



Chat Fri, 08 May, 14:00–15:45

Section: Recent development in GFD and remote sensing.

Nonlinear and turbulent processes under high wind conditions

Co-organized by AS2/NH1/OS4

Analytical solution technique in k-omega and k-epsilon turbulence parameterizations and their implementation in the OGCM

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Outline



Purpose.

The main goal of the work is **to advance the ocean general circulation model by improving description of the processes of vertical turbulent exchange** of heat, salt and momentum which significantly affect quality of reproducing the ocean circulation and thermohaline structure using the models based on a system of the ocean hydrothermodynamics primitive equations.

Methods and Results.

The main instrument of the research is **the sigma model of the oceanic and marine circulation INMOM (Institute of Numerical Mathematics Ocean Model)** developed at the Marchuk Institute of Numerical Mathematics, Russian Academy of Sciences. In the incompressibility, hydrostatics and Boussinesq approximations, the system of equations is supplemented with the $k - \omega$ and $k - \varepsilon$ parameterizations of the vertical turbulent exchange, the equations for which are solved by **the splitting method applied to the physical processes**. The equations are split into the stages describing **transport-diffusion** of the turbulence characteristics and their **generation-dissipation**. At the generation-dissipation stage, the equations for turbulent characteristics are solved **analytically**. At that, **the stability functions** resulted from application of the spectral algorithm are used. To assess quality of two parameterizations of the vertical turbulent exchange, **the North Atlantic–Arctic Ocean** circulation is numerically simulated and the upper ocean layer characteristics are studied.

Introduction



Most of Ocean general circulation models (OGCMs) are based on systems of primitive equations. Their attributes are as follows.

- 1) Rotation (Coriolis acceleration).
- 2) Simplification of the third equation of motion to a hydrostatic ratio.
- 3) “Artificial” vertical mixing till the achievement of stable stratification.
- 4) Free upper surface.
- 5) Complex coastal boundary, variable bottom topography.

The spatial-temporal scales of vertical turbulent exchange are
1–10 km horizontally
1–10 m vertically
and from seconds to several hours in time.

For primitive models, this is a subgrid process to be parameterized.

Turbulent mixing is often described in the OGCMs by a **second-order operator** with coefficients of turbulent exchange of momentum, heat and salt.



$$\frac{\partial f}{\partial t} = \frac{1}{H^2} \frac{\partial}{\partial \sigma} v_f \frac{\partial f}{\partial \sigma}, \quad f \text{ is temperature, salinity or a velocity component, } \sigma = \frac{z}{H}$$

The task is
to determine the exchange coefficients v_f

The models based on two equations can be used.

The first equation is written for the **turbulent kinetic energy (TKE) k** ,
the second one –
either for the **turbulence scale l ($k - l$ model)**
or for the **TKE specific dissipation rate ε ($k - \varepsilon$ model)**
or for the **dissipation frequency ω ($k - \omega$ model)**.

The characteristics **l , ε , ω** are related algebraically.

We develop an algorithm for solving **$k - \omega$** and **$k - \varepsilon$** turbulence equations.

The equations are split into the two main stages:

- 1) transport – diffusion;
- 2) generation – dissipation of the turbulent characteristics.

At the second stage of splitting, the turbulence equations **are solved analytically**.



The $k - \varepsilon$ model (σ -coordinate in vertical)

$$\begin{cases} \frac{dk}{dt} - \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_k^\varepsilon} \frac{\partial k}{\partial \sigma} - \Lambda k = \nu_u G^2 - \nu_\rho N^2 - \varepsilon, \\ \frac{d\varepsilon}{dt} - \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial \sigma} - \Lambda \varepsilon = \left[c_1^\varepsilon \nu_u G^2 - c_3^\varepsilon \nu_\rho N^2 - c_2^\varepsilon \varepsilon \right] \frac{\varepsilon}{k}, \end{cases}$$

k is turbulent kinetic energy (TKE)

$$\sigma_k^\varepsilon = 1, \quad \sigma_\varepsilon = 1.3$$

ε is TKE specific dissipation rate

$$c_1^\varepsilon = 1.44, c_2^\varepsilon = 1.92, c_3^\varepsilon = \begin{cases} -0.4, & N^2 > 0, \\ 1.0, & N^2 \leq 0. \end{cases}$$

Vertical viscosity and diffusivities: $\nu_u = (c_s^0)^3 c_s^u \frac{k^2}{\varepsilon}, \quad \nu_T = (c_s^0)^3 c_s^p \frac{k^2}{\varepsilon}. \quad c_s^0 = 0.5544$

The $k - \omega$ model

$$\begin{cases} \frac{dk}{dt} - \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_k^\omega} \frac{\partial k}{\partial \sigma} - \Lambda k = \nu_u G^2 - \nu_\rho N^2 - (c_s^0)^4 \omega k, \\ \frac{d\omega}{dt} - \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_\omega} \frac{\partial \omega}{\partial \sigma} - \Lambda \omega = \left[c_1^\omega \nu_u G^2 - c_3^\omega \nu_\rho N^2 - c_2^\omega (c_s^0)^4 k \omega \right] \frac{\omega}{k}. \end{cases}$$

k is turbulent kinetic energy (TKE)

$$\sigma_k^\omega = 2, \quad \sigma_\omega = 2$$

ω is TKE dissipation frequency

$$c_1^\omega = 0.555, c_2^\omega = 0.833, c_3^\omega = \begin{cases} -0.6, & N^2 > 0 \\ 1, & N^2 \leq 0 \end{cases}$$

Vertical viscosity and diffusivities:

$$\nu_u = \frac{c_s^u}{c_s^0} \frac{k}{\omega}, \quad \nu_T = \frac{c_s^p}{c_s^0} \frac{k}{\omega} \quad c_s^0 = 0.5562$$

$$G^2 = \left(\frac{1}{H} \frac{\partial u}{\partial \sigma} \right)^2 + \left(\frac{1}{H} \frac{\partial v}{\partial \sigma} \right)^2, \quad - \text{shear frequency}$$

$$N^2 = \frac{g}{\rho_0} \frac{1}{H} \frac{\partial \rho}{\partial \sigma} \quad - \text{buoyancy frequency}$$

Splitting of the turbulence equations

at the model time step $t_j < t < t_{j+1}$



1) 3-dimensional transport-diffusion

$$\begin{aligned}\frac{dk}{dt} &= \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_k^{\varepsilon, \omega}} \frac{\partial k}{\partial \sigma} + \Lambda k, \\ \frac{d\varepsilon}{dt} &= \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial \sigma} + \Lambda \varepsilon, \\ \frac{d\omega}{dt} &= \frac{1}{H^2} \frac{\partial}{\partial \sigma} \frac{\nu_u}{\sigma_\omega} \frac{\partial \omega}{\partial \sigma} + \Lambda \omega.\end{aligned}$$

upper boundary ($\sigma=0$) conditions

$$\begin{aligned}\frac{\nu_u}{\sigma_k} \frac{1}{H} \frac{\partial k}{\partial \sigma} &= -c_g (u_*^s)^3, \\ \frac{\nu_u}{\sigma_\varepsilon} \frac{1}{H} \frac{\partial \varepsilon}{\partial \sigma} &= Q_\varepsilon^0, \\ \frac{\nu_u}{\sigma_\omega} \frac{1}{H} \frac{\partial \omega}{\partial \sigma} &= Q_\omega^0,\end{aligned}$$

lower boundary ($\sigma=1$) conditions

$$\begin{aligned}\frac{\nu_u}{\sigma_k} \frac{1}{H} \frac{\partial k}{\partial \sigma} &= 0, \\ \frac{\nu_u}{\sigma_\varepsilon} \frac{1}{H} \frac{\partial \varepsilon}{\partial \sigma} &= 0, \\ \frac{\nu_u}{\sigma_\omega} \frac{1}{H} \frac{\partial \omega}{\partial \sigma} &= 0.\end{aligned}$$

2) generation-dissipation

$k - \varepsilon$

$$\begin{cases} \frac{\partial k}{\partial t} = \left[(c_s^0)^3 c_s^u \left(G^2 - \frac{N^2}{Pr} \right) \frac{k}{\varepsilon} - \frac{\varepsilon}{k} \right] k, \\ \frac{\partial \varepsilon}{\partial t} = \left[(c_s^0)^3 c_s^u \left(c_1^\varepsilon G^2 - c_3^\varepsilon \frac{N^2}{Pr} \right) \frac{k}{\varepsilon} - c_2^\varepsilon \frac{\varepsilon}{k} \right] \varepsilon \end{cases}$$

$k - \omega$

$$\begin{cases} \frac{\partial k}{\partial t} = \left[\frac{c_s^u}{c_s^0} \left(G^2 - \frac{N^2}{Pr} \right) \frac{1}{\omega} - (c_s^0)^4 \omega \right] k \\ \frac{\partial \omega}{\partial t} = \left[\frac{c_s^u}{c_s^0} \left(c_1^\omega G^2 - c_3^\omega \frac{N^2}{Pr} \right) \frac{1}{\omega} - c_2^\omega (c_s^0)^4 \omega \right] \omega \end{cases}$$

$$\begin{aligned}v_\rho &\equiv v_T \equiv v_s = \frac{\nu_u}{Pr}, \\ Pr &= c_s^u / c_s^T\end{aligned}$$

k - ω model

The analytical solution has the form

$$\omega = -r_d(r_m + r_p a_r) / (r_m - r_p a_r),$$

$$k = k^0 \left(\frac{E_1}{E_2} \right)^{\frac{A}{2B}} \left(\frac{E_3}{E_4} \right)^{\frac{D}{2C}},$$

$$r_d = \sqrt{B/C}, \quad r_m = \omega^0 - r_d,$$

$$r_p = \omega^0 + r_d, \quad a_r = \exp(2\sqrt{BC} \cdot t),$$

$$E_1 = (r_m + r_p a_r)^2, \quad E_2 = 4(\omega^0)^2 a_r,$$

$$E_3 = 4r_d^2 a_r, \quad E_4 = (r_m - r_p a_r)^2.$$

If $t \rightarrow \infty$ then $\omega \rightarrow \sqrt{B/C}$

The asymptotic solution is

$$\omega = \sqrt{B/C}, \quad k = k^0 \exp(\gamma t),$$

$$\gamma = \frac{A}{\omega} - D\omega = \frac{AC - BD}{\sqrt{BC}}$$

$$\begin{cases} \frac{\partial \omega}{\partial t} = B - C\omega^2, \\ \frac{\partial k}{\partial t} = \left(\frac{A}{\omega} - D\omega \right) k, \end{cases}$$

$$A = \frac{1}{c_s^0} (c_s^U G^2 - c_s^\rho N^2) = \frac{c_s^U}{c_s^0} \left(G^2 - \frac{1}{\text{Pr}} N^2 \right),$$

$$B = \frac{1}{c_s^0} (c_1^\omega c_s^U G^2 - c_3^\omega c_s^\rho N^2) = \frac{c_s^U}{c_s^0} \left(c_1^\omega G^2 - \frac{c_3^\omega}{\text{Pr}} N^2 \right),$$

$$C = c_2^\omega (c_s^0)^4, \quad D = (c_s^0)^4,$$

$$t^j < t < t^{j+1},$$

$$\omega(t = t^j) = \omega^0,$$

$$k(t = t^j) = k^0.$$

$$B, C, D > 0$$



$$\frac{\partial \ln k}{\partial t} = A / \tilde{\omega} - D \tilde{\omega},$$

$$\frac{\partial \ln \tilde{\omega}}{\partial t} = B / \tilde{\omega} - C \tilde{\omega},$$

$$\tilde{\omega} = \varepsilon / k / \left(c_s^0\right)^3 / c_s^u,$$

$$A = G^2 - N^2 / Pr, \quad D = \left(c_s^0\right)^3 c_s^u,$$

$$B = \left(c_1^\varepsilon - 1\right) G^2 - \left(c_3^\varepsilon - 1\right) N^2 / Pr,$$

$$C = \left(c_s^0\right)^4 \left(c_2^\varepsilon - 1\right),$$

$$t^j < t < t^{j+1},$$

$$\varepsilon(t = t^j) = \varepsilon^0,$$

$$k(t = t^j) = k^0.$$

The analytical solution has the form

$$k = k^0 \frac{\left[1 + \operatorname{th}(\sqrt{BC} t) \left(D \sqrt{B/C} k^0 / \varepsilon^0\right)\right]^{A/B}}{\left[1 + \operatorname{th}(\sqrt{BC} t) / \left(D \sqrt{B/C} k^0 / \varepsilon^0\right)\right]^{D/C}},$$

$$\varepsilon = \sqrt{\frac{B}{C}} \frac{\left(\varepsilon^0 / k^0\right) + D \sqrt{B/C} \operatorname{th}(\sqrt{BC} t)}{\left(\varepsilon^0 / k^0\right) \operatorname{th}(\sqrt{BC} t) + D \sqrt{B/C}} D k.$$

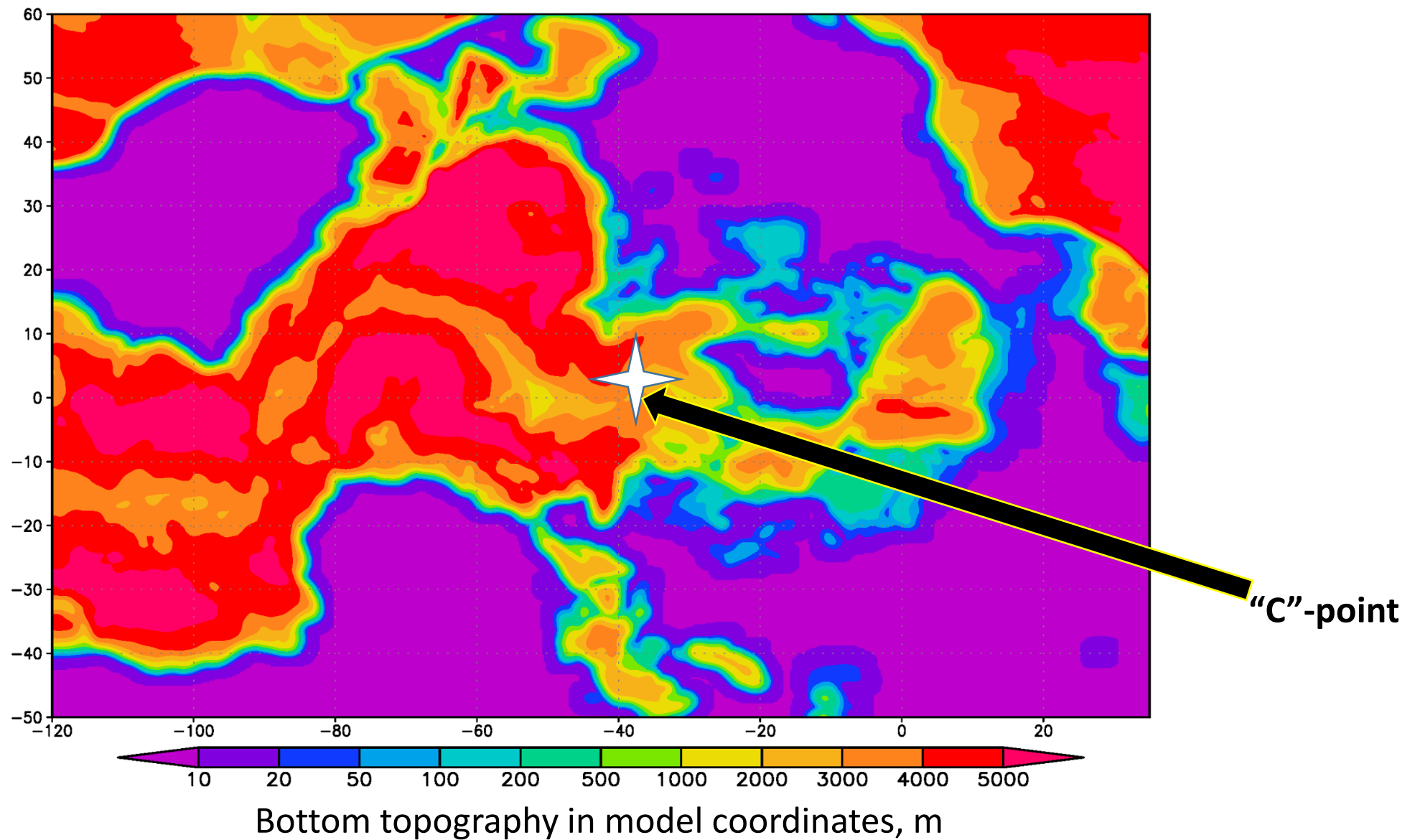
Numerical experiments and simulation results

The main purpose of numerical experiments is to assess the influence of the two turbulence parametrizations $k-\varepsilon$ and $k-\omega$ on the structure of modeled hydrophysical fields. The simulations were carried out using the INM RAS OGCM known as INMOM (Institute of Numerical Mathematics) with built-in $k-\varepsilon$ and $k-\omega$ subsystems and with the stability functions

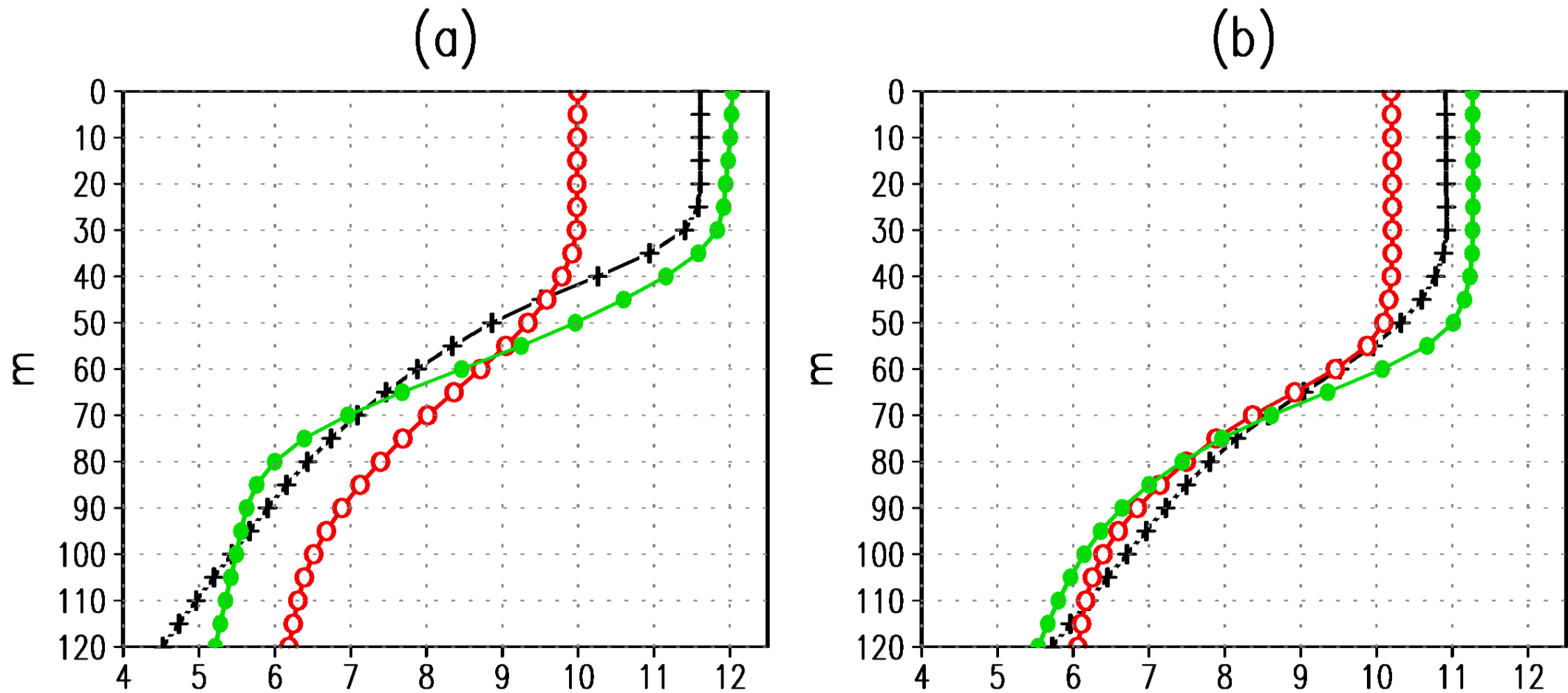
$$\frac{c_s^u}{c_s^0} = \frac{1 + 8\text{Ri}^2}{1 + 2.3\text{Ri} + 35\text{Ri}^2}, \quad \frac{c_s^T}{c_s^0} = \frac{1.4 - 0.01\text{Ri} + 1.29\text{Ri}^2}{1 + 2.44\text{Ri} + 19\text{Ri}^2}, \quad \text{Ri} = N^2 / G^2$$

Sukoriansky, S., Galperin, B. and Perov, V., 2005. Application of a New Spectral Theory of Stably Stratified Turbulence to the Atmospheric Boundary Layer over Sea Ice, Boundary-Layer Meteorology, 117(2), pp. 231-257. doi:10.1007/s10546-004-6848-4

- 1) The simulation area includes the Atlantic Ocean northward of 30° S, the Arctic Ocean and the Bering Sea.
- 2) Coordinate system is rotated sphere. Coordinates of poles: (0°N, 60°E) and (0°N, 120°W). The new “Equator” is aligned along the meridian 30°W and passes through the North Geographical Pole. This allows us to obtain a quasi-uniform grid along the whole simulated area.
- 3) The grid step is 0.25° in new “latitude” and “longitude”, 40 sigma levels are set in vertical with the refinement near the ocean surface.
- 4) Boundary conditions at the ocean surface were calculated using the atmospheric characteristics according to the CORE-II (Datasets for Common Ocean-ice Reference Experiments – Phase II) data [Large, W.G. and Yeager, S.G., 2009. The global climatology of an interannually varying air–sea flux data set. *Climate Dynamics*, 33(2–3), pp. 341-364. doi:10.1007/s00382-008-0441-3]
- 5) The simulations were carried out from January 1, 1976 to December 31, 1977.
- 6) January climatic fields of the ocean temperature and salinity, the rest state sea ice null are taken as initial conditions.
- 7) The turbulence equations at the generation–dissipation stage are solved analytically with the time step 1 hour which is equal to the OGCM main time step.

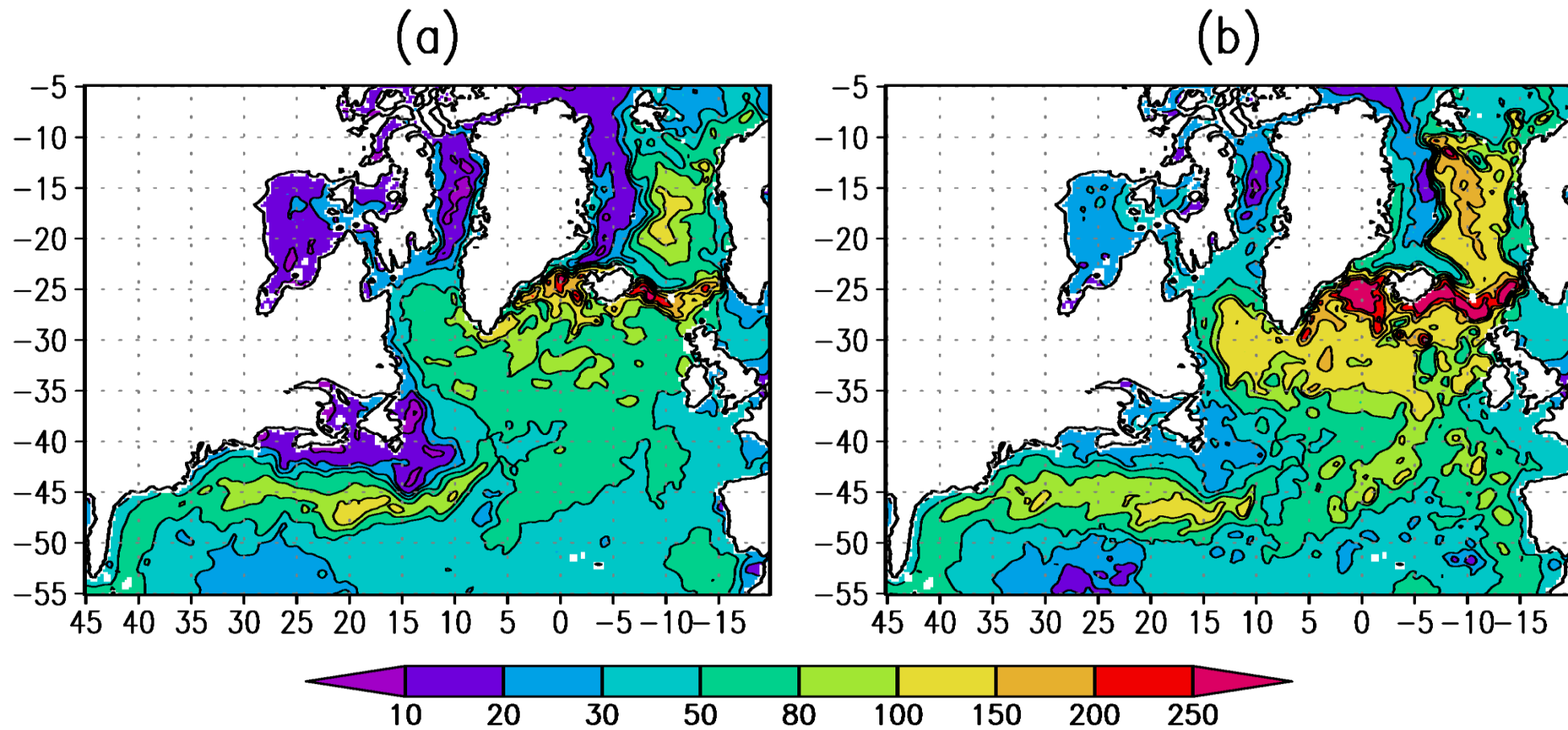


We compare the simulation results with the observational data for “C” ocean weather station (OWS) with coordinates 52.75°N, 35.5°W, where the upper layer probing was carried out 8 times a day [A collection of climatological and statistical data at the ocean station “C” (52° 45’ N, 35° 30’ W) for 1976–1980 period. Section 1. Oceanographic and hydrochemical observations. Obninsk, 1984. 338 p.]



Temperature profiles averaged for September, 1–10 (a) and October, 1–10, (b) 1977.
 Green line marks observational data from the oceanic weather station "C";
 black line marks the $k - \omega$ model results;
 red line marks the $k - \varepsilon$ model results.

For the month under consideration the observed temperature of the upper quasi-homogeneous layer (UQL) decreased by 1.75°C. The simulations indicate that the UQL deepening due to using the $k - \varepsilon$ model is noticeably larger in comparison with both the observational data and the $k - \omega$ model. The UQL temperature is reproduced better when using the $k - \omega$ parameterization.



Upper Quasihomogeneous Layer (UQL) thickness in the model coordinates (rotated counterclockwise for more convenient view) on October 10, 1977: North Atlantic, Norwegian and Greenland Seas.

a represents the $k - \omega$ model results;

b represents the $k - \varepsilon$ model results.

The water potential density in the UQL differs from the one at the ocean surface by less than 0.15 kg/m^3

All the qualitative features of the UQL thickness distribution coincide in both cases. However, the $k - \varepsilon$ parametrization (compared to the $k - \omega$) in most of the water area leads to stronger mixing. That is, in the initial period of free convection the UQL thickness, which is an important characteristic of vertical mixing, is sensitive to the choice of parameterization.

Conclusions



1. The application of the $k - \varepsilon$ and $k - \omega$ parameterizations of turbulent mixing in the INM RAS ocean general circulation model INMOM is described. When calculating the parameters of turbulence models, the stability functions are used based on the spectral algorithm.
2. The method of splitting with respect to physical processes is used for the numerical solution of turbulence equations. A feature of the method is the exact solution of the split equations at the generation – dissipation stage.
3. It is shown that the structure of large-scale fields of the North Atlantic – the Arctic Ocean is sensitive to the choice of vertical turbulence models. So, for example, in the $k - \varepsilon$ model the rate of seasonal pycnocline water involvement into the zone of developed turbulence is noticeably higher than when using the $k - \omega$ model.

References

Moshonkin, S., Zalesny, V. and Gusev, A. Simulation of the Arctic – North Atlantic Ocean Circulation with a Two-Equation K-Omega Turbulence Parameterization. *J. Mar. Sci. Eng.* 2018, V.6, article No. 95, doi:10.3390/jmse6010001

Zalesny, V.B., Moshonkin, S.N., Perov, V.L. and Gusev, A.V., 2019. Ocean circulation modeling with k-omega and k-epsilon parameterizations of vertical turbulent exchange. *Physical Oceanography*, [e-journal] 26(6), pp. 455-466.

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That's all !!!



Thanks!

