MULTISCALE MEASURES OF PHASE-SPACE TRAJECTORIES

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Q1: How to deal with multiscale fluctuations in nonlinear and nonstationary time series?

HILBERT HUANG TRANSFORM (HHT)

GENERALIZED FRACTAL DIMENSIONS (D_q)

Q2: How to characterize a fractal structure of phase-space manifolds?

A signal *s(t)* written as [Huang et al., 1998]

$$s(t) = \sum_{k=1}^{N_k} a_k(t) \cos\left[\varphi_k(t)\right] + r(t)$$

- $a_k(t)$ is the instantaneous amplitude
- $\varphi_k(t)$ is the instantaneous phase
- $c_k(t) = a_k(t) \cos[\varphi_k(t)]$ is named empirical mode since derived via an iterative adaptive algorithm, i.e., the *sifting proces*
- $c_k(t)$ oscillates on a **typical timescale**

$$\tau_k = \frac{1}{2\pi}\left\langle \frac{d}{dt}\varphi_k(t)\right\rangle$$

MULTISCALE GENERALIZED FRACTAL DIMENSIONS

- a signal manifests a **multiscale** behavior $\rightarrow s(t) = \langle s\rangle + \sum_\tau \delta s_\tau(t)$ with $\langle s\rangle$ being the steady-state value
- identify $\sum_\tau \delta s_\tau(t) = \sum_{k<k^*} c_k(t)$ as the fluctuations on $\tau_k \le \tau_{k^*}$
- introduce a **natural measure** $d\mu_\tau$ and a **partition function** $\Gamma_q(\mu_\tau, B_{s,\tau}(l))$

we define

$$D_{q,\tau} = \frac{1}{q-1}\lim_{\ell\to 0}\frac{\log \Gamma_q(\mu_\tau, B_{s,\tau}(\ell))}{\log \ell}$$

- let *s(t)* be a signal whose trajectory belongs to a *D*-dimensional space S^D
- define a **natural measure** $d\mu(\{s(t)\})$ and the **partition function** $\Gamma_q(\mu, B_s(l))$
- let $B_s(l)$ be the hypercube of size *l* centered at the point *s* of S^D

Hentschel and Procaccia (1983) defined

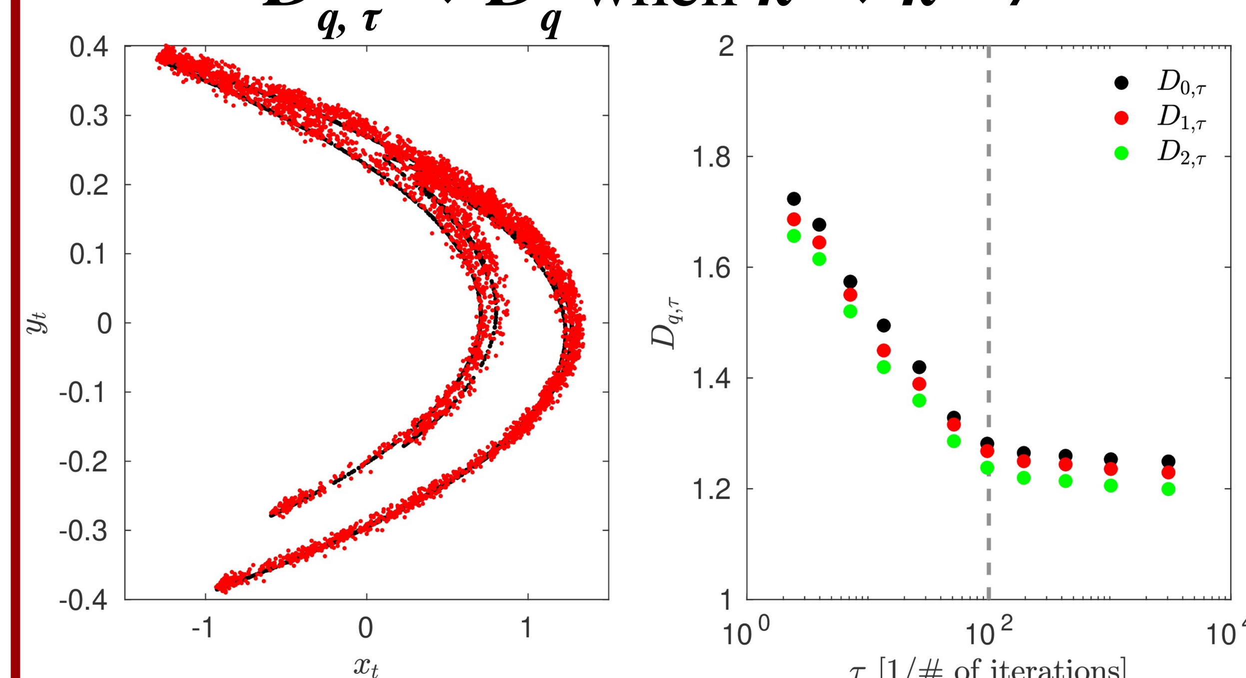
$$D_q = \frac{1}{q-1}\lim_{\ell\to 0}\frac{\log \Gamma_q(\mu, B_s(\ell))}{\log \ell} = \frac{1}{q-1}\lim_{\ell\to 0}\frac{\log \sum_i p_i^q}{\log \ell}$$

HENON MAP

$$x_{t+1} = 1 - a x_t^2 + y_t$$
$$y_{t+1} = b x_t$$

- Strange attractor: *a=1.4* and *b=0.3*
- $D_0 \simeq D_1 \simeq D_2 \simeq 1.24 \pm 0.03$
- EMD extracted out $N_k=11$ modes

$D_{q,\tau} \rightarrow D_q$ **when** $k \rightarrow k^*=7$



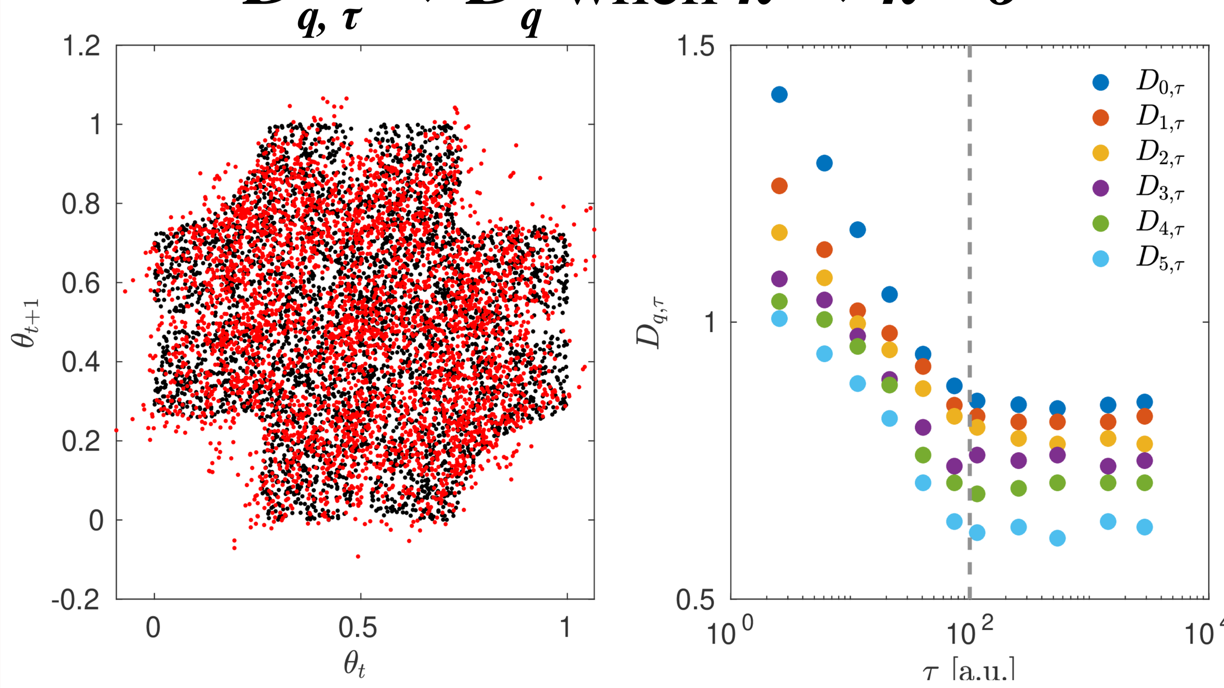
- The phase-space portrait (black dots) is reproduced by 7 modes (red dots)
- The geometrical and topological properties stored into a subset of "**informative**" empirical modes
- **Monofractal** nature at all timescales [Hentschel & Procaccia, 1983]

STANDARD MAP

$$p_{t+1} = p_t - K\sin(2\pi\theta_t)$$
$$\theta_{t+1} = \theta_t + p_{t+1}$$

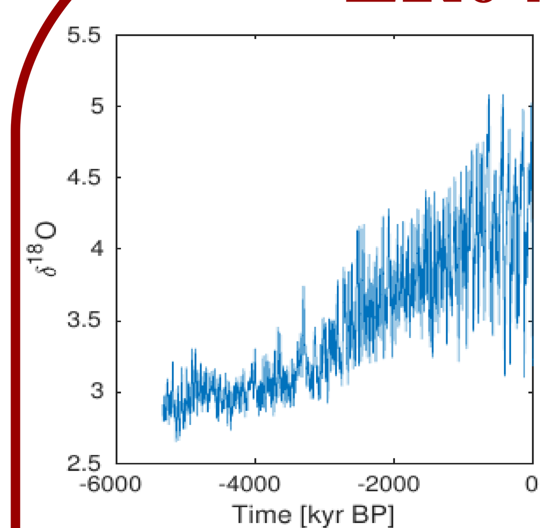
- Chaotic nature: $K \ge 1$
- $D_q \in [0.5, 0.924]$ and $D_0 \simeq 0.87$
- EMD extracted out $N_k=11$ modes

$D_{q,\tau} \rightarrow D_q$ **when** $k \rightarrow k^*=6$

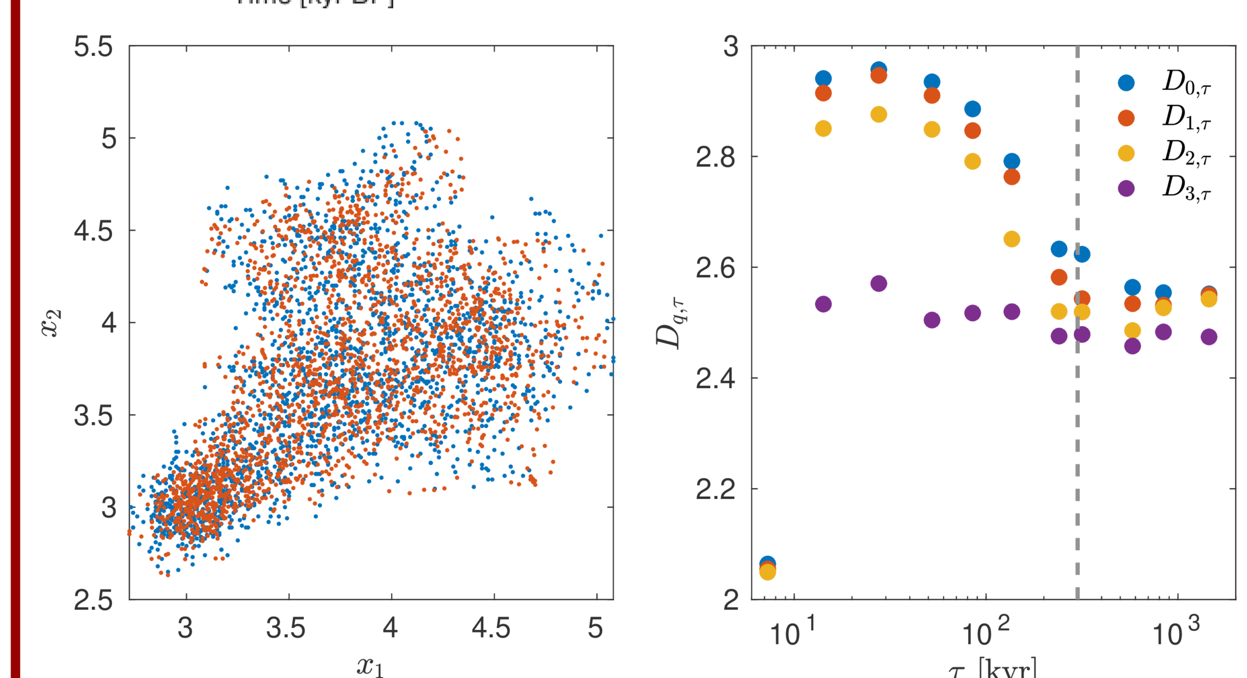


- The phase-space portrait (black dots) is reproduced by 6 modes (red dots)
- The geometrical and topological properties are stored into a subset of "**informative**" empirical modes
- **Multifractal** nature at all timescales [Hentschel & Procaccia, 1983]

LR04 RECORD

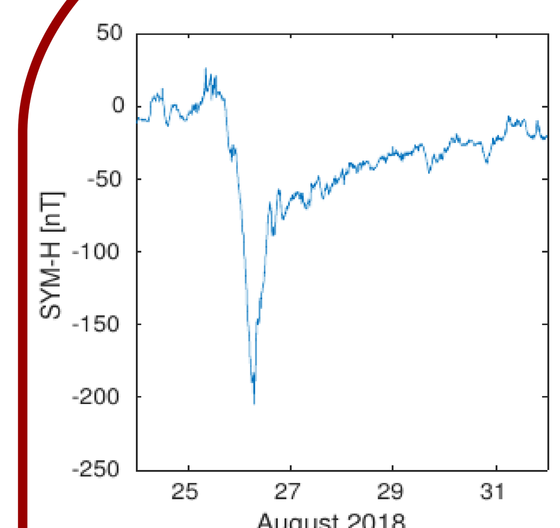


- 57 deep sea sediment cores [Lisiecki & Raymo, 2005]
- Paleoclimate variability during the last 5.3 Myr

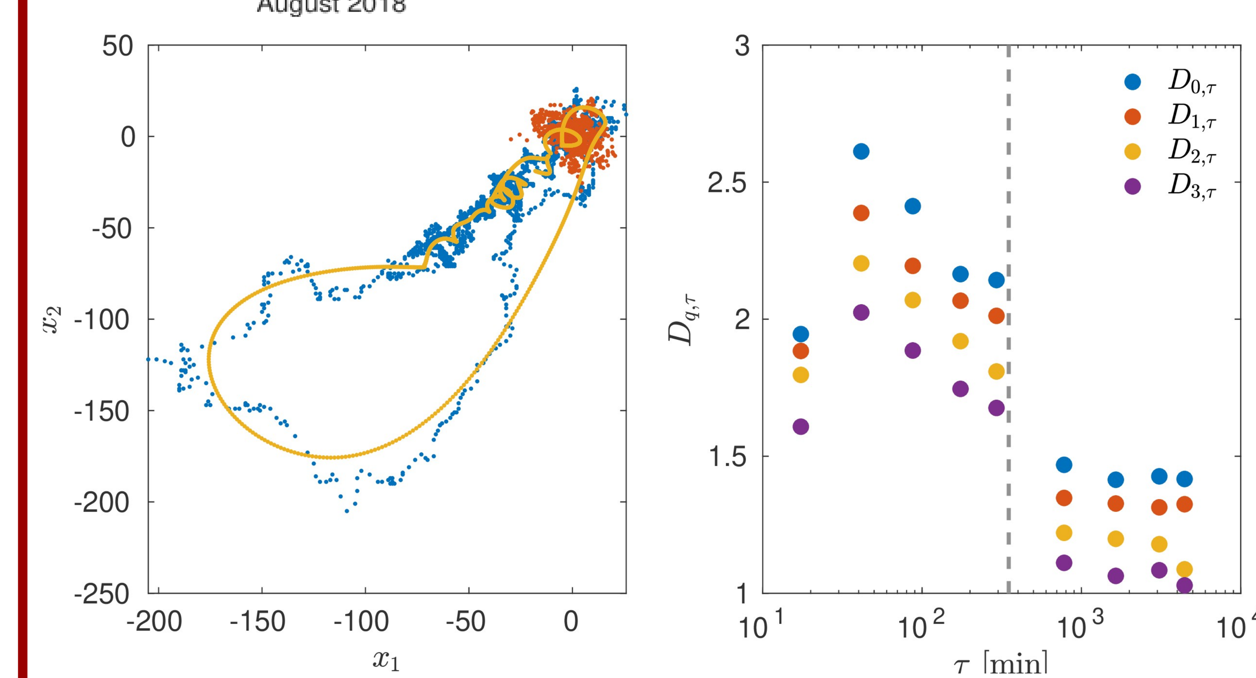


- $D_{q,\tau} \rightarrow D_q$ **when** $k \rightarrow k^*=7$
- $k=1$ $D_{q,\tau 1} \simeq 2 \rightarrow$ noise content
- $k \in [2,7] \rightarrow \tau \lesssim 300$ kyr $\rightarrow$ **multifractal**
- $k \ge 8 \rightarrow \tau \gtrsim 300$ kyr $\rightarrow$ **monofractal**
- Results consistent with Shao & Ditlevsen, Nat. Comm., 2016

SYM-H INDEX



- Low-latitude geomagentic activity [Iyemori, 1990]
- Occurrence of geomagnetic storm (peak value ~-200 nT)



- $D_{q,\tau} \rightarrow D_q$ **when** $k \rightarrow k^*=5$
- $D_{q,\tau} \rightarrow$ **multifractal** nature $\forall\tau$
- $k \ge 6 \rightarrow$ more predictable behavior
- **Internal** (more chaotic, $k \in [2,5]$) vs. **external** (more regular, $k \in [6,9]$) dynamics [Alberti et al., 2017]

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