Compatible finite element methods and parallel-in-time schemes for numerical weather prediction

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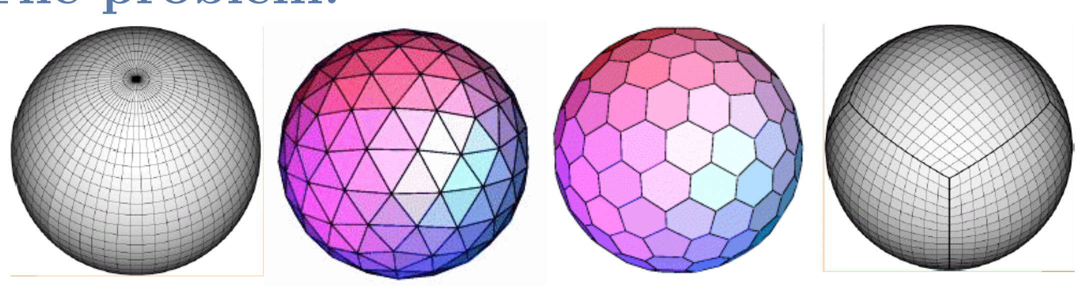
The Gung Ho Project (UK Met Office)

The problem:



The clustering of points at the pole of traditional lat-lon grids (left) is a bottleneck to parallel scaling. Can we use non-orthogonal grids such those on the right, while retaining the discretisation properties described in [1]?

The solution:

'mimetic' or 'compatible' finite element spatial discretisations that preserve discrete versions of the continuous vector calculus identities: div(curl)=0 and curl(grad)=0

Example: shallow water:

Prognostic variables velocity $\mathbf{u} \in \mathbb{V}_1$ and depth $D \in \mathbb{V}_2$. Vorticity $\zeta \in \mathbb{V}_0$ can be used to diagnose a consistent potential vorticity flux as in [2]

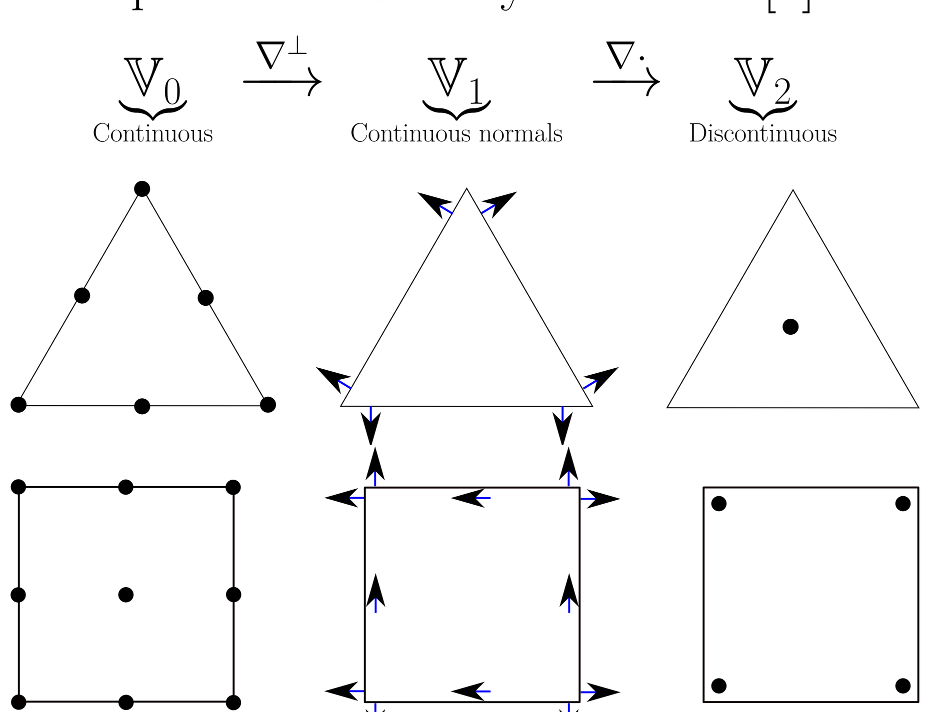


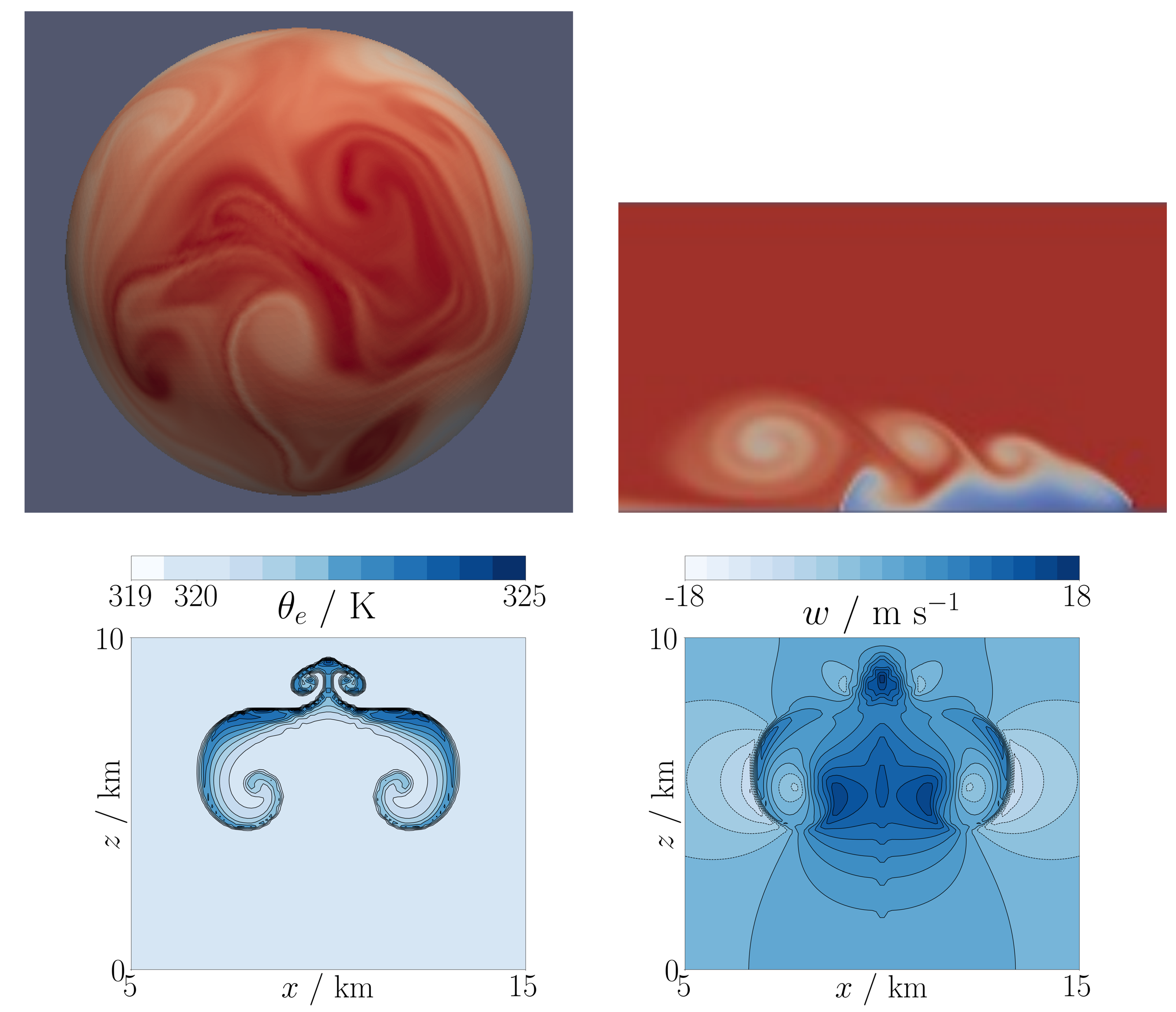
Figure 1:Top: lowest order function spaces on triangles; bottom: next-to-lowest-order function spaces on quadrilaterals

Gusto: the dynamical core toolkit

Gusto is dynamical core toolkit, built on top of the Firedrake finite element library, which enables rapid prototyping of algorithms based on the compatible finite element discretisations developed in the Gung Ho project.

- compatible function spaces dictated by the linear equations
- stable, accurate advection schemes available
- solves a range of GFD equations
- can be run in different geometries
- includes moist physics schemes

Some snapshots of simulation results can be seen on the right.



Top left: Potential vorticity from the shallow water flow over a mountain test case.
Top right: Potential temperature of a falling bubble that has spread across the bottom of the domain, developing a Kelvin-Helmholtz instability.
Bottom: Wet equivalent potential temperature and vertical velocity of a rising thermal in a cloudy atmosphere.

Parallel timestepping schemes

Motivation:

There is a limit to parallel speed up from spatial domain decomposition. Adding more processors leads to increased communication costs.

A solution:

Parallel exponential integrators The solution of the linear system is

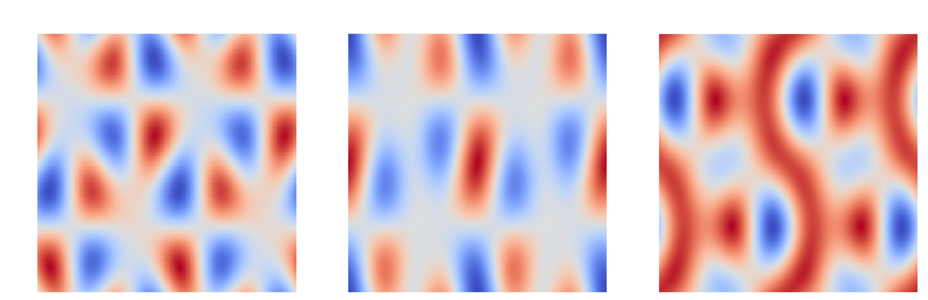
$$\boldsymbol{U}(t) = e^{-(t-t_0)\mathcal{L}}\boldsymbol{U}(t_0)$$

Compute the matrix exponential using the rational approximation

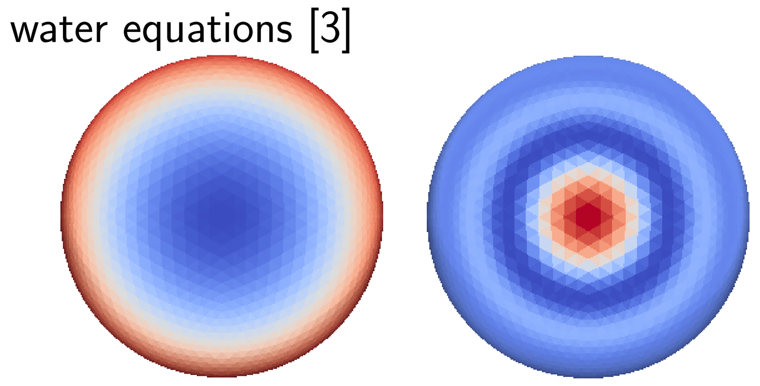
$$e^{\tau\mathcal{L}}\boldsymbol{U}(t_0) \approx \sum_{n=-N}^{N} \frac{\beta_n}{\tau\mathcal{L} + \alpha_n}\boldsymbol{U}(t_0)$$

where $\tau = t - t_0$, and $\alpha_n, \beta_n \in \mathbb{C}$ are given in []
Writing $\boldsymbol{V} = e^{\tau\mathcal{L}}\boldsymbol{U}(t_0)$ for each n we can solve ***in parallel***:

$$(\tau\mathcal{L} + \alpha_n)\boldsymbol{V}_n = \beta_n\boldsymbol{U}(t_0)$$



h, u and v for the wave benchmark for the linear f-plane shallow water equations [3]



left: h at $t = 36000s$, linear solid body rotation.
right: h at $t = 36000s$, wave propagation with initial conditions given by a Gaussian bump in h at north pole.

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Firedrake