Introduction into Dynamic Mode Decomposition (DMD)

time series $x_j$

trajectory-wise picture:

\[ \dot{x} = f(x), \quad x(0) = x_0 \]

Koopman operator: observable $\tilde{\phi}(x)$

\[ X \tilde{\phi}(x) = \tilde{\phi}(x(t)) \]

$X_t$: Koopman operator

Dynamic Mode Decomposition

\[ X = \left( \tilde{\phi}(x_t), \tilde{\phi}(x_{t+s}), ..., \tilde{\phi}(x_{t+\infty}) \right) \]

\[ \dot{X} = \left( \tilde{\phi}(x_{t+s}), \tilde{\phi}(x_{t+2s}), ..., \tilde{\phi}(x_{t+\infty}) \right) \]

\[ \dot{X} = K^{\infty} X \rightarrow K = XX^T \]

Reconstruction of dynamics

\[ x(t_{k+1}) = \sum_{s=1}^{r} \phi_{js} e^{\omega_s k \Delta t} b_j + \text{c.c.} \]

\[ K \phi_j = \lambda_j \phi_j \]

\[ \omega_s = \ln \lambda / \Delta t \]

- good for short times ("linear")
- needs good truncation $r$
- $St$ needs to be adapted to characteristic timescale

$K^{\infty}$ is a good approximation of the Koopman operator $X_{\infty}$ if:

i) data sufficiently sparse
ii) data sufficiently rich (eigenfunctions of $X$ lie in span of $\{\tilde{\phi}\}$)

ii) is hardly ever satisfied and also not a priori verifiable
Main Menage:

Fast transitory dynamics

Transitions typically occur on timescales faster than the timescale of the equilibrium dynamics. A sampling St chosen to compute K^st may not be sufficient to resolve the transient dynamics and hence DMD-reconstruction may be bad.

(Gottwald & Gugole, J Stat Phys. 2019)

tested for artificial data from the Kuramoto-Sivashinsky equation, and for NCEP reanalysis data.
Reconstruction error: \[ \mathcal{E}(t_k, r) = \frac{1}{m} \sum_{l=0}^{m-1} \| x(t_{k+l}) - \Phi \exp(\Omega l \Delta t) b_k - c.c \| \]

NCEP reanalysis data: Northern Hemisphere

Reanalysis

DMD reconstruction

large-scale regime change in the 1970s

transition from a predominantly negative NAO phase to a predominantly positive NAO phase
Reconstruction error:
\[ E(t_k, r) = \frac{1}{m} \sum_{l=0}^{m-1} \| x(t_{k+l}) - \Phi \exp(\Omega l \Delta t) b_k - c.c \| \]

NCEP reanalysis data: Southern Hemisphere

Reanalysis

DMD reconstruction

large-scale regime change in the 1970s

Significant decrease of frequency of blocking events around the mid 1970s and more zonal flow