On Determination of the Intensity and Size Frequency Distribution of Convective Vortices: Applications to Martian Dust Devils

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1. Introduction

Dust devils play a major role on Mars, providing a significant proportion of the total dust removal from the surface and its injection into the atmosphere, thus largely determining the overall radiative regime and the climatic state of the Martian atmosphere. Individual meteorological measurements (e.g. pressure time series) have been used to catalog vortex encounters on Mars. Lorenz (2016) explored how well the diameter $W$ and intensity of a dust-devil vortex (e.g. core pressure drop $\Delta P_0$), as well as encounter geometry parameters (e.g. miss distance), can be estimated from single-point measurements of pressure, wind speed and direction. This line of research has been continued in a recent study by Franzece et al. (2020). Since wind measurements are often less reliable compared to the pressure measurements, the mentioned problem cannot be completely solved and only the statistical approach to its solution is feasible. Jackson et al. (2018) proposed formalism for determining the joint statistical distribution of $\Delta P_0$ and $W$ values for a population of convective vortices based on the known from observations joint statistical distribution of $\Delta P$ and $\Gamma$ values. Here, $\Gamma$ is the full width of the recorded pressure depression at its half-maximum level, $\Delta P/2$, the so-called pressure “profile full width at half maximum” FWHM (Ellehoj et al., 2010). In order to transform $\Gamma$ from the time intervals recorded by the sensor into spatial dimensions, the velocity $U$ of the vortex translation relative to the sensor is used. Jackson et al. (2018) applied this methodology to Phoenix vortices using observational data published by Ellehoj et al. (2010).

Below, using the Abel transform, a two-step methodology has been developed to determine the marginal statistical distributions of convective vortices, including dust devils, on $\Delta P_0$ and $W$, based on statistics of $\Delta P$ and $\Gamma$. In a first step, if the pressure profile within the vortex is realistically modeled then the intensity-frequency distribution in the population of vortices can be inferred from the statistics of $\Delta P$ alone. In a second step and in a practically important case when the distribution of vortices on $\Delta P_0$ follows the power law, the problem of determining the vortex size-frequency distribution is solved from data obtained in pressure time-series surveys. This two-step technique has been applied with success to Mars Science Laboratory (MSL) convective vortices.

2. Theory: intensity-frequency distribution

As in Ellehoj et al. (2010), Kahanpää et al. (2016), and Jackson et al. (2018), the radial distribution of the pressure drop in convective vortices is described by the Lorentzian profile

$$\Delta P(r) = \Delta P_0 \left(1 + 4r^2/W^2\right)^{-1}. \quad (1)$$
Here, \( r \) is the distance to the center of the vortex. Both \( \Delta P(r) \) and \( \Delta P_0 \) are considered positive, since their absolute values are taken. We assume that the effective limit radius of the vortex corresponds to the minimum detection level \( \Delta P_{\text{min}} \) of the vortex signature (1) against the background pressure noise. Now, Eq. (1) shows that the effective dust-devil diameter, defined as twice the limit distance to the vortex center up to which \( \Delta P(r) > \Delta P_{\text{min}} \), equals to
\[
D_{\text{eff}} = W \sqrt{\frac{\Delta P_0}{\Delta P_{\text{min}}}} - 1.
\]

Given \( \Delta P_0 = \text{const} \), the probability to encounter a vortex with \( \Delta P > \Delta P \) among all the detected vortices is
\[
P\{\Delta P > \Delta P\} = \sqrt{\frac{\Delta P_0}{\Delta P - 1}} \sqrt{\frac{\Delta P_0}{\Delta P_{\text{min}} - 1}}.
\]
This relation corresponds to the complementary cumulative distribution function (CCDF). If \( \Delta P_0 \) is statistically distributed with the probability density function (PDF), \( \rho(\Delta P_0) \), then
\[
P\{\Delta P > \Delta P\} = \int_{\Delta P}^{\infty} \frac{\sqrt{\Delta P_0/\Delta P - 1}}{\Delta P_0/\Delta P_{\text{min}} - 1} \rho(\Delta P_0) d(\Delta P_0). \tag{3}
\]
The result is independent on \( W \)-values and holds for an arbitrary size-frequency distribution of dust devils. Equation (3) can be re-written in terms of the observed (inferred from measurements) PDF, \( w(\Delta P) \equiv -dP\{\Delta P > \Delta P\}/d(\Delta P) \):
\[
w(\Delta P) = \frac{\sqrt{\Delta P_0/\Delta P_{\text{min}}}}{2(\Delta P)^{3/2}} \int_{\Delta P}^{\infty} \frac{\Delta P_0}{\Delta P_0 - \Delta P_{\text{min}}} \sqrt{\frac{\Delta P_0}{\Delta P_0 - \Delta P}} \rho(\Delta P_0) d(\Delta P_0). \tag{4}
\]
This Abel integral equation can be inversed as follows (cf. Whittaker and Watson, 1996; Deans, 2000):
\[
\rho(\Delta P_0) = -2 \sqrt{\Delta P_0 - \Delta P_{\text{min}}} \int_{\Delta P}^{\infty} \frac{d(\Delta P)}{\Delta P_0 \sqrt{\Delta P_{\text{min}}}} \left[ w(\Delta P)(\Delta P)^{3/2} \right] \frac{d(\Delta P)}{-\sqrt{\Delta P - \Delta P_0}}. \tag{5}
\]
It is possible to arrive at (4) directly, without calculating (3) as an intermediate step.

However, there is a subtlety. The statistical differential distribution (PDF) of vortices \( \sigma(\sigma_{\text{eff}}(\Delta P)) \) on their effective diameter \( D_{\text{eff}} \) for the whole population of vortices differs from the differential distribution \( \sigma(D_{\text{eff}}) \) for vortices that have encounters with a sensor:
\[
\sigma(D_{\text{eff}}) = (D_{\text{eff}})^{3/2} \sigma(\sigma_{\text{eff}}(\Delta P)), \tag{8}
\]
where \( D_{\text{eff}} \) is the expectation value of \( D_{\text{eff}} \) for the whole population of vortices. A tendency toward detection of vortices with larger \( D_{\text{eff}} \) values is evident. We begin with Eq. (8) and assume no correlation between \( \Delta P_0 \) and \( W \) in \( D_{\text{eff}} = W \sqrt{\Delta P_0/\Delta P_{\text{min}} - 1} \) (cf. Jackson et al., 2018). We use that the expectation of a product of two statistically independent random variables is equal to the product of their expectations, i.e., in our case \( D_{\text{eff}} = \bar{W} \cdot \sqrt{\Delta P_0/\Delta P_{\text{min}} - 1} \). We get
\[
\rho(\Delta P_0) = \sqrt{\frac{\Delta P_0 - \Delta P_{\text{min}}}{\Delta P_0 - \Delta P_{\text{min}}}} \rho^*(\Delta P_0), \tag{9}
\]
where \( \rho^*(\Delta P_0) \) is the PDF for the whole population of vortices. From (4) and (9) one has that
\[
    w(\Delta P) = \frac{1}{2(\Delta P)^{3/2}} \frac{1}{\sqrt{\Delta P_0/\Delta P_{\min}}} \int_{-\Delta P}^{\Delta P} \frac{\rho^*(\Delta P_0)}{\Delta P_0} \Delta P_0 \, d(\Delta P_0). \tag{10}
\]

### 3. Results: intensity-frequency distribution

According to the prevailing concept of power distributions (Lorenz, 2014; Jackson et al., 2018), we take \( \rho^*(\Delta P_0) = A(\Delta P_{\min}/\Delta P_0)^m \) in (10), where \( m > 1 \) and \( A = (m-1)/\Delta P_{\min} \). After some algebra, we obtain

\[
    w(\Delta P) = A(\Delta P_{\min}/\Delta P)^m, \tag{11}
\]

i.e., the measurements provide an unbiased estimate of the pressure drop distribution.

In general cases, i.e., irrespective of whether or not correlation between \( \Delta P_0 \) and \( W \) does exist, it follows from (5) for the power differential distribution (11) and in the limit of \( \Delta P_0 >> \Delta P_{\min} \) that

\[
    \rho(\Delta P_0) \propto \left( \frac{\Delta P_{\min}}{\Delta P_0} \right)^{m-1/2}.
\]

The distribution of central pressure drops in the population of detected vortices \( \rho(\Delta P_0) \) has thus a less steep slope than the distribution of measured pressure drops, i.e. an underestimation of the number of larger pressure drop values takes place in the measured values. However, in the case of no correlation between \( \Delta P_0 \) and \( W \), the measurements provide still an unbiased estimate of the differential distribution \( \rho^*(\Delta P_0) \) of the whole population of vortices. If, alternatively, \( W \propto (\Delta P_0)^x \), \( x > 0 \), i.e., the more intense vortices have the larger pressure “profile full width at half maximum”, then \( D \propto (\Delta P_0)^{x+1/2} \), \( \rho(\Delta P_0) \propto \left( \Delta P_0 \right)^{x+1/2} \rho^*(\Delta P_0) \) and \( \rho^*(\Delta P_0) \propto (\Delta P_0)^{-(m+x)} \). The choice \( x = 1/2 \) leads to the result which is close to the inferences by Lorenz (2014), who however assumed no correlation between \( \Delta P_0 \) and \( W \).

### Table 1

Inferred value of exponents in the power-law complementary cumulative distribution of pressure drops in Martian convective vortices, including dust devils. In the last column, the first values are derived under an assumption of no correlation between the vortex width \( W \) and the central pressure drop \( \Delta P_0 \); the values in parentheses correspond to assumed proportionality between the vortex width squared and the central pressure drop (see more in the text).

<table>
<thead>
<tr>
<th></th>
<th>Observed distributions of ( \Delta P ) (Steakley and Murphy, 2016)</th>
<th>Inferred distributions of ( \Delta P_0 ) for encountered vortices</th>
<th>Inferred distributions of ( \Delta P_0 ) for the whole population of vortices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pathfinder vortices</td>
<td>$-1.73$</td>
<td>$-1.23$</td>
<td>$-1.73 \left( -2.23 \right)$</td>
</tr>
<tr>
<td>Phoenix vortices</td>
<td>$-2.48$</td>
<td>$-1.98$</td>
<td>$-2.48 \left( -2.98 \right)$</td>
</tr>
<tr>
<td>MSL vortices</td>
<td>$-2.77$</td>
<td>$-2.27$</td>
<td>$-2.77 \left( -3.27 \right)$</td>
</tr>
</tbody>
</table>
Table 1 summarizes our findings by using the estimates by Steakley and Murphy (2016) of the exponents in the power-law CCDF of pressure drops in the three sets of observations of Martian convective vortices, including dust devils; see also Fig. 2. Model (1) is used to infer a CCDF, corresponding to $\rho(\Delta P_0)$ for a subpopulation of encountered convective vortices (the third column), and to make inferences about a CCDF, corresponding to $\rho^*(\Delta P_0)$ for the whole population of convective vortices (the last column).

**Fig. 2.** Cumulative distribution of pressure drop magnitudes. Three distributions of pressure drop magnitudes from three separate Mars missions are shown as the total number of vortices detected above a given pressure drop magnitude in pascals. MSL vortices are open circles, Pathfinder vortices are plus signs, and Phoenix vortices are triangles. The small filled squares show the MSL detected vortices plus the candidates which were eliminated because they were classified as too ambiguous to be confident detections. Power law fits were applied to the cumulative distributions of each mission (solid lines) and the magnitude of each power law slope, $\alpha$, is shown in the legend (MSL $\alpha = -2.77$, Pathfinder $\alpha = -1.73$, and Phoenix $\alpha = -2.48$ (from Steakley and Murphy, 2016; their Fig. 9)).

Above, the Abel transform has been applied to determine the relationship between marginal statistical distributions of convective vortices on $\Delta P_0$ и $\Delta P$ values, respectively. However, as mentioned in Introduction, barometric measurements make it possible to determine not only $\Delta P$ but also $\Gamma$ values expressed in units of time. The question arises as to what additional information on the statistical properties of a population of convective vortices can be by a method similar to that used above extracted from the known statistical distribution of $\Gamma$ values? Can this knowledge help obtaining useful information about the statistical distribution of the diameters of convective vortices? The following considerations constitute an attempt to answer these questions, at least partially.

4. **Theory: size-frequency distribution**

If the vortex passes at the closest approach distance $r$ to the pressure sensor, then according to (1) the measured pressure drop profile takes the form
\[ \Delta P(r,x) = \Delta P_0 \left[1 + 4 \left( \frac{r^2 + x^2}{W^2} \right) \right]^{-1}, \]  

(12)

where \( x \) is the distance from the position where the vortex would be located when it is closest to the pressure sensor. Actually, the pressure sensor records the corresponding time durations and in order to convert them into the spatial dimensions, like in (12), they must be multiplied by the vortex translational speed \( U \) relative to the ground. In practice, this speed remains unknown exactly and can vary within certain limits, which adds uncertainty in the problem (see more below). Obviously, \( \Delta P(r) \equiv \Delta P(r,0) \) and by eliminating \( \Delta P_0 \) between (1) and (12), we have

\[ \Delta P(r,x) = \Delta P(r,0) \left(1 + 4 \frac{x^2}{\Gamma^2}\right)^{-1}, \]

where \( \Gamma = \sqrt{4r^2 + W^2} \) specifies the measured pressure profile FWHM, at \( \Delta P(r,0)/2 \) level, which is expressed in spatial units. As above, we assume that there is a minimal pressure drop \( \Delta P_{\text{min}} \) in the vortex flow, which is measurable against the background. In order to be able to speak on the measured pressure profile FWHM, it should be \( \Delta P(r,0) \geq 2\Delta P(r,x_{\text{max}}) = 2\Delta P_{\text{min}} \), where \( x_{\text{max}} \) stands for the maximum possible value of half-width of the recorded pressure profile at its half-maximum level \( \Delta P(r,0)/2 \) that is equal to \( \Delta P_{\text{min}} \). Taking the equality sign in the above-written non-strict inequality, we come from (1) and (12) to the relations

\[ 4r^2 = W^2 \left( \frac{\Delta P_0}{2\Delta P_{\text{min}}} - 1 \right), \quad 4r^2 + x_{\text{max}}^2 = W^2 \left( \frac{\Delta P_0}{\Delta P_{\text{min}}} - 1 \right). \]

Applying the identification \( x_{\text{max}} = \Gamma_{\text{max}}/2 \), we obtain that \( \Gamma_{\text{max}}^2 = W^2 \frac{\Delta P_0}{2\Delta P_{\text{min}}} \) and the pressure profile FWHM can be inferred from pressure records if only the vortex passes from the pressure sensor at a distance not greater than the limit distance \( r_{\text{max}} \), determined by the formula

\[ D \equiv 2r_{\text{max}} = W \sqrt{\frac{\Delta P_0}{2\Delta P_{\text{min}}} - 1}. \]  

(13)

The above variables are illustrated in Fig. 3. The pressure sensor location at point \( P \) corresponds to that maximum distance from the vortex center, \( r_{\text{max}} = D/2 \), which allows to determine FWHM (denoted \( \Gamma \) above) from the pressure records. Geometric proportions of Fig. 3 correspond to \( \Delta P_0 = 5\Delta P_{\text{min}} \), e.g., to \( \Delta P_0 = 2.5 \text{ Pa} \) and \( \Delta P_{\text{min}} = 0.5 \text{ Pa} \), and the radial pressure profile is described by (1). As a result, \( D_{\text{eff}} = 2W \) and \( D = W \sqrt{3/2} \approx 1.22 \times W \). By the Pythagorean theorem (see, a right triangle \( OPQ \) in Fig. 3) the maximum FWHM obtainable from the pressure records is \( \Gamma_{\text{max}} = 2x_{\text{max}} = \sqrt{D_{\text{eff}}^2 - D^2} = W \sqrt{5/2} \approx 1.58 \times W \); note that \( x_{\text{max}} \) corresponds to \( PQ \) and it can be proved that \( PQ = PR \); see, Fig. 3. When the distance between the dust devil and the sensor is smaller than \( r_{\text{max}} \), e.g., the sensor is located at point \( P_1 \) at the distance \( W/2 \) from the vortex center, then the recorded FWHM is \( \Gamma = 2P_1R = W \sqrt{2} \approx 1.41 \times W \). For an arbitrary position, \( P_2, P_3, \ldots \), of the sensor on the line segment \( OP \) the corresponding \( \Gamma \)-values would be given by the length of line segments \( 2P_2R, 2P_3R, \ldots \).
Fig. 3. Schematic of a dust devil encounter with the pressure sensor. The vortex center is at point $O$ and the central pressure drop is $\Delta P_0$. Points $P$ and $P_1$ mark two particular positions of the pressure sensor relative to the vortex center; see the text. The inner circle of radius $W/2$ corresponds to the pressure drop $\Delta P/2$ and $W$ is the vortex pressure profile FWHM, see (1). The outer circle of diameter $D_{\text{eff}}$ encircles the area where the pressure perturbation due to the vortex flow exceeds $\Delta P_{\text{max}}$. The intermediate dashed-lined circle of diameter $D$ corresponds to the pressure drop $2\Delta P_{\text{max}}$. Additional construction using point $R$ is discussed in the text. Geometric proportions of the drawing correspond to $\Delta P_0 = 5\Delta P_{\text{max}}$ and the radial pressure profile is given by (1).

We begin solving the problem by assuming that all vortices have the same maximum pressure drop $\Delta P_0$ at their center but different random values of the width $W$. The $\Delta P_{\text{max}}$-value is fixed. Assume that $\rho(W)dW$ is the probability that a vortex detected by the sensor has a width $W$ falling within the range $(W, W + dW)$. The probability that the sensor will record a value $\Gamma$ from the interval $(\Gamma, \Gamma + d\Gamma)$ equals the sum over all $W$-values of the product of $\rho(W)dW$ by the conditional probability $\rho(\Gamma|W)d\Gamma$. We have

$$w(\Gamma)d\Gamma = \int_0^\Gamma \rho(W)\rho(\Gamma|W)d\Gamma dW,$$

where it is used that according to the relationship $\Gamma^2 = 4r^2 + W^2$ the $\Gamma$-values always exceed the $W$-values. Obviously, $\rho(\Gamma|W)d\Gamma$ coincides with the probability that the sensor is placed relative to the center of the vortex at a “miss distance” $r$ falling within the interval $(r, r + dr)$ provided that the vortex width is $W$: $\rho(\Gamma|W)d\Gamma = \rho(r|W)dr$. By following previous papers dedicated to practical applications of the Abel integral transform, we assume that all miss distances $r$ are uniformly (evenly) distributed on the segment $[0, r_{\text{max}}]$ and, therefore,
\[
\rho(r \mid W) = \frac{1}{r_{\text{max}}} \left( \frac{2}{W} \left( \frac{\Delta P_0}{2\Delta P_{\text{min}}} - 1 \right) \right)^{-1/2}; \text{ see (13). In our work context, the notion of “miss distance” has close resemblance to the notion of “impact parameter” used in astronomical, technical and other applications. Returning to our calculus, we have that on the other hand}
\]
\[
\rho(\Gamma \mid W) d\Gamma = \rho(r \mid W) \left( \frac{\Delta P_0}{2\Delta P_{\text{min}}} - 1 \right)^{-1/2} \frac{\Gamma^2 - W^2}{2} d\Gamma .
\]
Collecting all formulas and dividing both sides of the resulting equation by \(d\Gamma\), we get the Abel integral equation
\[
w(\Gamma) = \Gamma \int_0^\Gamma \frac{1}{\sqrt{2\Delta P_{\text{min}} - 1}} \frac{\rho(W)}{W \sqrt{\Gamma^2 - W^2}} dW . \tag{14}
\]

More realistic is the case of vortices with continuously varying random \(\Delta P_0\)-values, characterized by the probability density function (PDF) \(\rho(\Delta P_0)\). Since \(\Delta P_0 \geq 2\Delta P_{\text{min}} \Gamma^2 / W^2\), equation (14) is generalized onto the equation
\[
w(\Gamma) = \Gamma \int_{\Delta P_{\text{min}}}^\Gamma \int_0^{\Delta P_{\text{min}}} \frac{\rho(\Delta P_0)}{\sqrt{2\Delta P_{\text{min}} - 1}} \frac{\rho(W)}{W \sqrt{\Gamma^2 - W^2}} d\Delta P_0 dW . \tag{15}
\]
So far, we considered not the whole population of the vortices but only that part (subpopulation) which is recorded by the pressure sensor. Due to the selective nature of pressure measurements, which favor detection of vortices with larger \(D\)-values, see (13), the above-introduced differential distributions \(\rho(\Delta P_0)\) and \(\rho(W)\) will, generally speaking, be skewed towards larger \(\Delta P_0\) and \(W\) values compared to the differential distributions \(\rho^*(\Delta P_0)\) and \(\rho^*(W)\) which refer to the whole population of vortices. Assume, again, that \(W\) and \(\Delta P_0\) are independent random variables when the corresponding joint PDF factorizes: \(\rho^*(W, \Delta P_0) = \rho^*(W) \rho^*(\Delta P_0)\). Using (3), and since the mathematical expectation (marked with an overbar) of the product of two independent random variables equals to the product of their mathematical expectations, \(\overline{D} = \overline{W} \cdot \sqrt{\frac{\Delta P_0}{2\Delta P_{\text{min}}} - 1}\), we get
\[
\rho(W) = \frac{W}{W} \rho^*(W), \quad \rho(\Delta P_0) = \frac{\sqrt{\Delta P_0 - 2\Delta P_{\text{min}}}}{\sqrt{\Delta P_0 - 2\Delta P_{\text{min}}}} \rho^*(\Delta P_0).
\]
In terms of \(\rho^*(W)\) and \(\rho^*(\Delta P_0)\), equation (15) acquires a more compact form
\[
w(\Gamma) = \Gamma \int_0^{\Delta P_{\text{min}}} \frac{1}{\overline{D}} \frac{\rho^*(\Delta P_0)}{\sqrt{\Gamma^2 - W^2}} \rho^*(W) \overline{D} d\Delta P_0 dW . \tag{16}
\]

The explicit Abel inverse to equations (15) and (16) does exist in a practically important case of the power law distribution \(\rho^*(\Delta P_0) = C \left( \frac{2\Delta P_{\text{min}}}{\Delta P_0} \right)^k\). Here, \(k > 1\) и \(C = (k - 1)/(2\Delta P_{\text{min}})\). Indeed, from equation (16) we have in this case that
and the Abel inverse to equation (17) is
\[
\frac{\rho^*(W)W^{2k-2}}{D} = \frac{2}{\pi} \frac{d}{dW} \int_0^W w(\Gamma) \Gamma^{2k-2} \frac{d\Gamma}{\sqrt{\Gamma^2 - W^2}}.
\] (18)

Analytical examples of the direct Abel transform (17) can be provided for the power law size-frequency distribution \( \rho^*(W) = A(W_{\text{min}}/W)^m \) (cf. Lorenz, 2009, 2011), where \( m > 1 \) and \( A = (m-1)/W_{\text{min}} \). In these examples, the differential distribution \( w(\Gamma) \) appears flatter than the differential distribution \( \rho^*(W) \) and the indices of these distributions differ by one or even by two, the latter for \( m - 2k + 2 > 0 \).

5. Results: size-frequency distribution; applications to MSL vortices

Hereafter, we supplement considerations of Sect. 4 with numerical calculations of the distribution \( \rho^*(W) \) from the known distribution \( w(\Gamma) \) for MSL vortices (Steakley and Murphy, 2016), using that the differential distribution \( \rho^*(\Delta P) \) for MSL vortices can be approximated by a power law with the exponent \( k = 3.77 \) (ibid; cf. Table 1). We follow an approximate matrix method (see, a classical paper by Wicksell (1925) and further references in, e.g., Pretzler et al. (1992) and De Micheli (2017)) which is used when \( w(\Gamma) \)-values are given as a table of \( N \) numbers. For the time being, assume that \( \Gamma \)-values have the dimension of time and correspond to time intervals measured in whole seconds. The values of \( W \) are calculated in the same time units. Let the interval of values of \( \Gamma \) and \( W \) spanning from 0 to \( N \) be divided into \( N \) subintervals with boundaries \( \Gamma_i \equiv i \) and \( \Gamma_{i+1} \equiv i + 1 \), where \( i = 0, ..., N - 1 \). Values of \( w(\Gamma) \) and \( \rho^*(W) \) are presumed constant in each subinterval and appear explicitly in the middle of it:
\[
w(\Gamma_{i+1/2}) \equiv w_{i+1/2} \text{ and } \rho^*(W_{i+1/2}) \equiv \rho^*_{i+1/2}.
\]
After integrating both sides of (17) over \( \Gamma \) from \( \Gamma_i \) to \( \Gamma_{i+1} \) and changing to finite sums we get
\[
\frac{1}{D} \rho^*_{i+1/2} (i + 1/2)^{2k-2} \left( \left( i + 1/2 \right)^2 - \left( j + 1/2 \right)^2 \right) - \left( \left( i + 1/2 \right)^2 - \left( j + 1/2 \right)^2 \right) + \frac{1}{D} \rho^*_{i+1/2} (i + 1/2)^{2k-2} \left( \left( i + 1/2 \right)^2 - \left( j + 1/2 \right)^2 \right).
\] (20)

Introducing the notations \( W_{i+1/2} = w_{i+1/2} (i + 1/2)^{2k-2} \) and \( R^*_{i+1/2} = \frac{1}{D} \rho^*_{i+1/2} (i + 1/2)^{2k-2} \) we rewrite (20) in matrix form \( \mathbf{W} = \mathbf{AR}^* \), where \( \mathbf{W} \) and \( \mathbf{R}^* \) are the column vectors of dimension \( N \) and \( \mathbf{A} \) is the lower triangular matrix of dimension \( N \times N \). Inversing this equation, we have \( \mathbf{R} = \mathbf{A}^{-1} \mathbf{W} \), where the lower triangular inverse matrix \( \mathbf{A}^{-1} \) has also dimension \( N \times N \). A factor of proportionality \( \left( \frac{1}{D} \right)^{2k-2} \) is immaterial and will disappear in the final reductions to relative frequencies.
Below we apply the matrix method to MSL vortices. From a separate catalog of MSL vortex parameters given in the supplementary material to (Steakley and Murphy, 2016) we determined the number $N(>\Gamma)$ of vortices (out of 245 detected, as a total) that have a pressure “profile full width at half maximum” exceeding the fixed $\Gamma$-value and presented this data in the form of a table with $\Gamma = 0,...,15$ s (Table 2).

**Table 2**
The number $N(>\Gamma)$ of vortices that have a pressure “profile full width at half maximum” exceeding the fixed $\Gamma$-value (in seconds).

<table>
<thead>
<tr>
<th>$\Gamma$ (sec)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(&gt;\Gamma)$</td>
<td>245</td>
<td>243</td>
<td>241</td>
<td>206</td>
<td>166</td>
<td>127</td>
<td>100</td>
<td>87</td>
<td>66</td>
<td>45</td>
<td>31</td>
<td>24</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

The complementary cumulative function $P(>\Gamma) = N(>\Gamma)/N(>0)$ is estimated from data in Table 2 and its graph is shown in Fig. 4.

![Fig. 4](image)

**Fig. 4.** The complementary cumulative function $P(>\Gamma)$ inferred from data in Table 2 (small black squares) and the exponential fit to it (solid line). The values of $\Gamma$ are in seconds.

To reduce the computational errors during the numerical differentiation of $P(>\Gamma)$ required to calculate the differential distribution $w(\Gamma)$ we make an exponential fit to $P(>\Gamma)$ at $2s \leq \Gamma \leq 15s$. As a result, in the specified $\Gamma$-range

$$w(\Gamma) = \frac{-dP(>\Gamma)}{d\Gamma} = 0.48671 \exp\left(-\Gamma/3.85362\right), \quad (21)$$

where $\Gamma$-values are in seconds, and the values of $w_{1/2}$ starting from $w_{2.5}$ are taken according to (21). Minor values of $w_{0.5}$ and $w_{1.5}$ are neglected and assumed to be zero. The result of calculations based on (20) with $k = 3.77$ is shown in Fig. 5.
At $2 \leq W \leq 15$ s, the normalized to unity function $\rho^*(W)$ is well fitted by the exponent

$$\rho^*(W) = 0.8655 \exp(-W/2.40805),$$

where $W$-values are in seconds. Small variations in the $k$-value in (20) change the result only continuously; therefore, a stable solution to the Abel inverse problem is obtained. The resulting exponential-like form of $\rho^*(W)$ does not contradict the analytical formulations previously discussed in the literature (Kurgansky, 2006; Pathare et al., 2010). As it, for example, follows from comparison of equations (21) and (22) and as one would expect a priori, the distribution $\rho^*(W)$ is steeper than $w(\Gamma)$.

A separate difficult problem is the recalculation of $\Gamma$-values, and consequently of $W$-values, from time intervals to spatial dimensions. In principle, it is necessary knowing the translational velocity $U$ of the vortex relative to the pressure sensor for each detectable vortex. Unfortunately, such data are not available for all MSL vortices and we use for the recalculation that $U = 7.6 \text{ m s}^{-1}$ which is an approximate average value of the median wind speed (Kahanpää et al., 2016). Now, the above-obtained value of $\bar{W} = 4.12924 \text{ s}$ in time units corresponds to $\bar{W} = 31.3822 \text{ m} \approx 31 \text{ m}$ in spatial units. This estimate is higher than the average value $D_{\text{eff}} \approx 21 \text{ m}$ calculated in Kahanpää et al. (2016) but is close to their determination of the diameter, corresponding to the average area of pressure depression (deeper than the critical value of 0.5 Pa) induced by the vortex. Nevertheless, our estimate can be calibrated, based on the relation

$$D_{\text{eff}} = \bar{W} \cdot \sqrt{\Delta P_0/\Delta P_{\text{min}} - 1}$$

that is valid when $\bar{W}$ and $\Delta P_0$ are independent random variables. We calculate $\sqrt{\Delta P_0/\Delta P_{\text{min}} - 1}$ for the power distribution $\rho^*(\Delta P_0) = C^*(\Delta P_{\text{min}}/\Delta P_0)^k$, with $k > 1$ and $C^* = (k-1)/\Delta P_{\text{min}}$, in order to get
\[
\sqrt{\frac{\Delta P_0}{\Delta P_{\text{min}}}} \approx (k-1)B\left(\frac{3}{2}, k-\frac{3}{2}\right),
\]

(23)

where \(B\) is the beta function. For MSL vortices \(k \approx 3.77\) and we obtain \(\sqrt{\Delta P_0/\Delta P_{\text{min}}} \approx 0.62\), i.e. at first glance paradoxically, the expected (average) value of the “perturbation diameter”, \(D_{\text{eff}}\), is less than the expected (average) value of the vortex “physical” width \(\bar{W}\). This occurs since the overwhelming majority of vortices in a vortex population have \(\Delta P_0\)-values that only slightly exceed the lower threshold \(\Delta P_{\text{min}}\)-value and, therefore, will formally possess \(W\)-values, see (1), which are greater than the corresponding \(D_{\text{eff}}\)-values. This interesting model effect is illustrated in Table 3, which shows that only for \(k < 2.5\), when the distribution \(\rho^* (\Delta P_0)\) is quite flat, the \(D_{\text{eff}}\)-value exceeds \(\bar{W}\)-value, but when \(k > 2.5\) then the opposite is true.

Table 3

<table>
<thead>
<tr>
<th>(k)</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>((k-1)B\left(\frac{3}{2}, k-\frac{3}{2}\right))</td>
<td>0.5(\pi) (\approx) 1.57</td>
<td>1</td>
<td>0.25(\pi) (\approx) 0.79</td>
<td>0.67</td>
<td>0.59</td>
</tr>
</tbody>
</table>

If we proceed from \(D_{\text{eff}} \approx 21\) m (Kahanpää et al., 2016) then for \(k = 3.77\) an estimate \(\bar{W} \approx 33.7\) m emerges which is close to that obtained above. This provides additional confidence in the correctness of our model calculations.

6. Discussion and summary

Using the Abel transform, a two-step method has been developed to determine the marginal statistical distributions of convective vortices, including dust devils, on their intensity (pressure drop in the vortex center) and size (diameter), based on statistics of transient pressure drops recorded when the vortices pass near a pressure sensor placed on the planet’s surface. In a first step, if the pressure profile within the vortex is realistically modeled then the intensity-frequency distribution in the population of vortices can be inferred from the statistics of peak pressure drops recorded alone. If the observed statistics can be approximated with a truncated power-law distribution and in the absence of an apparent correlation between the vortex diameter and the maximum pressure drop at its center, then the measurements provide an unbiased power-law estimate of the actual intensity-frequency distribution. In a second step and in a practically important case when the distribution of vortices on their intensity follows the power law, the problem of determining the vortex size-frequency distribution is solved from data obtained in pressure time-series surveys. This two-step technique has been applied with success to Mars Science Laboratory (MSL) convective vortices.

Returning back to the method developed in this contribution, we note how the Abel inverse transform is mathematically classified as a “mildly” ill-posed problem (cf. De Micheli,
2017), but a stable solution can be obtained after regularization of the problem, particularly for monotonic differential distributions, as shown in Sect. 5 of this paper.

The proposed method can also be used for post-processing the data obtained in pressure time-series surveys for dust devils in arid and semi-arid locations on Earth (Lorenz and Lanagan, 2014; Jackson and Lorenz, 2015) and, more generally, for inferring statistical properties of populations of atmospheric vortices of convective origin based on meteorological in situ measurements.

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References


