Langmuir circulation without wind or surface waves

Shear flow interacting with wavy topography

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One-slide summary

- Streamwise vortices occur in fluid flow over surface with crossing-wave pattern.
- Kinematically equivalent to Langmuir circulation (the «CL1» mechanism, Craik 1970).
- Analytical theory presented for Langmuir rolls during
  - the early onset, and
  - the final, steady-state.
- Explicit expressions when velocity profile is a power law $U(z) = z^q$.
- Confirmed by Lattice-Boltzmann simulations.
Introduction

- Full details of this work may be found here: arXiv:2005.00317v1 [physics.flu-dyn]
Introduction

• Conventional Langmuir circulation
  • Occurs due to interaction of waves and near-surface shear\(^1\)
  • Observable as «windrows» on surface, where foam etc gathers in downwelling areas.
  • Important contributor to mixing in the upper ocean\(^2\).

• Neither wind nor surface waves are necessary
  • The Craik-Leibovich mechanisms are purely kinematic interaction between mean shear and wavy fluid motion.

Theory (outline)

- Follows roughly the procedure of Craik 1970\(^1\). See our manuscript for full details.

- The wavy boundary introduces **perturbations** to the mean shear profile, treated up to **second order** in the wall corrugation steepness.

- Steps:
  - Assume known background flow \(U(z)\).
  - Derive (approximate) **linear order** perturbation based on a simplifying model.
  - There are 4 second-order modes due to self-advection.
  - One of these modes is resonant, growing linearly with time. It has the form of longitudinal vortices, or «rolls».

- Some more details on the theory and derivation are found on later slides.

Theory step 1: approximate linear solution

• Model: model the real (no-slip) wall by a displacement thickness of the same shape, creating a free slip, impermeable boundary.
  • Captures all essentials of this kinematic effect
• Treat first-order perturbation as steady and inviscid
  • Viscosity’s primary effect is to create mean shear $U(z)$ and displacement.
• Linearising Euler equation w.r.t. perturbations and eliminating velocities gives a Rayleigh equation for the first-order pressure perturbation $p_1(k, z) e^{i k \cdot \mathbf{r}}$:

$$p_1'' - 2 \frac{U'}{U} p_1' - k^2 p_1 = 0$$

and boundary conditions at bottom/top of domain.
• Easily solved numerically. Analytical solution for power law case $U = z^q$
• 1st order velocities given by

$$u = -\frac{p_1}{U} - \frac{U'}{k_x^2 U^2}, \quad v = -\frac{k_y p_1}{k_x U}, \quad w = \frac{ip_1'}{k_x U}.$$
Theory step 2: resonant 2\textsuperscript{nd} order mode

- Navier-Stokes equation at 2nd order reduces to

\[
(\partial_t - Re^{-1} \nabla^2) \nabla^2 w = \mathcal{R}(z);
\]

with

\[
\mathcal{R}(z) = 8 \frac{k_{1y}^2}{k_{1x}^2} \frac{U'}{U^3} \left[ (k_{1x}^2 - k_{1y}^2) p_1^2 + (p_1^2)^2 \right].
\]

- Second order harmonics from sums & differences of wave vectors (±k\textsubscript{x}, ±k\textsubscript{y}).

- Modes with purely spanwise wave number are resonant. Other 2\textsuperscript{nd} order harmonics are negligible.

- Solutions can be found in two cases:

1. Early onset, transient growth
   - Initially the 2nd order motion is transient and inviscid.

2. Ultimate steady state
   - Eventually vortices are stabilised by viscosity, reaching a viscous and steady state.
Theory result: early onset transient growth

- Initial growth assumed essentially inviscid; set $\text{Re}^{-1}=0$.
- Results in

$$w = \sum \pm d \pm e^{\pm \kappa z} + w_x(z);$$

with particular solution growing linearly in time:

$$w_x(z) = \frac{t}{\kappa} \int_0^z d \xi \mathcal{R}(\xi) \sinh \kappa(z - \xi).$$

($d^\pm$ chosen to satisfy boundary conditions)

- Streamwise velocity increases quadratically with $t$:

$$u(z, t) = -\frac{t}{2} U'(z) w(z, t).$$

- Spanwise velocity:

$$v = i w' / \kappa y$$

Downwelling towards crest/trough-line, upwelling from saddlepoint-line
Theory result: ultimate, steady state solution

- Set transient term in Navier-Stokes to zero. Again a simple solution:

\[ w = \sum_{\pm} (d_0^{\pm} + z d_1^{\pm}) e^{\pm \kappa z} + w_\infty(z); \]

\[ w_\infty(z) = \frac{Re}{2\kappa^3} \int_0^z d\xi \mathcal{R}(\xi) G[\kappa(z - \xi)]; \]

\[ G(Z) = \sinh(Z) - Z \cosh(Z). \]

- As before, \( v = i w'/\kappa_y \), and we find the streamwise 2nd order velocity:

\[ u(z) = \sum_{\pm} d_u^{\pm} e^{\pm \kappa z} + \frac{Re}{\kappa} \int_0^z d\xi U'(\xi) w(\xi) \sinh \kappa(z - \xi). \]

Note that spanwise and vertical velocities \( u, w \) scale as \( Re \), but the streamwise velocity \( u \) scales as \( Re^2 \).

- Initial growth closer to the wall; vortices moving towards the bulk before steady state.

**Example:** «deep» water (upper boundary far away):

- Initial growth rate
- Ultimate steady state

* see manuscript for explicit expressions for the \( d \)-coefficients
Theory result: dependence on crossing angle $\theta$

- Circulation strongest for «protracted eggcarton», $\theta \sim 10^\circ - 20^\circ$

![Initial growth rate greatest around 20°](image1)

- Vortices can change sign at large $\theta$, but are then weaker

![Ultimate vortex strength peaks around 10°](image2)

- Ultimate steady state around $\theta = 10^\circ$

- Relatively weak circulation at $\theta = 45^\circ$

- Vortices can change sign at large $\theta$, but are then weaker

- Initial growth rate greatest around $\theta = 20^\circ$

- Ultimate steady state around $\theta = 10^\circ$

- Different velocity profiles, same general trend.
Confirmation by numerical simulation

• The phenomenon is confirmed by simulation (Lattice-Boltzmann method) of laminar flow between two plates with crossing-wave pattern.

• Tested effect of amplitude, Re and spanwise phase shift.

θ = 22.5°, Re ≈ 400.
The phenomenon is confirmed by simulation (Lattice-Boltzmann method) of laminar flow between two plates with crossing-wave pattern.

Tested effect of amplitude, Re and spanwise phase shift.

Vortices become unstable at high Reynolds numbers. Here Re = 1600; Geometry: , , , , .