Langmuir circulation without wind or surface waves

Shear flow interacting with wavy topography

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One-slide summary

• Streamwise vortices occur in fluid flow over surface with crossing-wave pattern.

• Kinematically equivalent to Langmuir circulation (the «CL1» mechanism, Craik 1970).

• Analytical theory presented for Langmuir rolls during
  • the early onset, and
  • the final, steady-state.

• Explicit expressions when velocity profile is a power law \( U(z) = z^q \).

• Confirmed by Lattice-Boltzmann simulations.
Introduction

- Full details of this work may be found in the forthcoming paper (arXiv)

Langmuir-type vortices in boundary layers driven by a criss-cross wavy wall topography

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Introduction

• Conventional Langmuir circulation
  • Occurs due to interaction of waves and near-surface shear\(^1\)
  • Observable as «windrows» on surface, where foam etc gathers in downwelling areas.
  • Important contributor to mixing in the upper ocean\(^2\).

• Neither wind nor surface waves are necessary
  • The Craik-Leibovich mechanisms are purely kinematic interaction between mean shear and wavy fluid motion.

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Theory (outline)

• Follows roughly the procedure of Craik 1970\(^1\). See our manuscript for full details.

• The wavy boundary introduces **perturbations** to the mean shear profile, treated up to **second order** in the wall corrugation steepness.

• Steps:
  • Assume known background flow \(U(z)\).
  • Derive (approximate) **linear order** perturbation based on a simplifying model
  • There are 4 second-order modes due to self-advection.
  • One of these modes is resonant, growing linearly with time. It has the form of longitudinal vortices, or «rolls».

• Some more details on the theory and derivation are found on later slides.

Theory step 1: approximate linear solution

- Model: model the real (no-slip) wall by a displacement thickness of the same shape, creating a free slip, impermeable boundary.
  - Captures all essentials of this kinematic effect
- Treat first-order perturbation as steady and inviscid
  - Viscosity’s primary effect is to create mean shear $U(z)$ and displacement.
- Linearising Euler equation w.r.t. perturbations and eliminating velocities gives a Rayleigh equation for the first-order pressure perturbation $p_1(k, z)e^{i k \cdot r}$:

$$p_1'' - 2 \frac{U'}{U} p_1' - k^2 p_1 = 0$$

and boundary conditions at bottom/top of domain.
- Easily solved numerically. Analytical solution for power law case $U = z^q$
- 1st order velocities given by

$$u = -\frac{p_1}{U} - \frac{U' p_1'}{k_x U^2}, \quad v = -\frac{k_y p_1}{k_x U}, \quad w = \frac{i p_1'}{k_x U}. $$
Navier-Stokes equation at 2nd order reduces to

\[ (\partial_t - Re^{-1} \nabla^2) \nabla^2 w = \mathcal{R}(z) ; \]

with

\[ \mathcal{R}(z) = 8 \frac{k_{1x}^2 U'}{k_{1x}^2 U_0^3} \left[ \left( k_{1x}^2 - k_{1y}^2 \right) p_1^2 + (p_1')^2 \right]. \]

Second order harmonics from sums & differences of wave vectors (±\(k_x\), ±\(k_y\)).

Modes with purely spanwise wave number are resonant. Other 2\textsuperscript{nd} order harmonics are negligible.

Solutions can be found in two cases:

1. Early onset, transient growth
   - Initially the 2nd order motion is transient and inviscid.

2. Ultimate steady state
   - Eventually vortices are stabilised by viscosity, reaching a viscous and steady state.
**Theory result: early onset transient growth**

- Initial growth assumed essentially inviscid; set $\text{Re}^{-1}=0$.
- Results in
  \[ w = \sum_{\pm} d^\pm e^{\pm \kappa z} + w_x(z); \]

  with particular solution growing **linearly** in time:
  \[ w_x(z) = \frac{t}{\kappa} \int_0^z d\xi \mathcal{R}(\xi) \sinh \kappa(z - \xi). \]

  ($d^\pm$ chosen to satisfy boundary conditions)

- Streamwise velocity increases **quadratically** with $t$:
  \[ u(z, t) = -\frac{t}{2} U'(z) w(z, t). \]

- Spanwise velocity:
  \[ v = \frac{i w'}{\kappa_y} \]

Downwelling towards crest/trough-line, upwelling from saddlepoint-line
Theory result: ultimate, steady state solution

- Set transient term in Navier-Stokes to zero. Again a simple solution:

\[ w = \sum_{\pm} (d_0^\pm + zd_1^\pm) e^{\pm \kappa z} + w_x(z); \]

\[ w_x(z) = \frac{Re}{2\kappa^3} \int_0^z d\xi \, R(\xi) G(\kappa(z-\xi)); \]

\[ G(Z) = \sinh(Z) - Z \cosh(Z). \]

- As before, \( v = i w'/\kappa_y \), and we find the streamwise 2nd order velocity*

\[ u(z) = \sum_{\pm} d_u^\pm e^{\pm \kappa z} + \frac{Re}{\kappa} \int_0^z d\xi \, U'(\xi) w(\xi) \sinh(\kappa(z-\xi)). \]

Note that spanwise and vertical velocities \( u,w \) scale as \( Re \), but the streamwise velocity \( u \) scales as \( Re^2 \).

- Initial growth closer to the wall; vortices moving towards the bulk before steady state.

Example: «deep» water (upper boundary far away):

Initial growth rate

Ultimate steady state

* see manuscript for explicit expressions for the \( d \)-coefficients
Theory result: dependence on crossing angle $\theta$

- Circulation strongest for "protracted eggcarton", $\theta \sim 10^\circ - 20^\circ$

![Graphs showing initial and ultimate vortex circulation growth rate and velocity profiles](image-url)

- Initial growth rate greatest around $20^\circ$.
- Vortices can change sign at large $\theta$, but are then weaker.
- Ultimate vortex strength peaks around $10^\circ$.
- Relatively weak circulation at $45^\circ$.
- Vortices can change sign at large $\theta$, but are then weaker.

Different velocity profiles, same general trend.
Confirmation by numerical simulation

- The phenomenon is confirmed by simulation (Lattice-Boltzmann method) of laminar flow between two plates with crossing-wave pattern.

- Tested effect of amplitude, Re and spanwise phase shift.

\[ \theta = 22.5^\circ, \, Re \approx 400. \]
Confirmation by numerical simulation

- The phenomenon is confirmed by simulation (Lattice-Boltzmann method) of laminar flow between two plates with crossing-wave pattern.

- Tested effect of amplitude, Re and spanwise phase shift.

Vortices become unstable at high Reynolds numbers. Here Re = 1600; Geometry: \( \theta = \pi/2, \Theta = 22.5^\circ, h = \lambda/2 \).