GM and the rotational velocity ω. The value of $|W_N|$ on the level ellipsoid is W_0.

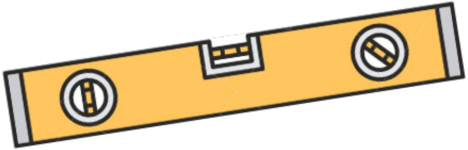
$$W_N(R,\Theta) = -\frac{GM}{R}\sum_{l=0}^{\infty} J_{2l}\left(\frac{a}{R}\right)^{2l} P_{2l}(\cos\Theta) - \frac{1}{2}\omega^2 R^2 \sin^2(\Theta) \tag{7}$$

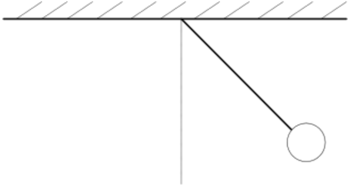
$$J_0 = 1\,, \quad J_{2l} = (-1)^l \frac{3(E/a)^{2l}}{(2l+1)(2l+3)}\left(1 - l - 5l\frac{a^2}{E^2}J_2\right)\,, \quad \forall\, l > 1 \tag{8}$$

Height Measurements and Leveling

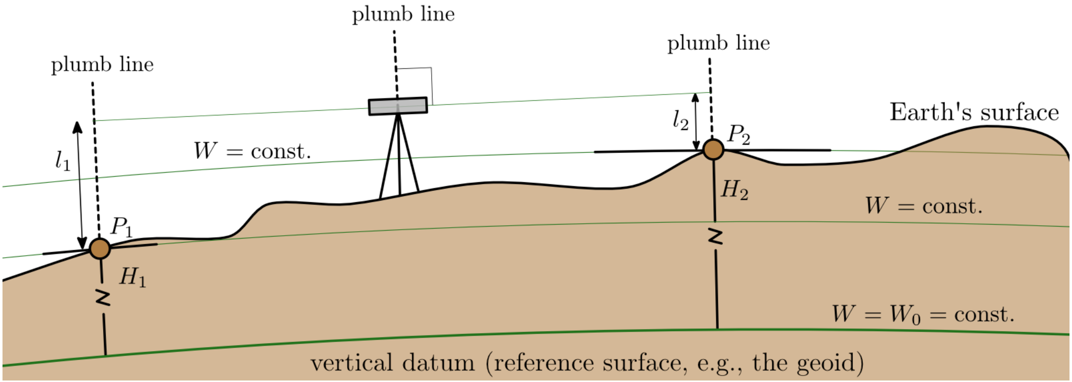
- How do you define and determine height (differences)?
- Which measurement devices do we need?







Height Measurements and Leveling



- Let $\vec{n}$ be the direction of the local plumb line, then

$$\int_{P_1}^{P_2} \mathrm{d}W = W_2 - W_1 = \int_{P_1}^{P_2} g \, \mathrm{d}n \,, \quad \oint g \, \mathrm{d}n = 0 \,, \quad \oint \mathrm{d}n \neq 0 \,. \tag{9}$$

- Hence, the sum of "height" increments along a leveling path is **not** a good height measure.

Height Measurements and Leveling

- For a point P on or above the Earth's surface, we define the geopotential number C_P by

$$\int_{P_0}^{P} g \, \mathrm{d}n = W_P - W_{P_0} = W_P + W_0 = W_0 - |W_P| =: C_P \, , \tag{10}$$

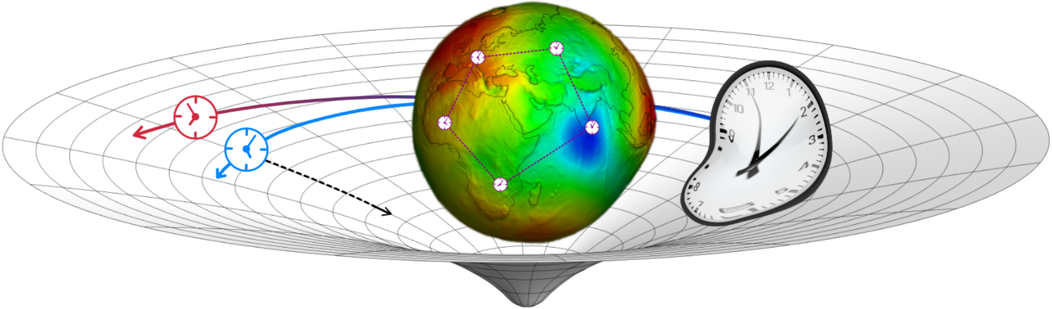
  where P_0 is a point on the geoid.

- For the orthometric height H (height above the geoid) we obtain

$$C_P = H_P \, \frac{1}{H_P} \int_0^{H_P} g \, \mathrm{d}H =: H_P \, \bar{g} \quad \Rightarrow \quad H_P = \frac{C_P}{\bar{g}} \, , \tag{11}$$

  where $\bar{g}$ is the average gravity along the plumb line from P_0 to P.
- Be aware of our (physical) sign convention!

Geodetic Concepts in Relativistic Gravity



Relativistic geodesy is deeply related to the physics of timelike Killing congruences.

Geodetic Concepts in Relativistic Gravity

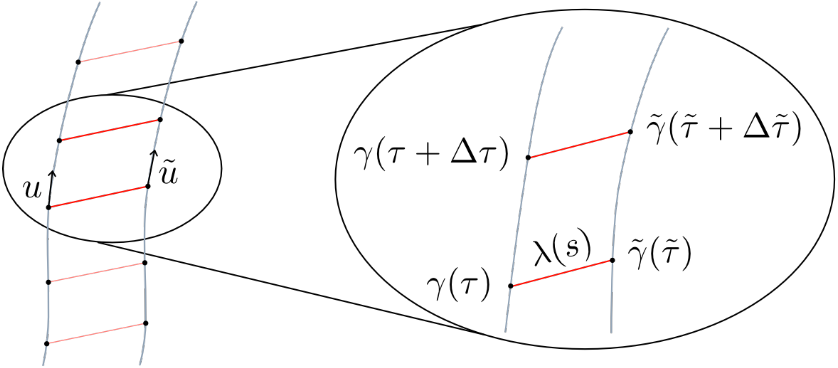
First Approach and Previous Results

"The relativistic geoid is the surface nearest to mean sea level on which precise clocks run with the same speed."

[Bjerhammar 1985, 1986]

- Needs to be clarified: "precise clocks" $\rightarrow$ standard clocks, "run with the same speed" $\rightarrow$ vanishing redshift
- Post-Newtonian approaches: [Soffel et al. 1988], [Kopeikin et al. 2015, 2016], [Shen, 2011]
- Exact approach by [Oltean et al. 2015], using quasi-local frames
- In [Soffel et al. 1988]: introduced the notions of a u-geoid and an a-geoid in the pN framework.
- Here, the results of Bjerhammar, Kopeikin, and Soffel et al. are used as inspiration and starting point to construct a general framework.

The Redshift in General Relativity



$$z + 1 := \frac{\nu}{\tilde{\nu}} = \frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta\tilde{\tau}}{\Delta\tau} = \frac{\left(g_{\mu\nu} \dfrac{d\lambda^\mu}{ds} \dfrac{d\gamma^\nu}{d\tau} \right)\Big|_{\gamma(\tau)}}{\left(g_{\rho\sigma} \dfrac{d\lambda^\rho}{ds} \dfrac{d\tilde{\gamma}^\sigma}{d\tilde{\tau}} \right)\Big|_{\tilde{\gamma}(\tilde{\tau})}} = \frac{k(u)|_{\gamma(\tau)}}{k(\tilde{u})|_{\tilde{\gamma}}(\tilde{\tau})} \tag{12}$$

[Kermack et al. 1934, Brill 1972]

The Redshift Potential

- We call a scalar function ϕ a redshift potential for an observer congruence u if for any two integral curves γ and $\tilde{\gamma}$ of u we have

$$\log(z+1) = \phi\big(\tilde{\gamma}(\tilde{\tau})\big) - \phi\big(\gamma(\tau)\big)\,. \tag{13}$$

- ϕ is a time-independent redshift potential iff [Hasse, Perlick 1988]

$$\exp(\phi)u =: \xi \tag{14}$$

  is a Killing vector field. Hence, the spacetime must be stationary.
- The general metric can then be written as

$$g = e^{2\phi(x)}\left[-(c\,\mathrm{d}t + \alpha_a(x)\mathrm{d}x^a)^2 + \alpha_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b\right]\,. \tag{15}$$

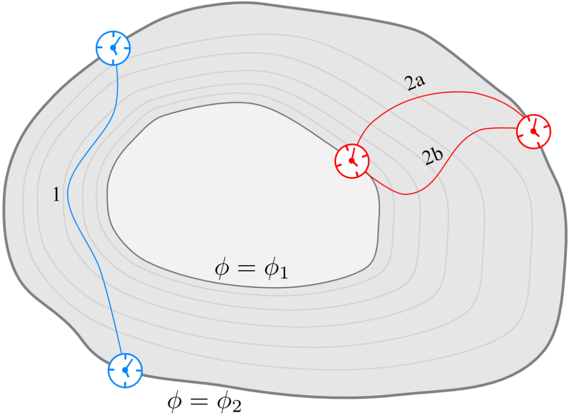
- Level surfaces of ϕ are called isochronometric and the redshift is

$$z + 1 = \frac{\nu}{\tilde{\nu}} = e^{\phi|_{\tilde{\gamma}} - \phi|_{\gamma}} =: e^{\Delta\phi}\,. \tag{16}$$

Clock Comparison via Fiber Links

- Use the optical metric [DP *et al.* 2017]

$$g = e^{2\phi(x)}\left[-n(x)^{-2}(c\,\mathrm{d}t + \alpha_a(x)\mathrm{d}x^a)^2 + \alpha_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b\right]\,, \tag{17}$$



- independent of the fiber's shape

$$z + 1 = \frac{\nu}{\tilde{\nu}} = \frac{e^{\phi}|_{\tilde{\gamma}}}{e^{\phi}|_{\gamma}}\frac{n|_{\gamma}}{n|_{\tilde{\gamma}}}\,. \tag{18}$$

Some Redshift Experiments

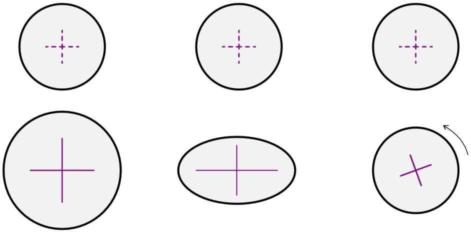
Year	Experiment	Details
1960	Pound-Rebka	22.56 m tower, $z = 2.46 \times 10^{-15}$, error ≈ 10 %
1976	GPA	Scout rocket 10000 km with H-maser, 1.4×10^{-4}
2018	RELAGAL	Using Galileo satellites 5 & 6, 2.67×10^{-5}
2020	ACES	Clocks on the ISS, 10^{-6}

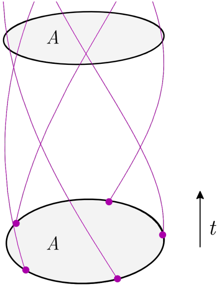
Observer Congruences

- The motion of a congruence with tangent vector field u can be decomposed into

$$D_\nu u_\mu = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu} - \frac{1}{c^2}\, u_\nu a_\mu \,, \quad P^\mu_\nu = \delta^\mu_\nu + \frac{1}{c^2}\, u^\mu u_\nu \,, \tag{19}$$

$$\omega_{\mu\nu} := P^\rho_\mu\, P^\sigma_\nu\, D_{[\sigma} u_{\rho]} \,, \quad \sigma_{\mu\nu} := P^\rho_\mu\, P^\sigma_\nu\, D_{(\sigma} u_{\rho)} - \frac{1}{3}\theta P_{\mu\nu} \,, \quad \theta := D_\mu u^\mu \,.$$





Observer Congruences II

- We can define the rotation vector (twist) [Ehlers 1961]

$$\omega^\mu := \frac{1}{2c}\eta^{\mu\nu\sigma\lambda}u_\nu\omega_{\sigma\lambda} = \frac{1}{2c}\,\eta^{\mu\nu\sigma\lambda}u_\nu\partial_\lambda u_\sigma\,. \tag{20}$$

- For a Born-rigid congruence, we have [Ehlers 1961, 1991]

$$P^\mu_\nu\dot{\omega}^\nu = 0\,, \quad P^\mu_\nu\dot{a}^\nu = \omega^\mu{}_\nu a^\nu \quad \Leftrightarrow \quad D_{[\nu}a_{\mu]} = 0\,. \tag{21}$$

- This means there exists an acceleration potential $\hat{\phi}$ such that $a_\mu = c^2\partial_\mu\hat{\phi}$
- This is true for a Born-rigid congruence iff [Salzmann, Taub 1954] $e^{\hat{\phi}}u = \xi$

$$\Rightarrow \quad \hat{\phi} \equiv \phi \quad \text{and} \quad a = c^2\mathrm{d}\phi\,. \tag{22}$$

- Generalizes [Soffel et al. 1988]

Relativistic Gravity Potential

- We now define a new artificial relativistic gravity potential U^* by [DP *et al.* 2019, 2020]

$$e^\phi =: 1 + \frac{U^*}{c^2} \quad \Leftrightarrow \quad U^* = c^2\left(e^\phi - 1\right) = c^2\left(\sqrt{-g_{00}} - 1\right) . \tag{23}$$

- We have $[U^*] = [c^2] = \mathrm{m}^2/\mathrm{s}^2$ and the weak-field limit is

$$U^* \xrightarrow[c\to\infty]{} W\,, \quad U^*_{\mathrm{pN}} = W + \frac{1}{2}\frac{U^2}{c^2} \quad \text{[Soffel et al. 1988]}\,. \tag{24}$$

- Then, redshift and acceleration are given by

$$1 + z = \frac{\nu_1}{\nu_2} = e^{\phi_2 - \phi_1} = \frac{1 + U_2^*/c^2}{1 + U_1^*/c^2} = 1 + \frac{U_2^* - U_1^*}{c^2} + \mathcal{O}(c^4)\,, \tag{25a}$$

$$a = -c^2 \mathrm{d}\phi = -c^2 \frac{\partial\phi}{\partial U^*}\mathrm{d}U^* = \frac{-\mathrm{d}U^*}{1 + U^*/c^2}\,. \tag{25b}$$

The Relativistic Geoid

For a spacetime equipped with a metric in the form below and a congruence of observers on t-lines, *the relativistic geoid* is a particular level surface of the relativistic gravity potential U^* such that $U^*\big|_{\text{geoid}} = U_0^* = \text{const.}$ [DP *et al.* 2020].

For the value U_0^* there are multiple choices, e.g.,

(i) Choose the Newtonian value such that $U_0^* = -W_0$

(ii) Define a master clock which is, by definition, situated on the geoid and singles out one isochronometric surface

$$g = \left(1 + \frac{U^*}{c^2}\right)^2 \left[-(c\,\mathrm{d}t + \alpha_a(x)\mathrm{d}x^a)^2 + \alpha_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b\right]$$

IAU Resolutions and U_0^*

- We start with the metric in co-rotating coordinates such that observers move on integral curves of ∂_t,

$$g = e^{2\phi(x)} \left[-(c\,\mathrm{d}t + \alpha_a(x)\mathrm{d}x^a)^2 + \alpha_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \right] . \tag{26}$$

- The relation between coordinate time t and proper time on the geoid τ_g is [DP 2019]

$$\mathrm{d}\tau_g = e^{\phi_0(x)}\mathrm{d}t = \left(1 + \frac{U_0^*}{c^2}\right)\mathrm{d}t =: \left(1 - L_g^*\right)\mathrm{d}t \quad \Rightarrow U_0^* = -L_g^* c^2 . \tag{27}$$

- L_g^*, which fixes the relation between geocentric coordinate time TCG and time on the geoid (TAI), is adopted as a defining constant by IAU resolution.
- Since for 1pN: $U^* \approx W$ this also fixes a value for W_0, which is then a derived constant.
- According to resolution B1.9 (2000): $L_g = 6.969290134\ 10^{-10}$ [IAU 2000].

Parametrized post-Newtonian Spacetime

- In harmonic non-rotating coordinates, we have [Will 2014]

$$g_{00}(\mathbf{x}) = -\left(1 + \frac{2U(x,y,z)}{c^2} + \frac{2\beta U(x,y,z)^2}{c^4}\right) + \mathcal{O}(c^{-6})\,, \tag{28a}$$

$$g_{0i}(\mathbf{x}) = -\frac{2(\gamma+1)\,|U^i(x,y,z)|}{c^3} + \mathcal{O}(c^{-5})\,, \tag{28b}$$

$$g_{ij}(\mathbf{x}) = \delta_{ij}\left(1 - \frac{2\gamma U(x,y,z)}{c^2}\right) + \mathcal{O}(c^{-4})\,. \tag{28c}$$

- with $\phi \to \phi' = \phi - \omega t$ we transform into a rotating reference system and obtain

$$U^*_{\text{ppN}}\Big|_{\text{geoid}} = W + \frac{U^2(\beta - 1/2)}{c^2}\Bigg|_{\text{geoid}} = U^*_0 = \text{const.} \tag{29}$$

$$1 + z = \frac{\nu_1}{\nu_2} = 1 + \frac{W_2 - W_1}{c^2} + \mathcal{O}(c^{-4}) \tag{30}$$

Asympt. & elementary flat Weyl solutions

- In spheroidal coordinates (x, y), the metric is [Quevedo 1989]

$$g = -e^{2\psi}c^2dt^2 + m^2e^{-2\psi}(x^2-1)(1-y^2)d\varphi^2 \tag{31}$$
$$+ m^2e^{-2\psi}e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right), \quad \text{where} \quad \psi = \sum_{l=0}^{\infty}(-1)^{l+1}q_l\,Q_l(x)\,P_l(y)$$

- Quevedo moments q_l in the Newtonian limit [Quevedo 1989, Ehlers 1997]

$$q_l = (-1)^l\frac{(2l+1)!!}{l!\,m^l}\,R^l_{\text{ref}}J_l\,. \tag{32}$$

- Redshift potential for rigidly co-rotating observers [DP *et al.* 2017]

$$e^{\phi(x,y)} = \sqrt{e^{2\psi(x,y)} - \frac{\omega^2}{c^2}m^2(x^2-1)(1-y^2)e^{-2\psi(x,y)}}\,. \tag{33}$$

Schwarzschild-Droste, Erez-Rosen, & Kerr spacetime

$$\frac{U^*_{\text{Schwarzschild}}}{c^2} = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{\omega^2}{c^2} r^2 \sin^2\vartheta} - 1\,, \tag{34}$$

$$\frac{U^*_{\text{ER}}}{c^2} = \sqrt{e^{2\psi_{\text{ER}}(x,y)} - \frac{\omega^2}{c^2}(GM/c)^2(x^2-1)(1-y^2)e^{-2\psi_{\text{ER}(x,y)}}} - 1 \tag{35}$$

with $\psi_{\text{ER}(x,y)} = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + q_2 \frac{(3y^2-1)}{2}\left(\frac{(3x^2-1)}{4}\log\left(\frac{x-1}{x+1}\right) + \frac{3}{2}x\right)\,,$

$$\frac{U^*_{\text{Kerr}}}{c^2} = \sqrt{1 - \frac{2mr}{\rho(r,\vartheta)^2} + 4\,\frac{\omega}{c}\,\frac{amr\sin^2\vartheta}{\rho(r,\vartheta)^2} - \frac{\omega^2}{c^2}\,\sin^2\vartheta\left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\rho(r,\vartheta)^2}\right)} - 1\,. \tag{36}$$

See [DP et al. 2020] for details

Relativistic Normal Gravity I

- For the Newtonian normal gravity field W_{N} the moments are

$$J_0 = 1\,, \quad J_{2l} = f(J_2, E, a, l)\,, \quad \forall\, l > 1\,.$$

- Construct a general relativistic spacetime, which is a vacuum solution, asymptotically + elementary flat, static, axisymmetric, and yields W_{N} in the Newtonian limit $\to$ Weyl solution [Weyl 1916]. For details see [DP 2019].
- In spheroidal coordinates (x, y), the metric is [Quevedo 1989]

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -e^{2\psi}c^2dt^2 + m^2e^{-2\psi}(x^2-1)(1-y^2)d\varphi^2 \\ &\quad + m^2e^{-2\psi}e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)\,, \end{aligned} \tag{37}$$

$$\text{where} \quad \psi = \sum_{l=0}^{\infty}(-1)^{l+1}q_l\, Q_l(x)\, P_l(y) \tag{38}$$

Relativistic Normal Gravity II

- Now, we use a Weyl spacetime for which we determine the Quevedo moments in the Newtonian limit [Quevedo 1989, Ehlers 1997]

$$q_l = (-1)^l \frac{(2l+1)!!}{l!\, m^l} R_{\text{ref}}^l J_l \,. \tag{39}$$

- With the J_{2l} above and only even terms in the expansion for ψ:

$$\begin{aligned}
\psi_N &= -\sum_{k=0}^{\infty} \frac{(4k+1)!!}{(2k)!} \left(\frac{R_{\text{ref}}}{m}\right)^{2k} J_{2k}\, Q_{2k}(x)\, P_{2k}(y) \\
&= -\sum_{k=0}^{\infty} \left[\frac{(4k+1)!!}{(2k)!} \left(\frac{R_{\text{ref}}}{m}\right)^{2k} J_{2k}\, P_{2k}(y) \right. \\
&\quad \left. \times \left(\log\left(\frac{x+1}{x-1}\right) P_{2k}(x) - 2\sum_{i=0}^{k-1} \frac{4k-4i-1}{(2k-i)(2i+1)} P_{2k-2i-1}(x) \right) \right] , \\
&= -Q_0(x) - \frac{15}{2}\left(\frac{R_{\text{ref}}}{m}\right)^2 J_2 Q_2(x) P_2(y) - \sum_{k=2}^{\infty} \ldots
\end{aligned} \tag{40}$$

Relativistic (Chronometric) Height Definition

- The orthometric height is $H_P = C_P/\bar{g}$ and $C_P = W_P + W_0$ can be determined by comparing clocks on the geoid and at P:

$$z + 1 = \frac{\nu_0}{\nu_P} = 1 + \frac{W_P + W_0}{c^2} + \mathcal{O}\left(c^{-4}\right) \quad \Rightarrow H = \frac{c^2 z}{\bar{g}} \,. \tag{41}$$

- However, it is a mixture of Newtonian and relativistic concepts. **Better**: define redshift potential numbers [DP 2019]

$$C_P^* := c^2\left(e^{\phi_P} - e^{\phi_0}\right) = U_P^* - U_0^* = c^2 e^{\phi_0} z \,, \tag{42}$$

- where $e^{\phi_0} = 1 + U_0^*/c^2$. Then we define the chronometric height

$$H_P^* = \frac{C_P^*}{\bar{a}} = \frac{c^2 e^{\phi_0} z}{\bar{a}} \,. \tag{43}$$

- Here, $\bar{a}$ is the average of $a = c^2 \mathrm{d}\phi$ along the normal w.r.t. isochronometric surfaces between the geoid and P.

Applications

Let us consider the following two applications:

- The difference between the relativistic and Newtonian geoids for the choice $|U_0^*| = W_0$. The relativistic geoid is isometrically embedded into $\mathbb{R}^3$ to enable the comparison. The embedding is necessary to overcome any coordinate ambiguities. See [DP *et al.* 2020] for details.
- Isochronometric surfaces in strong gravity regimes: a Kerr black hole with $a/m = 0.99$ close to the ergosphere. Results for relativistic geodesy can also be applied to the strong gravity regime.

Application I

Difference of the relativistic and Newtonian geoid after isometrically embedding into Euclidean space $\mathbb{R}^3$

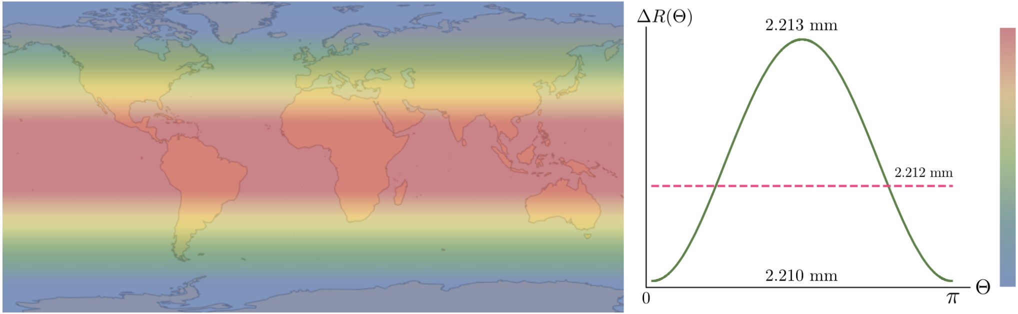
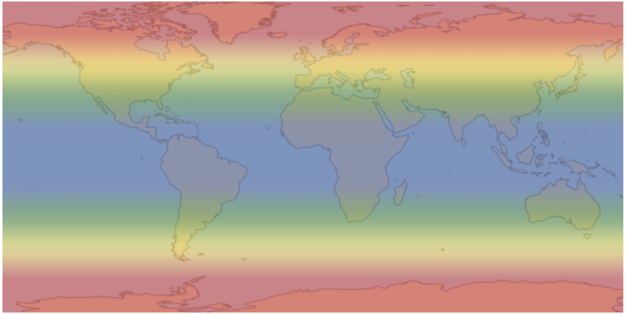


Figure from [DP *et al.* 2020]

Application I

Difference of the relativistic and Newtonian geoid **without** embedding



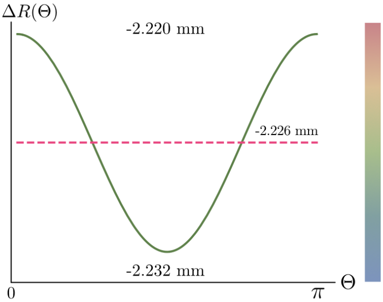
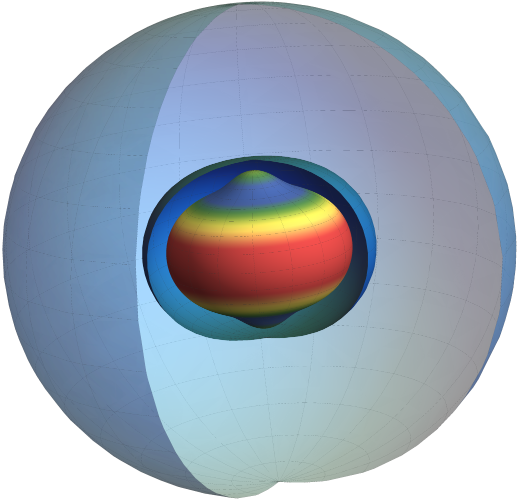


Figure from [DP *et al.* 2020]

Application II



Isochronometric surfaces: $a/m = 0.99$ at $r_0/m = 2.2, 3, 7$

Summary and Outlook

Summary

- It was shown how Newtonian concepts in conventional geodesy can be generalized and lifted to the framework of General Relativity.
- The leading order difference between the conventional geoid an its relativistic generalization is about $2\,\mathrm{mm}$.

Outlook

- Not all information is encoded in the geoid surface.
- If the vacuum field equation is fulfilled, $\exists$ a potential for the twist of the congruence

$$w_\mu = \partial_\mu \psi, \quad \omega^\mu = \eta^{\mu\nu\sigma\lambda} \xi_\nu \partial_\lambda \xi_\sigma \tag{44}$$

- The gradient of the twist potential causes the Sagnac effect $\Delta t = \int_S \epsilon_{\mu\nu\rho\sigma} u^\nu \omega^\mu \mathrm{d}S^{\rho\sigma}$
- Redshift potential, twist potential, and spatial metric are needed, see [Bäckdahl 2006]
- Framework for relativistic Geodesy $\to$ Physics of timelike Killing congruences; norm and twist of Killing vector fields.

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