A NEW AND EFFICIENT PROCEDURE FOR DISPERSIVE TSUNAMI SIMULATIONS ON SPHERICAL COORDINATES BASED ON A HYPERBOLIC APPROACH

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J. Macías, S. Lorito, and F. Romano
**Goal**: Efficient model implementations for tsunami propagation and coastal-type computations.

**Requirements:**

- **Simple** models as the hydrostatic Shallow water equations (SW).
- Accounting for **dispersive** effects: Stokes - Airy theory

\[
Celerity_{Airy}^2 = gH \frac{\tanh (kH)}{kH}.
\]

Hydrostatic Shallow Water equations:

\[
Celerity_{SW}^2 = gH \quad \text{Inaccurate}.
\]

- **Faster Than Real Time (FTRT):**

**Tsunami-HySEA**: [Link]

- FTRT multi-GPU SW solver
- Hydrostatic - non dispersive
- Aim: include dispersive effects
Shallow water -non dispersive- results


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Hydrostatic models, such as SW, do not take into account dispersive effects: Wrong propagation speed. Inaccurate arrival amplitude and time of the wave

Dispersive models

Two prominent families of systems for the simulation of dispersive water waves:

**Boussinesq type:**
- Boussinesq (1872)
- Peregrine (1967)
- Madsen (1992)
- Usually contains **high-order derivatives** in the final model
- Unaffordable complexity for 2D domains.

**Non-hydrostatic pressure:**
- Casulli *et al* (1995)
- Bristeau *et al* (2008)
- Fernández-Nieto *et al* (2017)
- First order systems
- Simple systems

**Non-hydrostatic** solvers are capable of solving many relevant features of coastal water waves such as:

- Dispersive water waves **propagation**
- Shoaling
- Non-linearities
- Refraction
- Run-up
- Run-down
Non-Hydrostatic pressure model


\[
\begin{align*}
\partial_t h + \partial_x (hu) &= 0, \\
\partial_t (hu) + \partial_x \left( hu^2 + \frac{1}{2} gh^2 + hp \right) &= (gh + \gamma p) \partial_x H, \\
\partial_t (hw) + \partial_x (uhw) &= \gamma p, \\
\partial_x u + \frac{w - w_b}{h/2} &= 0, \quad w_b = -\partial_t H - u \partial_x H
\end{align*}
\]

- Improved dispersion relation valid from intermediate to shallow-waters
- For \( \gamma = 2 \), the model is similar to Yamazaki et al system (NEOWAVE code)
- For \( \gamma = 3/2 \), the model reduces to the Green-Naghdhi system

\[
C_{NH}^2 = gH \frac{1}{1 + \frac{1}{2\gamma} (kH)^2}.
\]

We choose \( \gamma = 2 \).
Non-Hydrostatic implementation. Escalante, Castro and Morales\textsuperscript{1}

Numerical implementation of the previous system.

Pros:

- Efficiently solved for 2D domains on GPU (Escalante\textit{ et al} 2018\textsuperscript{1}). Hybrid finite-volume finite-difference 2nd order scheme.
- **Computational cost**: 2 times more expensive than a one-GPU hydrostatic Shallow Water solver.

Cons:

- The system is not hyperbolic and the numerical scheme involves the solution of linear systems
- The computational cost increases for high-order schemes
- Not amenable for simple multi-GPU implementations

The difficulties arise from the imposition of the divergence condition:

\[ \partial_x u + \frac{w - w_b}{h/2} = 0, \quad w_b = -\partial_t H - u \partial_x H. \]

That will lead to developing implicit numerical schemes. Otherwise, a quite restrictive time-step CFL condition must be considered. The same difficulties arise for Boussinesq-type systems.

Solitary wave on a plane beach. Hydrostatic approach - SW

Solitary wave on a plane beach. Non-hydrostatic approach

Solitary wave impinging on a small-scale model of Seaside, Oregon

Benchmark problem. $2^{nd}$ order hybrid finite-volume finite-difference scheme

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Solitary wave impinging on a small-scale model of Seaside, Oregon

Benchmark problem. 2\textsuperscript{nd} order hybrid finite-volume finite-difference scheme

Solitary wave impinging on a small-scale model of Seaside, Oregon

- Benchmark problem. 2\textsuperscript{nd} order hybrid finite-volume finite-difference scheme

- Computational time: 2-3 times slower than a one-GPU SW simulation.

A novel hyperbolic relaxation non-hydrostatic pressure system

A more efficient approach is developed in Escalante, Castro and Dumbser, 2019\(^1\). The **key idea** is to replace the divergence condition:

\[
\partial_x u + \frac{w - w_b}{h/2} = 0, \quad w_b = -\partial_t H - u\partial_x H.
\]

by the relaxed equation:

\[
\partial_t (hp) + \partial_x (uhp) + hc^2 \left( \partial_x u + \frac{w - w_b}{h/2} \right) = 0, \quad w_b = -\partial_t H - u\partial_x H.
\]

Here, divergence errors are transported with a wave speed \(c = \alpha \sqrt{gH_0}\), \(H_0\) being the typical depth.

The resulting system is **hyperbolic** and the time-step CFL condition is similar to the SW system:

\[
\Delta t \leq \frac{\Delta x}{\lambda_{\text{max}}}, \quad \lambda_{\text{max}} = |u| + \sqrt{gh + p + c^2}.
\]

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A novel hyperbolic relaxation non-hydrostatic pressure system¹

Pros:

- The system is hyperbolic: No need of solving linear systems
- Can be discretized by any efficient high-order and explicit numerical scheme.
- Simple and straightforward multi-GPU implementations
- Preserves the dispersion relation from the original non-hyperbolic system
- The same model recovers hydrostatic results (classical SW equations) by setting $\alpha = 0$.

Relative error of the phase velocity with respect to the Airy theory for different values of the relaxation parameter: $\alpha = 3, 5, 10$ given in blue, green and magenta respectively. In practice we choose $\alpha = 3$.

Real-world Tsunami simulations

- The 2D implementation of the non-hyperbolic non-hydrostatic pressure system,
- and the 2D implementation of the hyperbolic non-hydrostatic pressure system,

may suffer the lack of Earth-curvature effects for simulations on bigger scenarios.

To account for curvature effects:
- We consider the non-hydrostatic pressure formulation for spherical coordinates,
- similar to the one given for the NEOWAVE code (Yamazaki et al, 2010).
- Then, we propose a similar hyperbolic approach technique to relax the divergence constraint, and obtaining a hyperbolic model.
The 2014 Iquique earthquake. Comparison with NEOWAVE

The real test case to solve using a one-GPU implementation

- The domain: west coast of northern Chile. Grid resolution: 1 arc-min.
- Size: $2880 \times 1800 = 5184000$ cells.
- Wall clock to be simulated: 10000 seconds $\approx 2$ hours 45 minutes.
- We are interested in compare the Green’s functions obtained in a subfault of $20 \times 20$ km size (corresponding to the 2014 Iquique earthquake) time series provided by Dart buoys in 3 locations against the computed numerical simulations from the new hyperbolic non-hydrostatic model.
The 2014 Iquique earthquake. Comparison with NEOWAVE

The real test case to solve using a one-GPU implementation

- Hydrostatic SW simulation: $\alpha = 0$
- Non-hydrostatic simulation: $\alpha = 3$
- Numerical scheme: Finite-Volume 3\textsuperscript{rd} order TVD Runge-Kutta in time and 3\textsuperscript{rd} order CWENO in space.

Free-surface at time 1500 s.
Hydrostatic SW simulation ($\alpha = 0$).

Free-surface at time 1500 s.
Non-Hydrostatic hyperbolic simulation ($\alpha = 3$).
The 2014 Iquique earthquake. Comparison with NEOWAVE

The real test case to solve using a one-GPU implementation

- Hydrostatic SW simulation: $\alpha = 0$
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Free-surface at time 4500 s.
Hydrostatic SW simulation ($\alpha = 0$.)

Free-surface at time 4500 s.
Non-Hydrostatic hyperbolic simulation ($\alpha = 3$.)
The 2014 Iquique earthquake Time series. Comparison with NEOWAVE

C. Escalante

Dispersion Water Waves. A Hyperbolic Approach

EGU 2020
The 2014 Iquique earthquake Time series. Comparison with NEOWAVE
The 2014 Iquique earthquake Time series. Comparison with NEOWAVE

\[ \eta (cm) \]

-2
0
2
0 1000 2000 3000 4000 5000 6000 7000 8000 9000 10000

\[ t (s) \]

\[ \eta (cm) \]

-1
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1
0 1000 2000 2500 3000 3500 4000 4500 5000 5500 6000

\[ t (s) \]

\[ \eta (cm) \]

-1
0
1
1500 2000 2500 3000 3500 4000 4500 5000 5500 6000

\[ t (s) \]
The 2014 Iquique earthquake. Computational effort

<table>
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<tr>
<th>Model</th>
<th>Comput. time</th>
<th># times FTRT</th>
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<tbody>
<tr>
<td>Hydrostatic SW ($\alpha = 0$)</td>
<td>659.29</td>
<td>15.17</td>
</tr>
<tr>
<td>Non-hydrostatic ($\alpha = 3$)</td>
<td>1271.92</td>
<td>7.86</td>
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Ratio computational times Non-Hydrostatic/SW: 1.93.

Computations performed with nVIDIA TESLA V100.
Conclusions and perspectives

- Towards an operational **FTRT** dispersive approach.
- Model includes **dispersive** effects and **curvature** effects,
- and efficient numerical implementations can be proposed due to its **hyperbolic** nature.
- Dispersive effects without solving linear systems.
- Validation: simple geometries (lab) and complex real cases.
- Computational times: around **two times** slower than a **SW** model. Single-GPU.
- Nested meshes and multi-**GPU** implementations can be straightforwardly implemented (Future work).
- Coupling with **landslides** models (Ongoing work).