Highly accurate analytical footprint model for general stratification of the atmosphere

Jean-Claude Krapez¹, Gregoire Ky¹, and Claire Sarrat²

¹ ONERA, DOTA, 13661 Salon de Provence, France
² ONERA, DTIS, Toulouse University, 31055 Toulouse, France

krapez@onera.fr

EGU 2020, May 6th 2020
The notion of footprint is used to describe the spatial extent and position of the surface area that is contributing to a turbulent concentration or flux measurement at a specific height for specific atmospheric conditions and surface characteristics.

Footprint relatively to the measurement of:

- a passive scalar (i.e.: CO₂ or pollutant concentration)
- a vertical flux (heat, humidity, pollutant …)

The footprint function corresponds to the convolution kernel

\[
\eta(x_m, y_m, z_m) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Q(x', y', z = 0) \, dx' \, dy'
\]

It then appears that \( f_\eta(x_m - x', y_m - y', z_m) \) is equal to the response \( \eta(x_m, y_m, z_m) \) at height \( z_m \) to a unitary point source at ground at \((0,0,0)\)

© 2020 Krapez, Ky, Sarrat. All rights reserved
Existing footprint models

The footprint calculation amounts to solve a 3D turbulent dispersion problem in the atmospheric boundary layer.

Often, the calculations are restricted to the surface layer.

**Lagrangian stochastic particle dispersion models**

(e.g. Hsieh, 2000, Kljun 2004, 2015)

- General purpose
- May be applied outside the surface layer, below the entrainment layer
- High computing time

Surrogate models have been proposed (analytical parameterizations)

- Very fast
- Biased results (fitting errors)
Existing footprint models

Tools based on the analytical solution of a (close) advection-diffusion problem

- Variable separation
  \[ f_\eta(x, y, z_m) = \chi_y(x, y) f_\eta^y(x, z_m) \]
  Gaussian crosswind dispersion

- 1\textsuperscript{st} order turbulence closure: K-theory
  \[ w' \chi' \approx -K(z) \frac{\partial \chi}{\partial z} \]
  Eddy diffusivity (height-dependent)

- Analytical modeling of the associated 2D advection-diffusion process:
  \[ u(z) \frac{\partial \chi}{\partial x} = \frac{\partial}{\partial z} \left( K(z) \frac{\partial \chi}{\partial z} \right) \]

\begin{itemize}
  \item Very fast
  \item Restricted to the surface layer
  \item Considered analytical models neglect longitudinal dispersion
  \item The underlying analytical models are restricted to \textit{power-law} profiles of \( u \) and \( K \)
\end{itemize}
New (semi-)analytic model

New method for the calculation of the crosswind-integrated footprint \( \bar{f}_{\eta}^{y}(x, z_m) \)

Semi-analytic method to solve the advection-diffusion equation:

\[
\frac{u(z)}{\partial x} = \frac{\partial}{\partial z} \left( K(z) \frac{\partial \chi}{\partial z} \right)
\]

based on an optimal adjustment of the relevant parameters with the actual turbulent profiles.

The adjustment is realized by:

- performing a Liouville transformation, which gives rise to a new parameter agglomerating \( u(z) \) and \( K(z) \), namely the atmosphere reluctance:
  \[
b(z) = \sqrt{u(z)K(z)}
\]

- splitting the height range in due number of sublayers and performing a (extended-type) power-law fit for the reluctance in each sublayer

- solving the advection-diffusion equation by adapting the quadrupole method (e.g. Maillet, 2000, Krapez, 2014) to the present extended-type power-law profiles

\[+\] Very fast (< 1s)

\[+\] Perfect adjustment with e.g. Monin-Obukhov profiles (or whatever else profiles)

\[+\] Not restricted to the surface layer (insofar as the input profiles \( u(z) \) and \( K(z) \) conform the real ones)
The Liouville Transformation consists of a change of the independent variable $z \rightarrow \xi$ (see e.g. Krapez, 2016)

$$\xi = \xi(z) := \int_{0}^{z} \sqrt{\frac{u(z')}{K(z')}} dz'$$

The new independent variable $\xi(z)$ can be interpreted as the square root of the longitudinal distance covered by a plume when reaching height $z$. It will be called the **Square Root of Plume Extension (SRPE)**.

Unstable

Stable
New paradigm: consider a change of space (Liouville transformation)

Liouville Transformation:

\[ z \rightarrow \xi ; \quad \xi = \xi(z) := \int_0^z \sqrt{\frac{u(z')}{K(z')}} \, dz' \]

Square Root of Plume Extension - SRPE
(new independent variable)

Dimensionless profiles of SRPE as inferred from MO similarity theory, and taking Businger/Högström universal similarity functions

Horizontal plume extension is higher in stable conditions

Dimensionless height \( z_0/L \) from \(-10^{-3}\) to \(+10^{-3}\)
In the Liouville space, the advection-diffusion equation becomes:

\[ b(\xi) \frac{\partial \chi}{\partial x} = \frac{\partial}{\partial z} \left( b(\xi) \frac{\partial \chi}{\partial z} \right) \quad \text{with} \quad b(\xi) = \sqrt{u(\xi)K(\xi)} \]

Instead of two profiles (wind speed and eddy diffusivity), it now features only one profile: \( b(\xi) \) which has been called the atmosphere reluctance (namely inertia of the atmosphere to concentration changes).

All the concentration and flux features (in particular the footprints) are entirely determined by the single profile of reluctance \( b(\xi) \) which is therefore a parameter of uttermost importance (yet overlooked).

Dimensionless profiles of reluctance as inferred from MO similarity theory, and taking Businger/Högström universal similarity functions.
New paradigm: consider a change of space (Liouville transformation)

The advection-diffusion equation \( b(\xi)\chi_{,x} = \left(b(\xi)\chi_{,z}\right)_{,z} \) has an analytical solution for power-law profiles of reluctance.

However, the MO reluctance profiles are not power-law profiles. They are actually unlikely to be analytically solvable profiles.

Profiles of reluctance deriving from the MO theory (or from parametrizations extending beyond the surface layer) will be **piecewise approximated by solvable profiles of high flexibility.**

It was shown that the family of extended-power-law profiles defined by

\[
b(\xi) = \hat{\xi} \left[ A_B \hat{\xi}^\nu + A_D \hat{\xi}^{-\nu} \right]^2 ; \quad \hat{\xi} = \xi + \xi_c
\]

are **analytically solvable and flexible (four adjustable free parameters) while suitable for high-range variations.**
Gooness of fit

\[
\frac{z_0}{L} = +4 \times 10^{-4}
\]

\[
\frac{z_0}{L} = -8 \times 10^{-4}
\]

\[
\hat{\xi} = \xi + \xi_c
\]

These profiles can be considered as solvable splines.

With the chosen discretization density, the fitting error is less than 0.1%

Their application is not restricted to the surface layer.

in **blue**: Reluctance profiles derived from **MO similarity theory**, while taking **Businger/Högström universal similarity functions**

in **black** (perfectly overlapping): fitted **extended-power-law profiles** defined by:

\[
b(\xi) = \hat{\xi} \left[ A_B \hat{\xi}^v + A_D \hat{\xi}^{-v} \right]^2;
\]

© 2020 Krapez, Ky, Sarrat. All rights reserved
Semi-analytic calculation

**Quadrupole** method (or **Transfer Matrix** method) in the Laplace domain (the classical case of homogeneous sublayers has been described in, e.g. Maillet, 2000, Krapez, 2014)

**Quadrupole matrices** have been defined for the present case of **extended-power-law profiles**.

Simple algebra with these matrices allows computing the concentration and the flux at any height of the composite boundary layer (see e.g. Krapez 2014, 2016).

Numerical inverse Laplace transform to get the concentration and the flux vs. downwind distance (e.g. Krapez 2014)

**Fast (< 1s) and highly accurate semi-analytic solution for concentration, flux, and footprint as a by-product**
Example for the **neutral case**

**Concentration footprint** (crosswind integrated)

**Flux footprint** (crosswind integrated)

Normalized measurement height $z_m/z_0 = 20, 40, 100, 200, 500, 1000, 3000, 10000$
Fetch in neutral case

Flux measured at height $z_m$

Weight function for flux

Cumulative footprint

Relative distance from border $x/z_m$

© 2020 Krapez, Ky, Sarrat. All rights reserved
Comparaison with other models (neutral case)

Flux footprint for two different heights of the sensor

\[ \frac{z_m}{z_0} = 100 \]

\[ \frac{z_m}{z_0} = 1000 \]

\[ \chi' \]

\[ u' \]

\( Kljun, 2015 \) (Lagrangian+metamodel)

\( Kljun, 2004 \) (Lagrangian+metamodel, \( z_0 = 0.04m \))

\( Hsieh, 2000 \) (Lagrangian+metamodel)

\( Kormann, 2001 \) (power-law profiles for \( u \) and \( K \))

present model (quasi exact diffusion model)

Large dispersion of results. Might be partly explained by the longitudinal turbulence \( u' \chi' \)

The Lagrangian model (Hsieh, 2000) gives a maximum that is
- closer in the case of unstable condition
- further downwind in the case of neutral or stable condition

$L$ : Obukhov length

$z_0/L$

-0.0008 (unstable)
0 (neutral)
+0.0004 (stable)

dashed lines : Hsieh, 2000

Illustration for $z_0=0.04m$, $z_m=4m$
$L=-50m$ (unstable), inf. (neutral), +100m (stable)

<table>
<thead>
<tr>
<th>$L$</th>
<th>Present method</th>
<th>Hsieh, 2000</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50 m</td>
<td>6.3</td>
<td>5.2</td>
<td>-17%</td>
</tr>
<tr>
<td>Inf.</td>
<td>7.7</td>
<td>11.0</td>
<td>+43%</td>
</tr>
<tr>
<td>+100 m</td>
<td>9.0</td>
<td>14.5</td>
<td>+62%</td>
</tr>
</tbody>
</table>

Cumulative flux footprint

$L$ : Obukhov length

$z_0/L$

-0.0008 (unstable)
0 (neutral)
+0.0004 (stable)

dashed lines : Hsieh, 2000

Illustration for $z_0=0.04m$, $z_m=4m$

$L=-50m$ (unstable), inf. (neutral), +100m (stable)

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$L$</th>
<th>Present method</th>
<th>Hsieh 2000</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-50 m</td>
<td>93.0</td>
<td>99.9</td>
<td>+7%</td>
</tr>
<tr>
<td>0.9</td>
<td>Inf.</td>
<td>161</td>
<td>208</td>
<td>+29%</td>
</tr>
<tr>
<td></td>
<td>+100 m</td>
<td>266</td>
<td>276</td>
<td>+4%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Threshold</th>
<th>$L$</th>
<th>Present method</th>
<th>Hsieh 2000</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>-50 m</td>
<td>39.3</td>
<td>36.6</td>
<td>-7%</td>
</tr>
<tr>
<td></td>
<td>Inf.</td>
<td>57.8</td>
<td>76.2</td>
<td>+32%</td>
</tr>
<tr>
<td></td>
<td>+100 m</td>
<td>80.8</td>
<td>101.2</td>
<td>+25%</td>
</tr>
</tbody>
</table>
The current model allows to solve in a short time (<1s) and with great accuracy the equation of the mean (turbulent) concentration provided that:

- parameterizations of the vertical profiles of wind velocity \( u(z) \) and \( K(z) := -w' \chi'/(\partial \chi/\partial z) \) are well representative of the turbulent structure of the atmosphere

- the term of longitudinal dispersion \( u' \chi' \) is negligible when compared to the advection term

- suitable for the surface layer by using MO similarity functions
- may be extended beyond the surface layer if due profiles of \( u(z) \) and \( K(z) \) are available

- the present model outperforms classical « analytical » footprint models (e.g. Korman, 2001) and LPD models that assume the same approximation (e.g. Hsieh, 2000)
- may be used for pollutant-dispersion modeling as a tradeoff between Gaussian models and more sophisticated (albeit more time consuming) models (e.g. Sarrat 2017)
- turbulent longitudinal dispersion will be added in the model in the next future.
References


