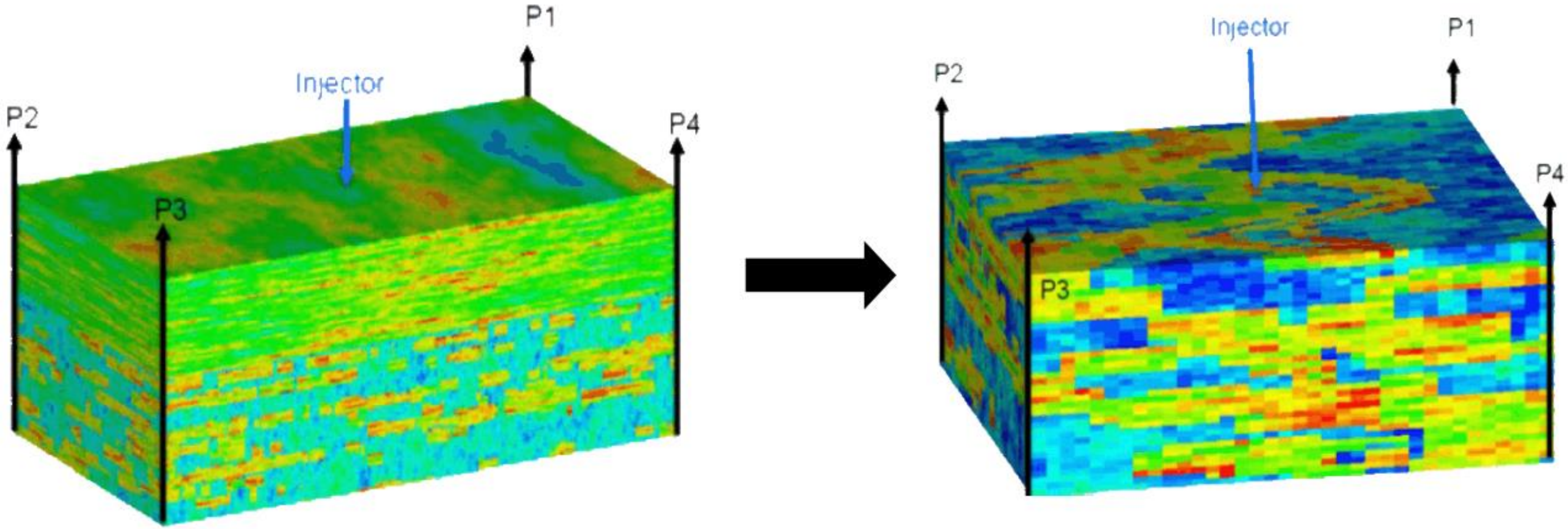
Efficient Analytical Upscaling of Conductivity Tensor for Three-dimensional Heterogeneous Anisotropic Formations

Qinzhuo Liao, Gang Lei, Shirish Patil

Department of Petroleum Engineering, CPG, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

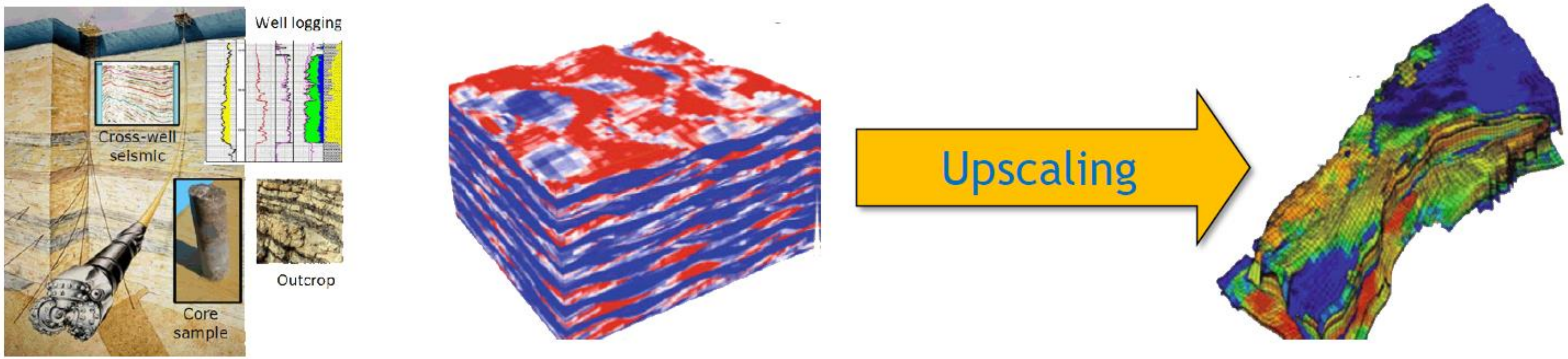
Introduction

- Model parameters: φ, k, k_r, p_c
- Upscaling: fine-scale (heterogeneous) → coarse-scale (homogeneous)



Introduction

- Reservoir development
  - ➢Data (core, well log, seismic, outcrops)
  - ➢Geophysical model (lithology, geostatistics): fine-scale $O(10^8)$ cells
  - ➢Reservoir model (black-oil, compositional): coarse-scale $O(10^6)$ cells

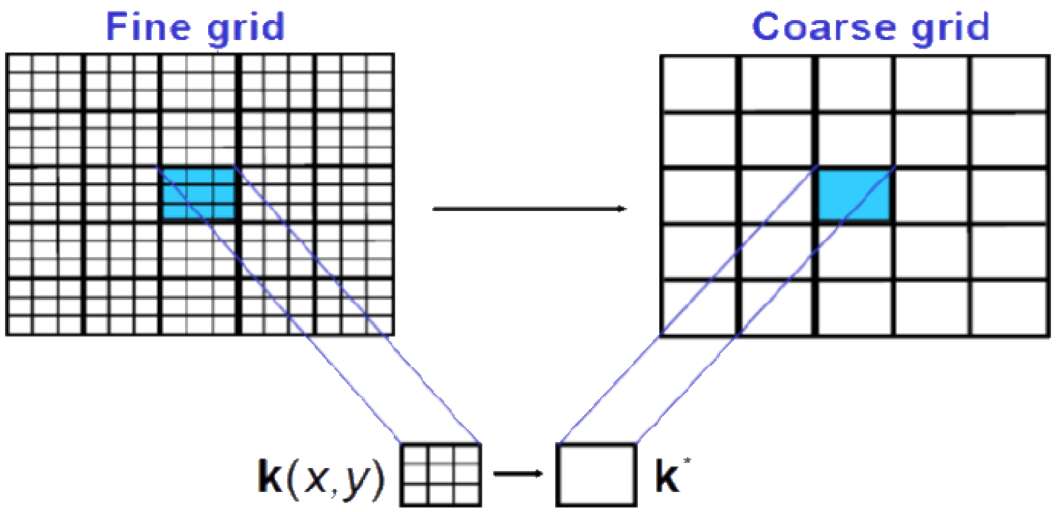


Single-phase flow upscaling

- Governing equation
  - ➢Mass conservation
  - ➢Darcy's law

$$\nabla \cdot \mathbf{v} = q$$

$$\mathbf{v} = -\frac{\mathbf{K}}{\mu} \nabla p$$



- Parameter: absolute permeability *K*
- Objective: same pressure *p* and velocity/flux ***v***

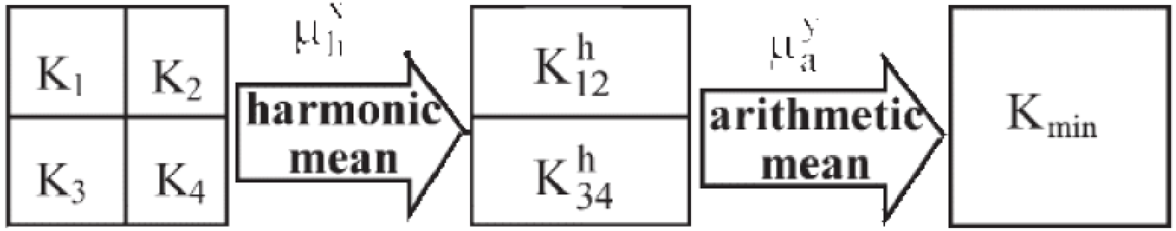
Analytical method

- Simple and fast
- Isotropic
  - Arithmetic mean $$K_a = \frac{1}{N}\sum_{i=1}^{N} K_i$$
  - Harmonic mean $$K_h = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i}\right)^{-1}$$
  - Geometric mean $$K_g = \left(\prod_{i=1}^{N} K_i\right)^{1/N}$$
  - Power mean $$K_p = \left(\frac{1}{N}\prod_{i=1}^{N} K_i^p\right)^{1/p}$$

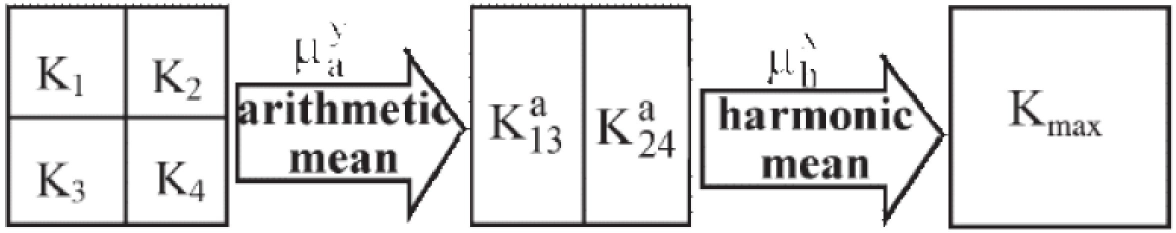
$$\mathrm{p} = 1,\ arithmetic,\ \max$$
$$\mathrm{p} = -1,\ harmonic,\ \min$$
$$\mathrm{p} \to 0,\ geometric$$

Analytical method

- Simple and fast
- Anisotropic
  - ➢ Harmonic-arithmetic mean



  - ➢ Arithmetic-harmonic mean



$$Kx_{\max} = \mu_h^x(\mu_a^y(\mu_a^z)) = \mu_h^x(\mu_a^z(\mu_a^y)) = \left[\frac{1}{n_x} \sum_{i=1}^{n_x} \left(\frac{1}{n_y n_z} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} Kx_{i,j,k} \right)^{-1} \right]^{-1}$$

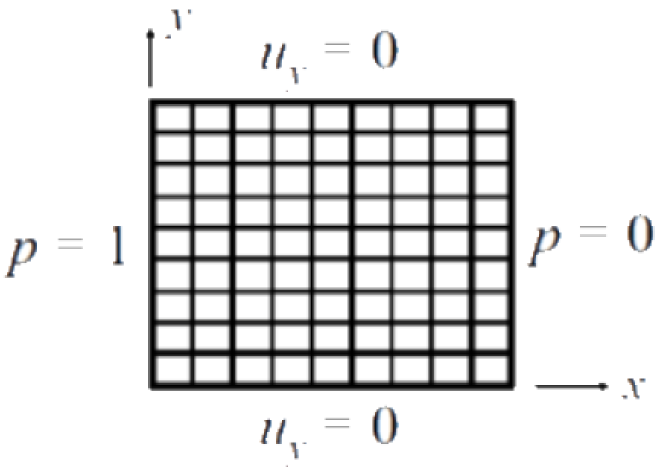
$$K_b = K_{max}^{\alpha} K_{min}^{1-\alpha}$$

$$Kx_{\min} = \mu_a^y(\mu_a^z(\mu_h^x)) = \mu_a^z(\mu_a^y(\mu_h^x)) = \frac{1}{n_y n_z} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} \left(\frac{1}{n_x} \sum_{i=1}^{n_x} Kx_{i,j,k}^{-1} \right)^{-1}$$

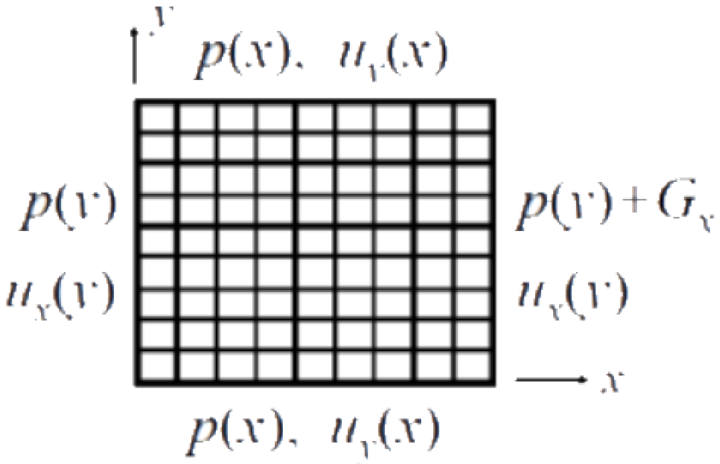
Numerical method

- Solve the flow equation and calculate permeability
  - ➢ Pressure-no flow boundaries
  - ➢ Periodic boundaries

Pressure - no flow bcs



Periodic bcs



Idea

- Optimal coefficient α such that $K_{analytical} = K_{numerical}$

$$K_b = {K_{max}}^{\alpha} {K_{min}}^{1-\alpha}$$

- Assume
  - ➢Permeability: Gaussian random field
  - ➢Covariance

$$\text{Matern}: C_Y(\mathbf{x}_1, \mathbf{x}_2) = \sigma_Y^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{2\nu} d\right)^{\nu} K_{\nu}\left(\sqrt{2\nu} d\right)$$

$$\text{Spherical: } C_Y(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} \sigma_Y^2 \left(1 - \frac{3d}{2} + \frac{d^3}{2}\right), d \le 1 \\ 0, d > 1 \end{cases}$$

$$d(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{\left(\frac{x_1' - x_2'}{\eta_x}\right)^2 + \left(\frac{y_1' - y_2'}{\eta_y}\right)^2 + \left(\frac{z_1' - z_2'}{\eta_z}\right)^2}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution

$$E[Kxx_{FD}] = E[Kx_{\max}]^{\alpha} E[Kx_{\min}]^{1-\alpha}$$

$$E[Kyx_{FD}] = \kappa_g \sigma_Y^2 \gamma_{yx}$$

$$\alpha_{xx} = \frac{\ln E[Kxx_{FD}] - \ln E[Kx_{\min}]}{\ln E[Kx_{\max}] - \ln E[Kx_{\min}]} = \frac{\gamma_{xx} - \left(-\frac{1}{2} + \hat{\rho}_x\right)}{\left(\frac{1}{2} - \hat{\rho}_{yz} + \hat{\rho}_{xyz}\right) - \left(-\frac{1}{2} + \hat{\rho}_x\right)} = \frac{\gamma_{xx} + \frac{1}{2} - \hat{\rho}_x}{1 - \hat{\rho}_x - \hat{\rho}_{yz} + \hat{\rho}_{xyz}}$$

$$\gamma_{yx} = \frac{A_y}{1 + A_y + A_z}\hat{\rho}_{xyz} - \sum_{k_1=1}^{n_x}\sum_{k_2=1}^{n_y}\sum_{k_3=1}^{n_z} \frac{A_y \sin\left(\frac{\pi k_1}{n_x}\right)\sin\left(\frac{\pi k_2}{n_y}\right)\cos\left(\frac{\pi k_1}{n_x}\right)\cos\left(\frac{\pi k_2}{n_y}\right)}{\sin^2\left(\frac{\pi k_1}{n_x}\right) + A_y \sin^2\left(\frac{\pi k_2}{n_y}\right) + A_z \sin^2\left(\frac{\pi k_3}{n_z}\right)}\tilde{\rho}_{k_1,k_2,k_3}$$

$$\gamma_{xx} = \frac{(n_x - 1)\rho(1,0,0) + \rho(n_x - 1,0,0)}{2n_x} + \frac{1}{1 + A_y + A_z}\hat{\rho}_{xyz} - \sum_{k_1=1}^{n_x}\sum_{k_2=1}^{n_y}\sum_{k_3=1}^{n_z} \frac{\sin^2\left(\frac{\pi k_1}{n_x}\right)\cos^2\left(\frac{\pi k_1}{n_x}\right)}{\sin^2\left(\frac{\pi k_1}{n_x}\right) + A_y \sin^2\left(\frac{\pi k_2}{n_y}\right) + A_z \sin^2\left(\frac{\pi k_3}{n_z}\right)}\tilde{\rho}_{k_1,k_2,k_3}$$

$$\tilde{\rho}_{k_1,k_2,k_3} = \frac{\sigma^2}{n_x n_y n_z}\sum_{x=-(n_x-1)}^{n_x-1}\sum_{y=-(n_y-1)}^{n_y-1}\sum_{z=-(n_z-1)}^{n_z-1} \exp\left[-\mathbf{i}2\pi\left(\frac{k_1 x}{n_x} + \frac{k_2 y}{n_y} + \frac{k_3 z}{n_z}\right)\right]\left(1 - \frac{|x|}{n_x}\right)\left(1 - \frac{|y|}{n_y}\right)\left(1 - \frac{|z|}{n_z}\right)\rho(x,y,z)$$

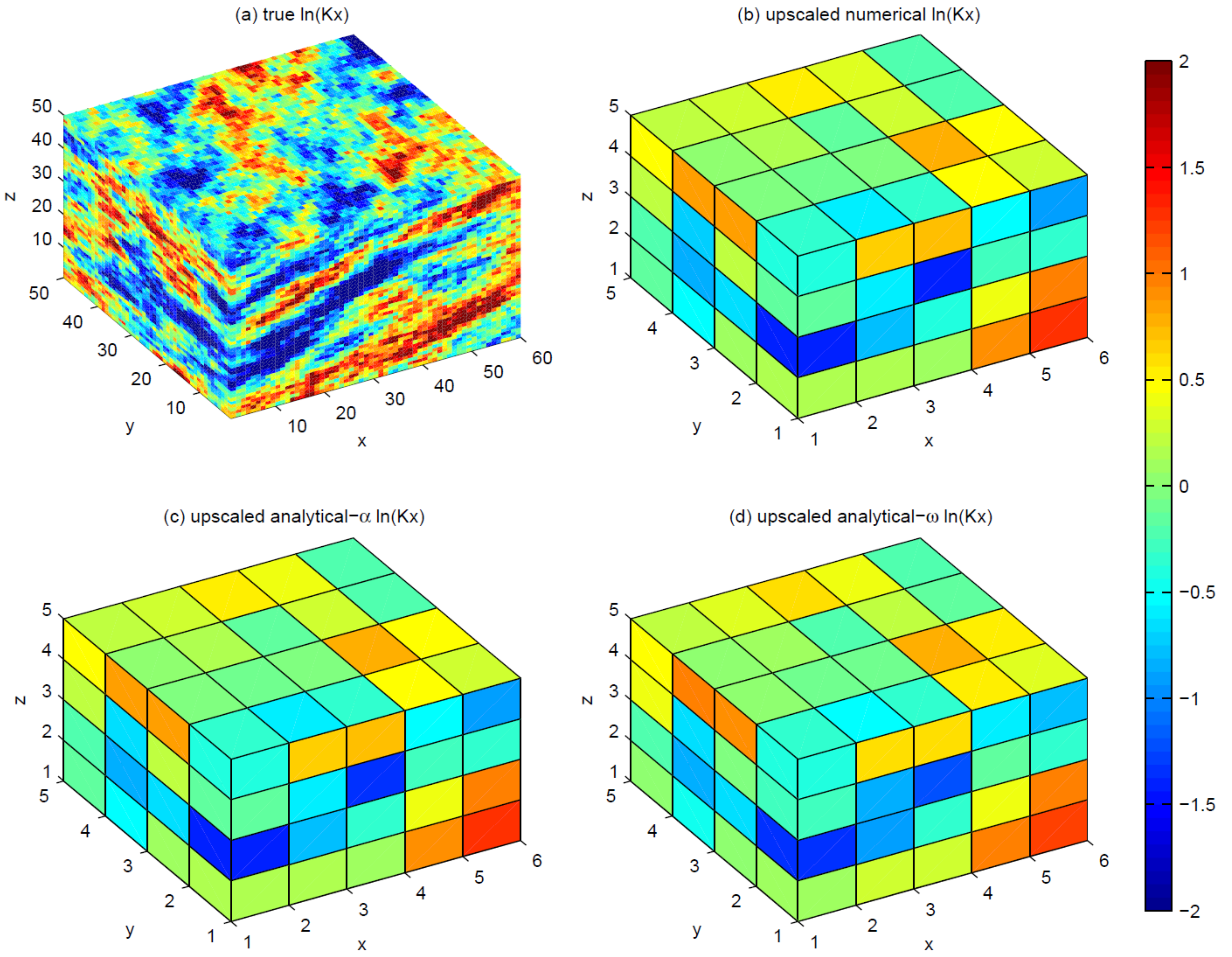
$$\hat{\rho}_{xyz} = \frac{1}{n_x^2 n_y^2 n_z^2}\sum_{i=1}^{n_x}\sum_{j=1}^{n_y}\sum_{k=1}^{n_z}\sum_{i'=1}^{n_x}\sum_{j'=1}^{n_y}\sum_{k'=1}^{n_z}\rho(i-i', j-j', k-k')$$

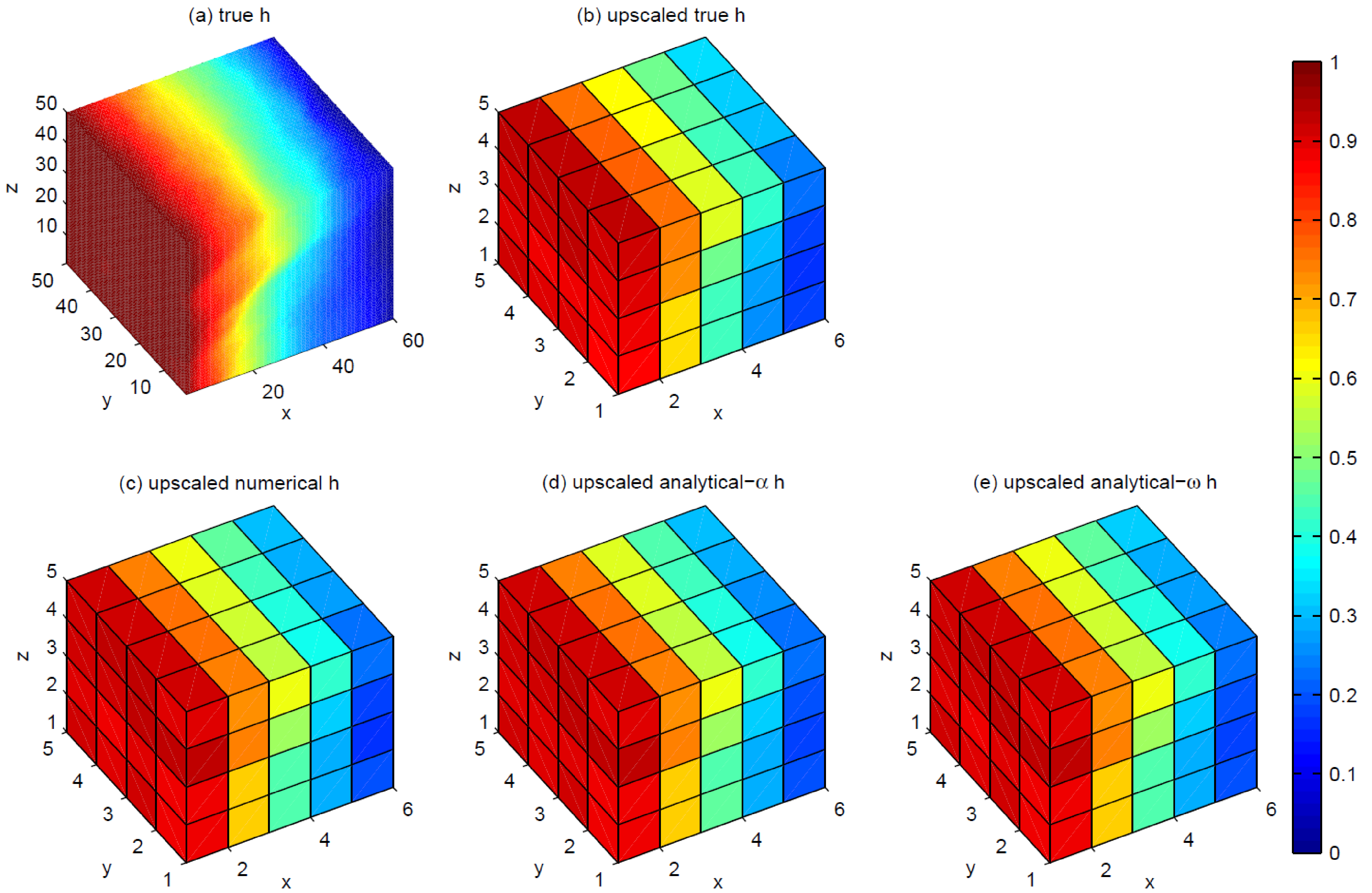
$$= \frac{1}{n_x^2 n_y^2 n_z^2}\sum_{i=1}^{n_x}\sum_{j=1}^{n_y}\sum_{k=1}^{n_z}\sum_{x=i-n_x}^{i-1}\sum_{y=j-n_y}^{j-1}\sum_{z=k-n_z}^{k-1}\rho(x,y,z)$$

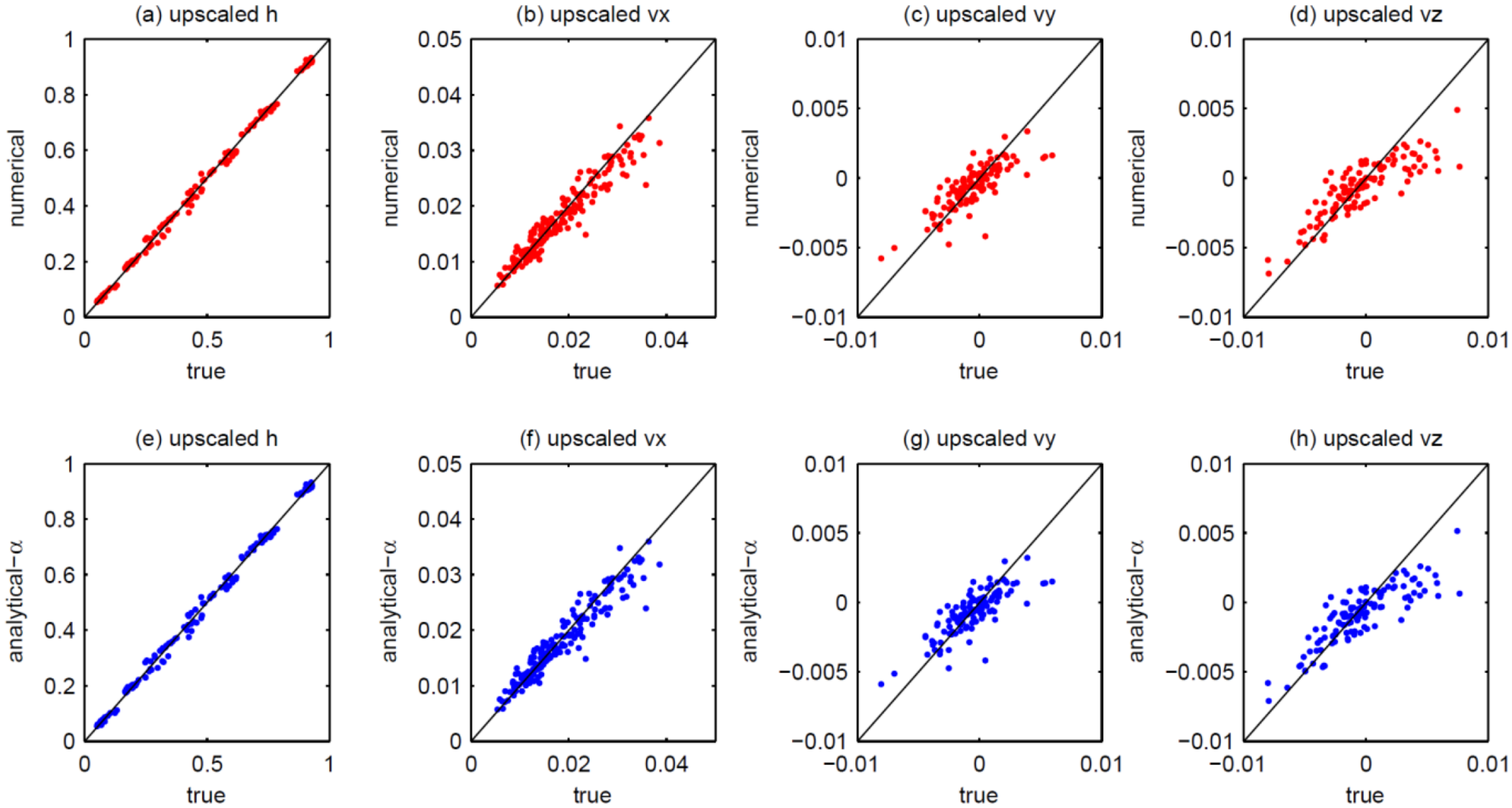
$$= \frac{1}{n_x n_y n_z}\sum_{x=-(n_x-1)}^{n_x-1}\sum_{y=-(n_y-1)}^{n_y-1}\sum_{z=-(n_z-1)}^{n_z-1}\left(1 - \frac{|x|}{n_x}\right)\left(1 - \frac{|y|}{n_y}\right)\left(1 - \frac{|z|}{n_z}\right)\rho(x,y,z),$$

with $\rho(x_1 - x_2, y_1 - y_2, z_1 - z_2) = C_Y(x_1, y_1, z_1; x_2, y_2, z_2) = C_Y(\mathbf{x}_1, \mathbf{x}_2)$

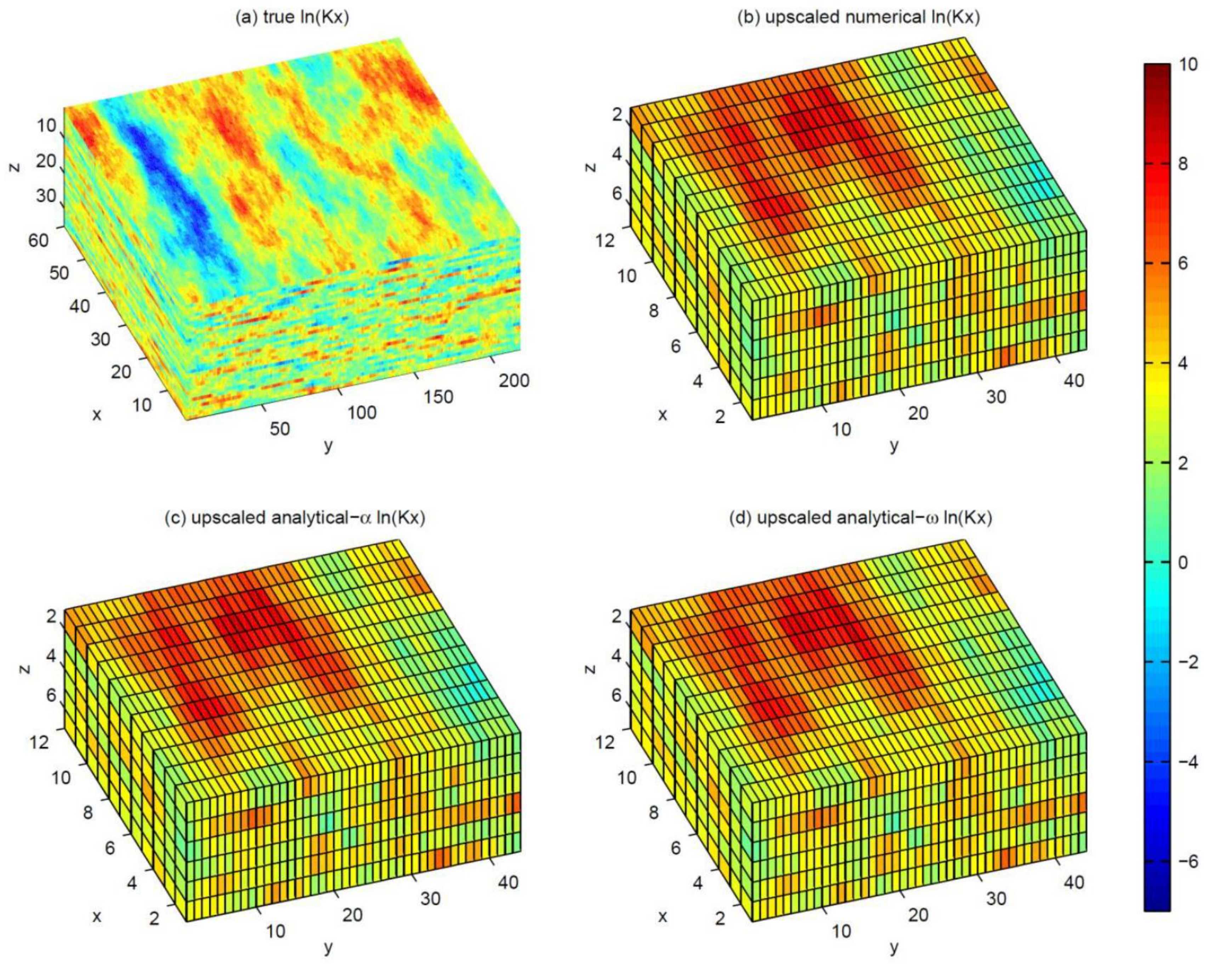
Case 1
Highly Heterogeneous

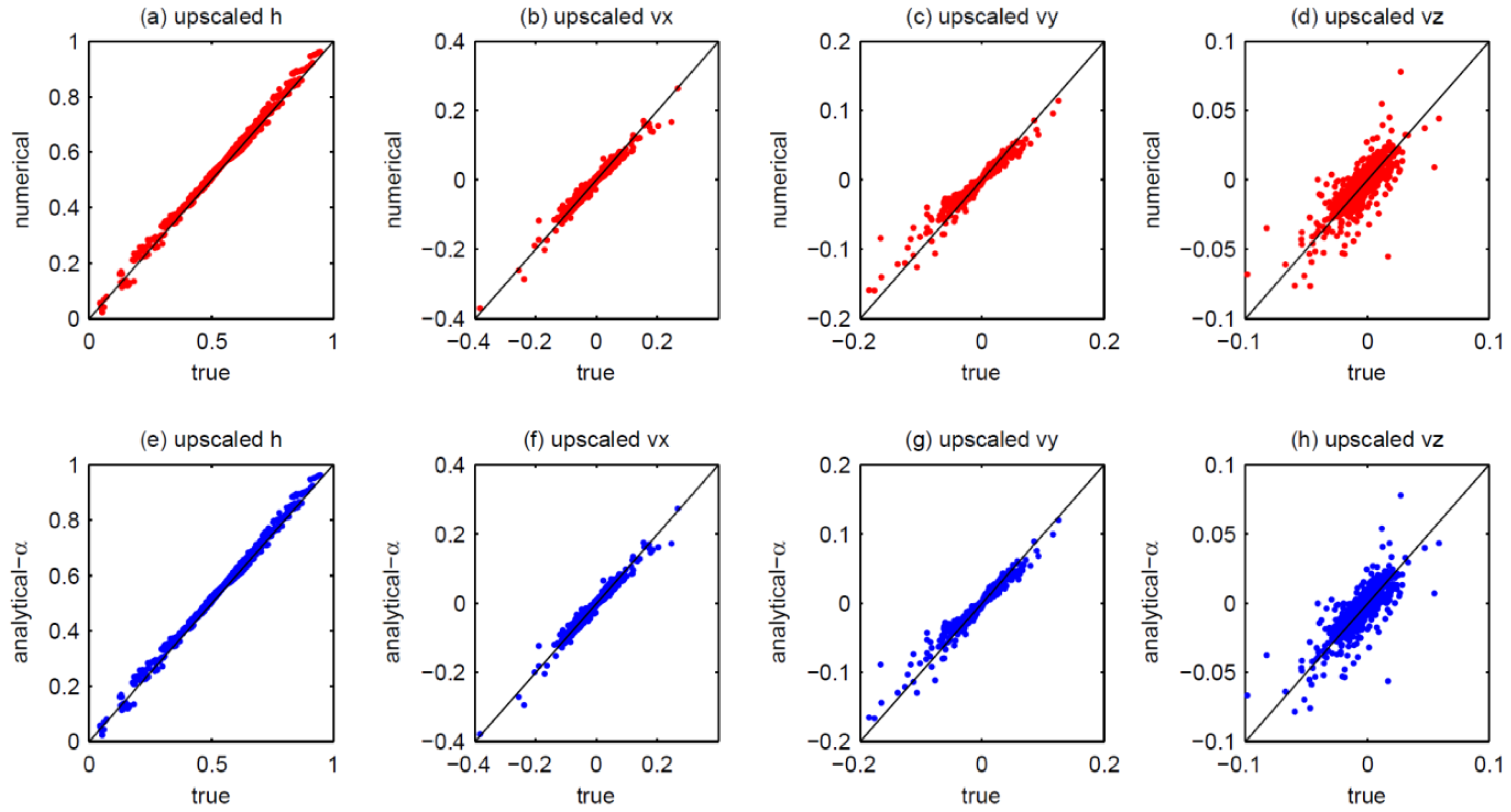



(a) true h
(b) upscaled true h
(c) upscaled numerical h
(d) upscaled analytical−α h
(e) upscaled analytical−ω h
x
y
z
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

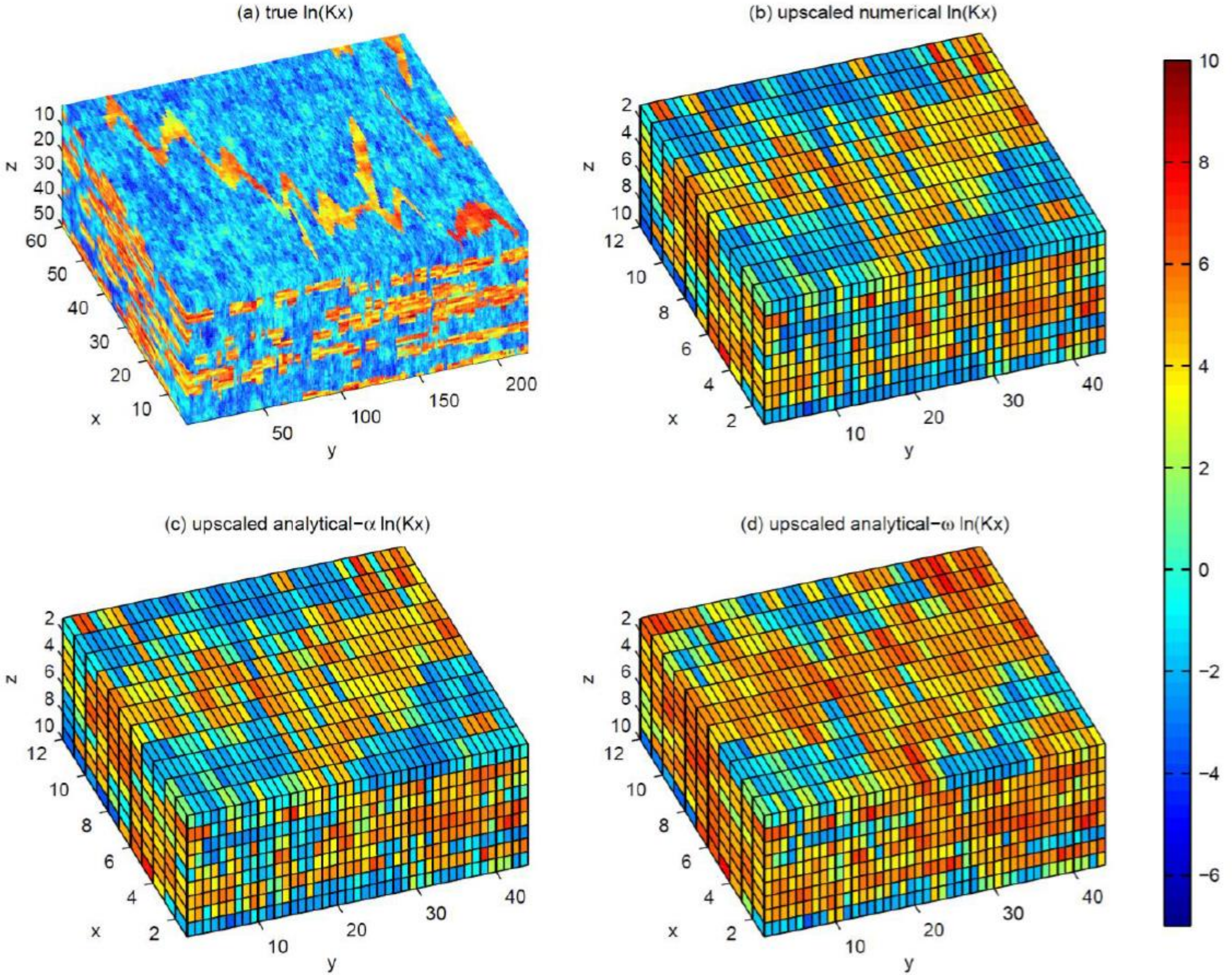

(a) upscaled h
numerical
true
(b) upscaled vx
numerical
true
(c) upscaled vy
numerical
true
(d) upscaled vz
numerical
true
(e) upscaled h
analytical-α
true
(f) upscaled vx
analytical-α
true
(g) upscaled vy
analytical-α
true
(h) upscaled vz
analytical-α
true

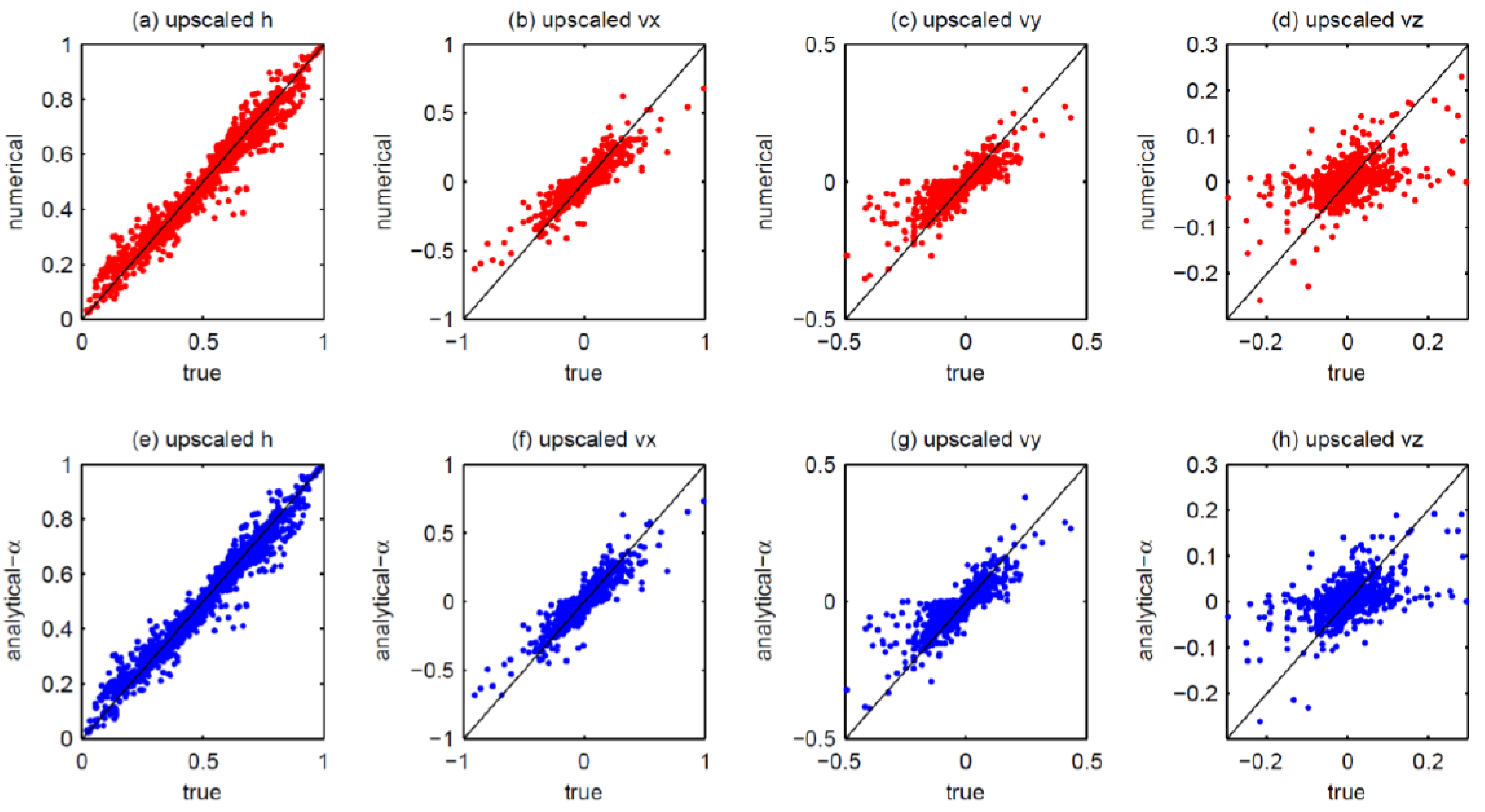
Case 2 (SPE 10 upper layers) large variance




(a) upscaled h
(b) upscaled vx
(c) upscaled vy
(d) upscaled vz
(e) upscaled h
(f) upscaled vx
(g) upscaled vy
(h) upscaled vz
numerical
analytical-α
true

Case 3 (SPE 10 bottom layers) Non-Gaussian





(a) upscaled h
(b) upscaled vx
(c) upscaled vy
(d) upscaled vz
(e) upscaled h
(f) upscaled vx
(g) upscaled vy
(h) upscaled vz
numerical
analytical-α
true

Summary

- **Pros**
  - Tensor conductivity (include off-diagonal terms)
  - Very accurate (reproduce numerical results)
  - Very fast (10-100 times faster than numerical methods)
  - General (Type, $\sigma^2_{\ln K}$, η_x, η_y, η_z, θ, φ, ψ, A_{ky}, A_{kz}, A_{ly}, A_{lz}, n_x, n_y, n_z)
- **Cons**
  - Need statistics of permeability/conductivity random field
- **More details**
  - Effect of rotation angle, large variance, upscaling ratio, correlation lengths
  - In Liao, Q., Lei, G., Wei, Z., Zhang, D. and Patil, S., 2020. Efficient analytical upscaling method for elliptic equations in three-dimensional heterogeneous anisotropic media. *Journal of Hydrology*, *583*, p.124560.