**HILBERT**

- **Q1:** How to deal with nonlinear and nonstationary time series?
  
  A signal $s(t)$ written as [Huang et al., 1998]
  
  $$s(t) = \sum_{k=1}^{N_s} a_k \cos \varphi_k(t) + r(t)$$
  
  - $a_k(t)$ is the instantaneous amplitude
  - $\varphi_k(t)$ is the instantaneous phase
  - $c_k(t) = a_k(t) \cos(\varphi_k(t))$ is named empirical mode since derived via an iterative adaptive algorithm, i.e., the **sifting process**
  - $c_k(t)$ oscillates on a **typical timescale**

  $$\tau_k = \frac{1}{2\pi} \left( \frac{d}{dt} \varphi_k(t) \right)$$

  **MULTISCALE GENERALIZED FRACTAL DIMENSIONS**
  
  - a signal manifests a **multiscale** behavior $s(t) = \langle s \rangle + \sum \delta s_{\tau}(t)$
  - identify a **natural measure** $d\mu_{\tau}$ and a **partition function** $\Gamma_{\mu, \tau}(l)$
  - introduce a natural measure $d\mu_{\tau}$ and a partition function $\Gamma_{\mu, \tau}(l)$
  - we define

  $$D_{q,\tau} = \frac{1}{q \rightarrow 0} \lim_{\ell \rightarrow 0} \frac{\log \Gamma_{\mu, \tau}(B_{s,\tau}(\ell))}{\log \ell}$$

  **HENON MAP**
  
  - $x_{n+1} = 1 -ax_n^2 + y_n$
  - $y_{n+1} = bx_n$
  - Strange attractor: $a=1.4$ and $b=0.3$
  - $D_0 \approx D_2 \approx 1.24 \pm 0.03$
  - EMD extracted out $N_k = 11$ modes
  - $D_{s,\tau} \rightarrow D_q$ when $k \rightarrow k^*=7$
  - The phase-space portrait (black dots) is reproduced by 7 modes (red dots)
  - The geometrical and topological properties stored into a subset of “informative” empirical modes
  - **Monofractal** nature at all timescales

  **STANDARD MAP**
  
  - $p_{n+1} = p_n - k \sin(2\pi \theta_n)$
  - $\theta_{n+1} = \theta_n + p_{n+1}$
  - Chaotic nature: $K=1$
  - $D_q \in [0.5, 0.924]$ and $D_q \approx 0.87$
  - EMD extracted out $N_k = 11$ modes
  - $D_{s,\tau} \rightarrow D_q$ when $k \rightarrow k^*=6$
  - The phase-space portrait (black dots) is reproduced by 6 modes (red dots)
  - The geometrical and topological properties stored into a subset of “informative” empirical modes
  - **Monofractal** nature at all timescales

  **LR04 RECORD**
  
  - 57 deep sea sediment cores [Lisiecki & Raymo, 2005]
  - Paleoclimatic variability during the last 5.3 Myr
  - $D_{s,\tau} \rightarrow D_q$ when $k \rightarrow k^*=7$
  - $k=1 \rightarrow$ noise content
  - $k \in [2,7] \rightarrow$ less than 300 kyr $\rightarrow$ **multifractal**
  - $k \in [2,8] \rightarrow$ more predictable behavior
  - $k \geq 8$ $\rightarrow$ more predictable behavior
  - Results consistent with Shao & Ditlevsen, Nat. Comm., 2016

  **SYM-H INDEX**
  
  - Low-latitude geomagnetic activity [Iyemori, 1990]
  - Occurrence of geomagnetic storms (peak value $\sim 200$ nT)
  - $D_{s,\tau} \rightarrow D_q$ when $k \rightarrow k^*=5$
  - $D_{s,\tau} \rightarrow$ **multifractal** nature
  - $k \geq 6$ $\rightarrow$ more predictable behavior
  - **Internal** (more chaotic, $k \in [2,5]$) vs. **external** (more regular, $k \in [6,9]$) dynamics [Alberti et al., 2017]

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**Q2:** How to characterize a fractal structure of phase-space manifolds?

- let $s(t)$ be a signal whose trajectory belongs to a $D$-dimensional space $S^D$
- define a natural measure $d\mu_{\tau}(s(t))$ and the partition function $\Gamma_{\mu, \tau}(l)$
- let $B(l)$ be the hypercube of size $l$ centered at the point $s$ of $S^D$

Hentschel and Procaccia (1983) defined

$$D_q = \frac{1}{q \rightarrow 0} \lim_{\ell \rightarrow 0} \frac{\log \Gamma_{\mu, \tau}(B_{s,\tau}(\ell))}{\log \ell} = \frac{1}{q \rightarrow 0} \lim_{\ell \rightarrow 0} \frac{\log \sum_{l^q} p_l^q}{\log \ell}$$