An Experimental Study of the Systematic Underestimation of Wave Crests Measured by Lagrangian Buoys, and a Retrospective Correction Method

Mark L. McAllister and Ton S. van den Bremer
University of Oxford
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Lagrangian Measurement of Waves by Buoys

- Motivation
- Second-Order Motion of a Wave-Following Measurement Buoy
- Statistical Properties of Directionally Spread Ocean WavesMeasured by Buoys
- Approximate retrospective correction method for crest heights
Motivation
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- Buoys avoid large crest
- Lack or mooring compliance drags buoys under crests
- Low sampling rate misses crests
- Lagrangian motion “linearises” crests
- Instrumentation and signal processing
Second-Order Motion of a Wave-Following Measurement Buoy
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \eta_E = \eta_E^{(1)} + \eta_E^{(2)} + O(3) \]
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \eta_E = \eta_E^{(1)} + \eta_E^{(2)} + O(3) \]

\[ \eta_L^{(2)} = \eta_E^{(2)} + \Delta \eta_L^{(2)} \]
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \rho = 0 \]

\[ \eta_E = \eta_E^{(1)} + \eta_E^{(2)} + O(3) \]

\[ \eta_L^{(2)} = \eta_E^{(2)} + \Delta \eta_L^{(2)} \]

\[ \Delta \eta_L^{(1)} = \Delta x_H^{(1)} \cdot \nabla_H \eta^{(1)} \]

\( \rho \)
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \rho = 0 \]

\[ \rho \]

\[ \eta_E = \eta_E^{(1)} + \eta_E^{(2)} + O(3) \]

\[ \eta_L^{(2)} = \eta_E^{(2)} + \Delta \eta_L^{(2)} \]

\[ \Delta \eta_L^{(1)} = \Delta x_H^{(1)} \cdot \nabla H \eta^{(1)} \]

\[ \eta_E^{(2)} = -\Delta \eta_L^{(2)} \text{ (Deep water)} \]
Second-Order Motion of a Wave-Following Measurement Buoy

Eulerian $\eta = \frac{t}{\Delta \eta}$

Lagrangian $\eta = \eta^{(2)} + \Delta \eta^{(2)}$

$\Delta \eta^{(1)} = \Delta x^{(1)} \cdot \nabla \eta^{(1)}$

$\eta^{(2)}_E = -\Delta \eta^{(2)}$ (Deep water)
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \eta_E^{(1)} \quad \eta_L^{(1)} \]

\[ \eta_E^{(2)} \quad \eta_{L_+}^{(2)} \]

\[ \eta_E^{(2)} \quad \eta_{L_-}^{(2)} \]

Eulerian \( \eta = \) \[ \quad \]
Lagrangian \( \eta = \) \[ \quad \]
Second-Order Motion of a Wave-Following Measurement Buoy

- Diameter 0.07m ($D/\lambda_0 \ll 1$)
- Density $\approx 0.5 \rho_w$
- Highly-flexible taught mooring
- Depth 2 m ($k_0 d = 3 - 4$)
- Wave amplitude $\approx 0.2$ m ($k_0 a_0/k_p H_s \approx 0.3$)
Second-Order Motion of a Wave-Following Measurement Buoy

\[ \sigma_\theta = 10^\circ, \Delta \theta = 0^\circ \]

\[ \sigma_\theta = 10^\circ, \Delta \theta = 90^\circ \]

\[ \sigma_\theta = 10^\circ, \Delta \theta = 180^\circ \]
Second-Order Motion of a Wave-Following Measurement Buoy
Second-Order Motion of a Wave-Following Measurement Buoy

\[
\sigma_\theta^* = 17.6^\circ \\
\sigma_\theta^* = 38.6^\circ \\
\sigma_\theta^* \rightarrow \infty
\]

\[
\eta (m) 
\]

\[
\begin{align*}
(a) & \quad \bullet \eta_E \quad -\bullet \eta_L \\
(b) & \quad \bullet \eta_E \quad -\bullet \eta_L \\
(c) & \quad \bullet \eta_E \quad -\bullet \eta_L
\end{align*}
\]
Second-Order Motion of a Wave-Following Measurement Buoy

\( \sigma_\theta^* = 0.0^\circ \)  \hspace{1cm} \( \sigma_\theta^* = 7.8^\circ \)  \hspace{1cm} \( \sigma_\theta^* = 15.7^\circ \)  \hspace{1cm} \( \sigma_\theta^* = 28.0^\circ \)  \hspace{1cm} \( \sigma_\theta^* = 37.9^\circ \)  \hspace{1cm} \( \sigma_\theta^* = 47.6^\circ \)

\( \eta_\pm^{(2)} \) vs. \( t \) (s)

- \( \eta_{L,M} \)
- \( \eta_{L,T} \)
Statistical Properties of Directionally Spread Ocean Waves Measured by Buoys

\[ \sigma_\theta = 10^\circ \quad \sigma_\theta = 20^\circ \quad \sigma_\theta = 30^\circ \]
Statistical Properties of Directionally Spread Ocean Waves Measured by Buoys

\[ \sigma_\theta = 10^\circ \]

\[ \sigma_\theta = 20^\circ \]

\[ \sigma_\theta = 30^\circ \]

\[ \sigma_\theta = 20^\circ, \Delta \theta = 90^\circ \]

Probability of exceedance

\[ 10^0 \]

\[ 10^{-2} \]

\[ 10^{-4} \]

\[ \eta_c/H_s \]

\[ \eta_c/H_s \]

\[ \eta_c/H_s \]

\[ \eta_c/H_s \]
Statistical Properties of Directionally Spread Ocean Waves Measured by Buoys
Approximate retrospective correction method for crest heights

![Graph showing probability of exceedance vs. \( \eta_c/H_s \)]
Approximate retrospective correction method for crest heights

\[ H_\eta \left( \sigma_J \right) = \frac{1}{2} \left( 1 + e^{3L} \right) \]

\[ \tilde{H}^{(2)}(\sigma_\theta) = \frac{1}{2} \left( 1 + e^{-\sigma_\theta^2} \right) \]

\[ \tilde{H}^{(2)}(\Delta\theta) = \frac{1}{2} + \frac{1}{4} \left( 1 + \cos(\Delta\theta) \right) \]
Approximate retrospective correction method for crest heights

A directionally spread wave group

Two crossing wave group

\[ \tilde{\alpha}(2) \]

\[ \sigma_\theta \text{ (.deg)} \]

\[ \Delta \theta \text{ (.deg)} \]

HJ16

\[ \tilde{\eta}_L^{(2)} \]
Approximate retrospective correction method for crest heights

\[ \eta_c \approx \eta_c^{(1)} \]

\[ \eta_{\text{corr},c} = \eta_c + \eta_c^{(2)} \]
Approximate retrospective correction method for crest heights

\[ \eta_c \approx \eta_c^{(1)} \]

\[ \eta_{\text{corr},c} = \eta_c + \eta_c^{(2)} \]

\[ \eta_c^{(2)} = \frac{1}{2} (\eta_c)^2 k_0 \tilde{\eta}^{(2)}(\sigma_\theta, \Delta \theta)(1 - \varepsilon_{\text{hpf}}) \]

\[ \tilde{\eta}^{(2)}(\sigma_\theta) = \frac{1}{2} \left( 1 + e^{-\sigma_\theta^2} \right) \]

\[ \tilde{\eta}^{(2)}(\Delta \theta) = \frac{1}{2} + \frac{1}{4} (1 + \cos(\Delta \theta)) \]
Approximate retrospective correction method for crest heights
Conclusions

• In deep water (ocean waves), second-order Lagrangian motion causes the cancelation of super-harmonic (sigma theta -> 0) and an increase in sub-harmonic contribution to crest height
• O(2) effects alone will not result in a change to crest height, however, this constitutes a shifting of bound energy from low to high
• For deterministic extreme (non breaking) wave groups buoy motion is essentially purely Lagrangian

• Spectral parameters (Hs, Tp. Etc.) are not significantly different between buoys and gauge measurements
• Filtering slightly affects measured Hs, and significantly reduces measured skewness λ³
• Wave and crest height measured by buoys and gauges follow the same distributions

• Simplified expressions for second-order contribution to crest height can be used to retrospectively correct measurements and remove the effects of filtering

• These experiments do not consider a realistic mooring configuration, however, if a lack of mooring compliance was to cause an underestimation of crests we believe this would also affect measured wave heights
Thanks for your attention!

Spectral Parameters
Wave height
Amplitude/Period

(a) $a_L$ (m) vs. $a_E$ (m)

(b) $a_L^{(1)} / a_E^{(1)}$ vs. $\sigma_\theta$, $\Delta \theta$ (deg)

- $a(\sigma_\theta)$, $a_0 = 0.15$ m
- $a(\sigma_\theta)$, $a_0 = 0.2$ m
- $a(\Delta \theta)$, $a_0 = 0.15$ m
- $a^{(1)}(\sigma_\theta)$, $a_0 = 0.15$ m
- $a^{(1)}(\sigma_\theta)$, $a_0 = 0.2$ m
- $a^{(1)}(\Delta \theta)$, $a_0 = 0.15$ m
Frequency attenuation

(a)

(b)