Topological waves in geophysical and astrophysical flows

Antoine Venaille

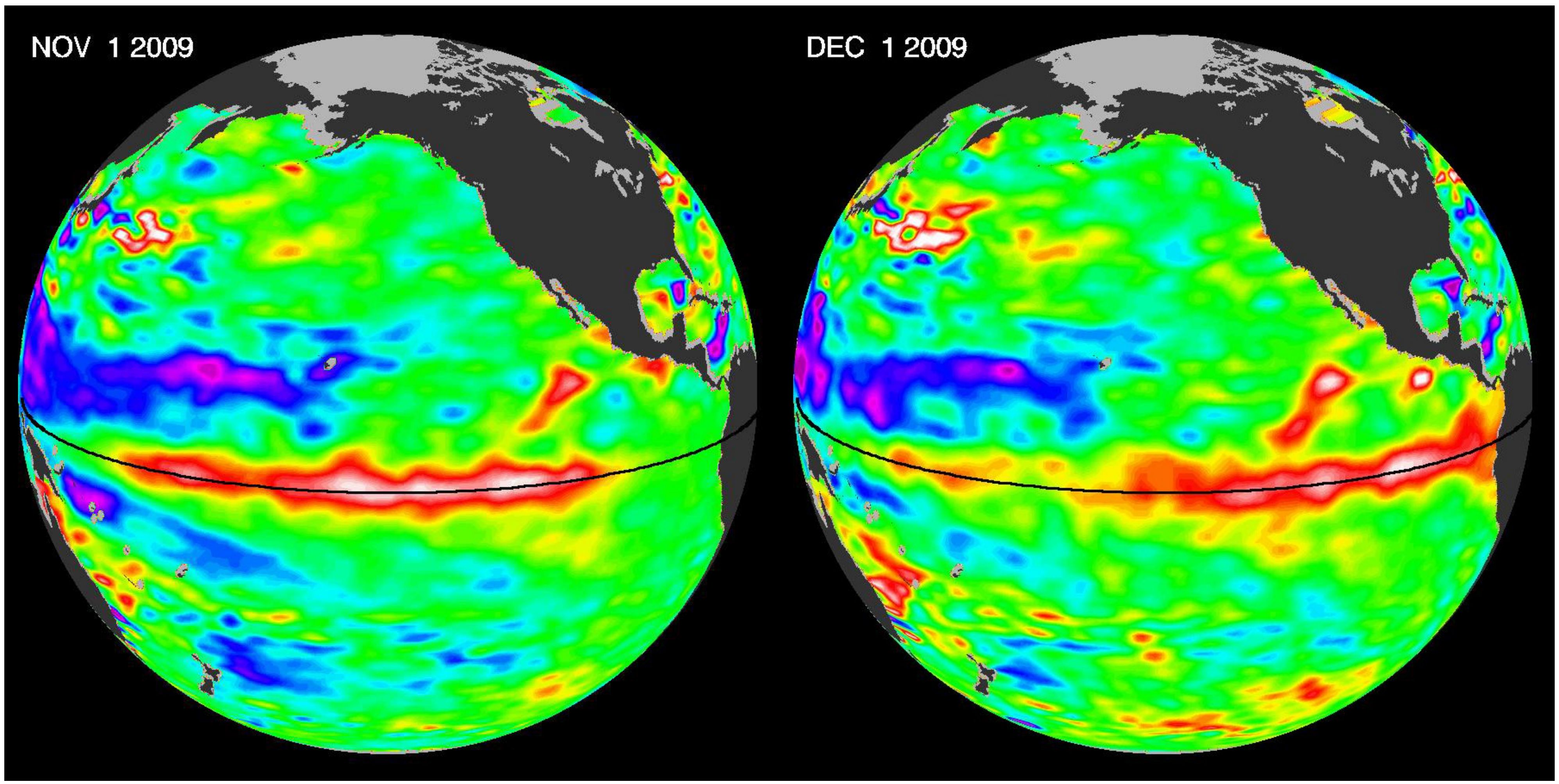
EGU, May 8 2020

Lab PHYS ENS de LYON

CNRS

ENS DE LYON

A precursor of El Nino



temperature anomaly

Eastward propagation of an equatorially trapped mode without backscattering. Why ?

Topological insulators

Exotic materials isolant in the bulk…

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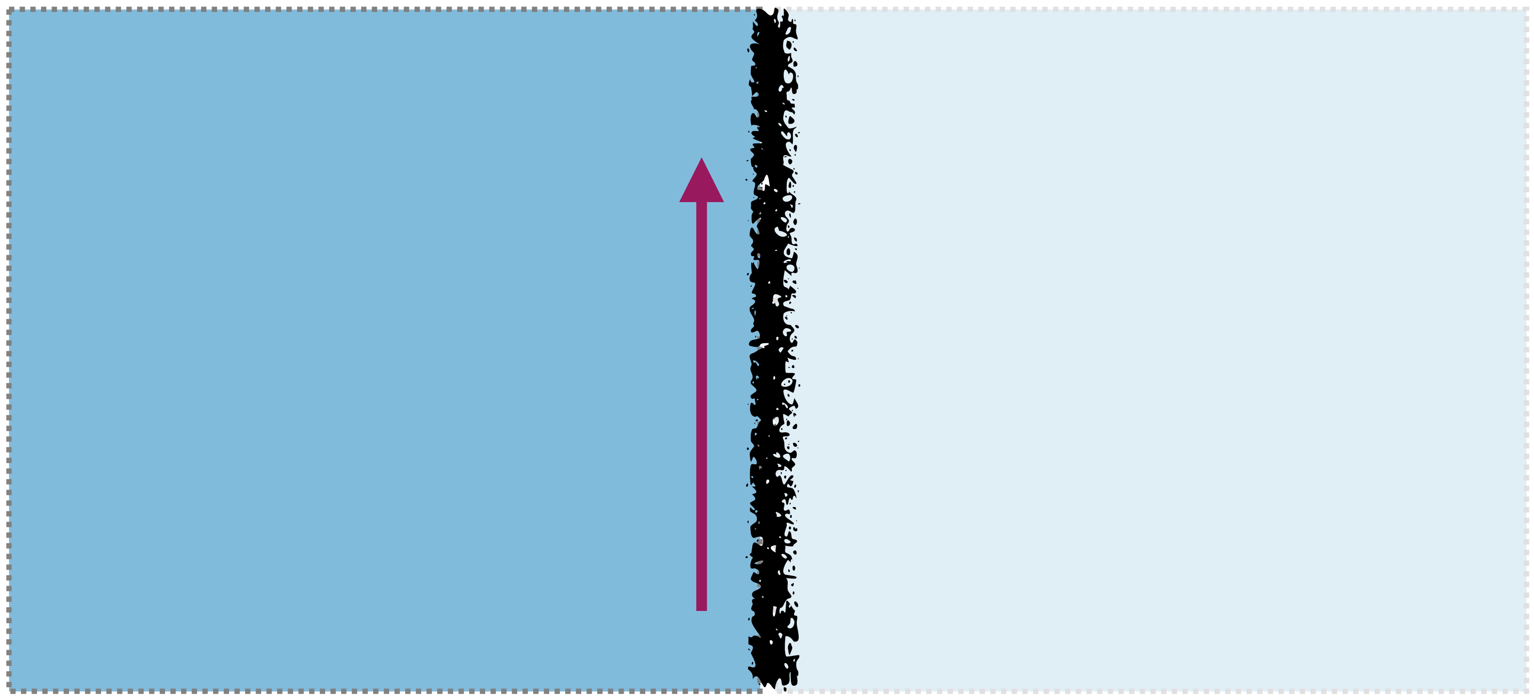

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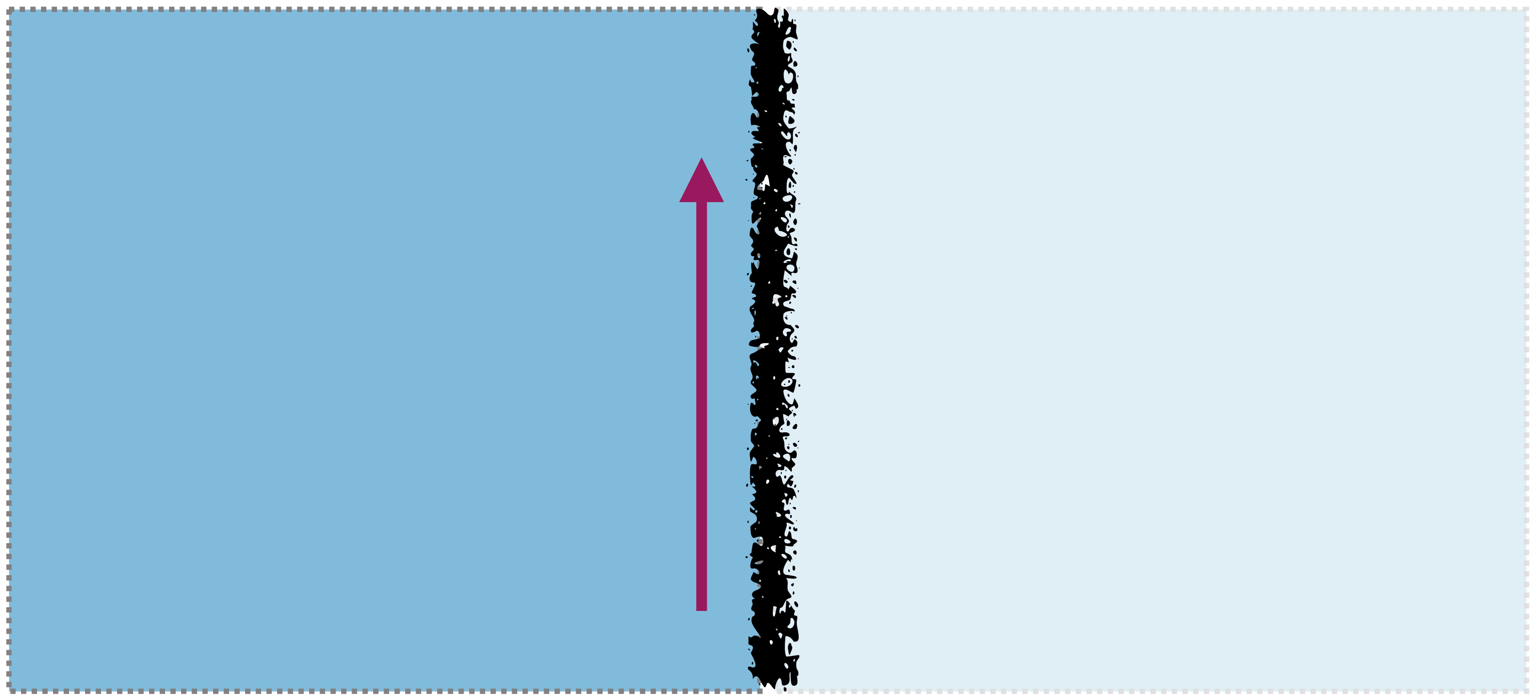

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- The number of unidirectional edge modes is related to a topological invariant
- Importance of discrete symmetries.

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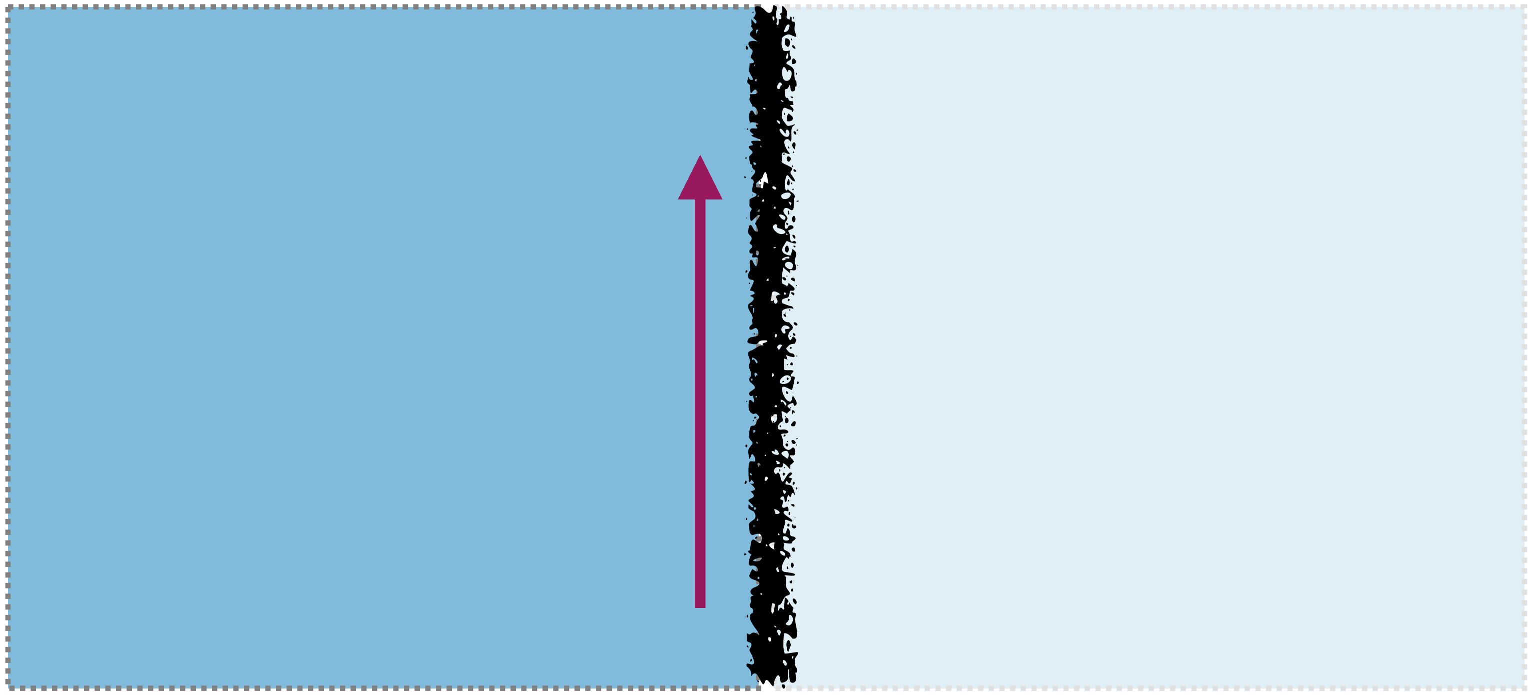

Pierre Delplace

...but excellent electronic conduction properties at the boundary

- The number of unidirectional edge modes is related to a topological invariant
- Importance of discrete symmetries.

Broken symmetries

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - g\mathbf{e}_z - 2\boldsymbol{\Omega} \times \mathbf{u}$$

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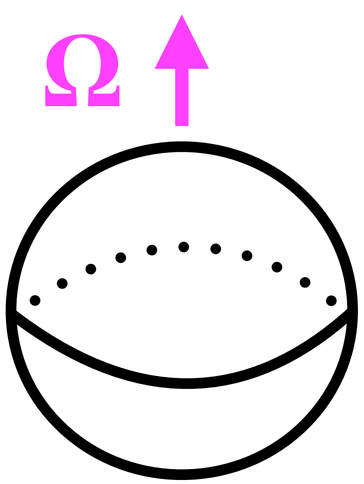
Breaks time-reversal symmetry

$$t \to -t$$



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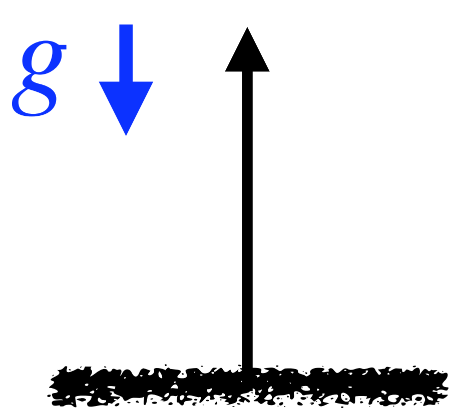

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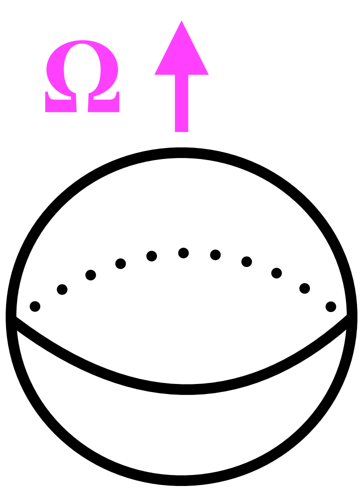

Broken symmetries

Is there a relation with the emergence of peculiar trapped waves?

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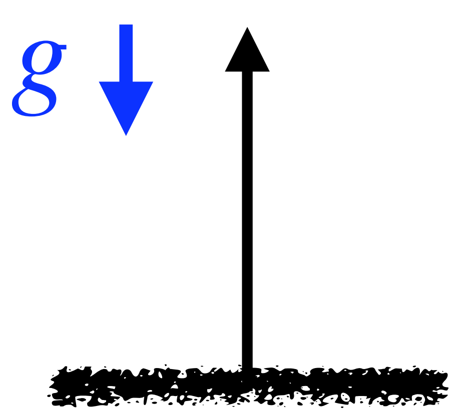

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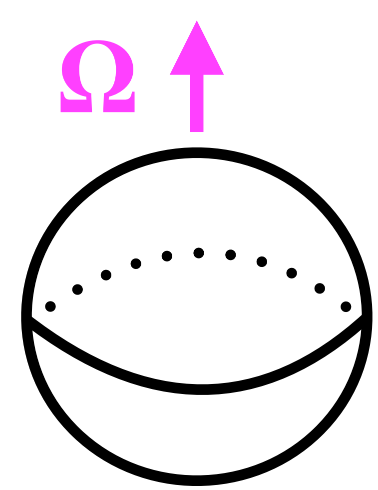

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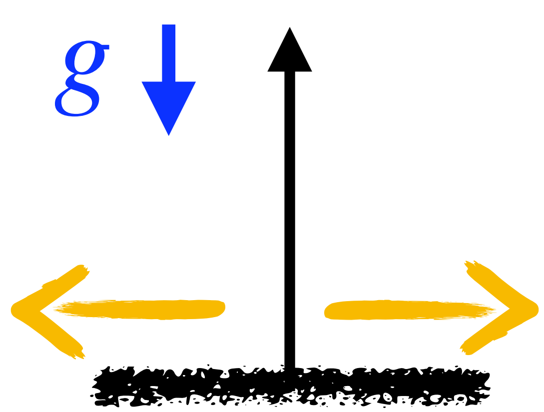

Lamb waves

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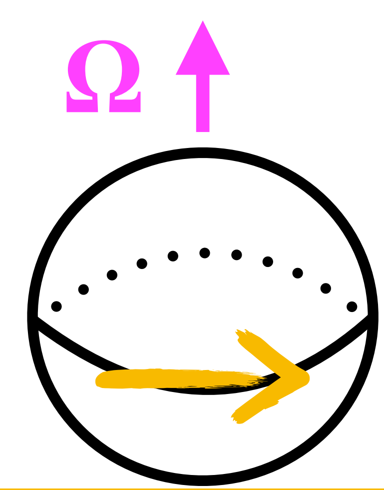
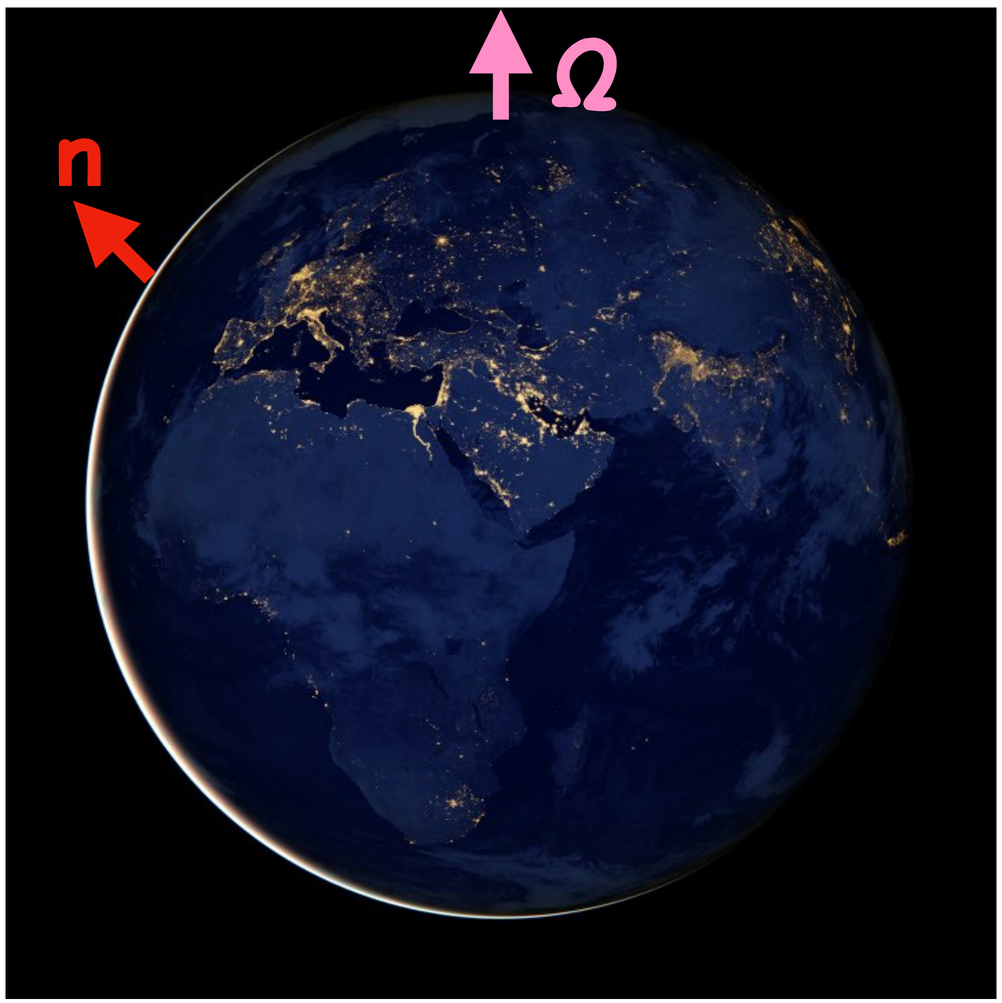
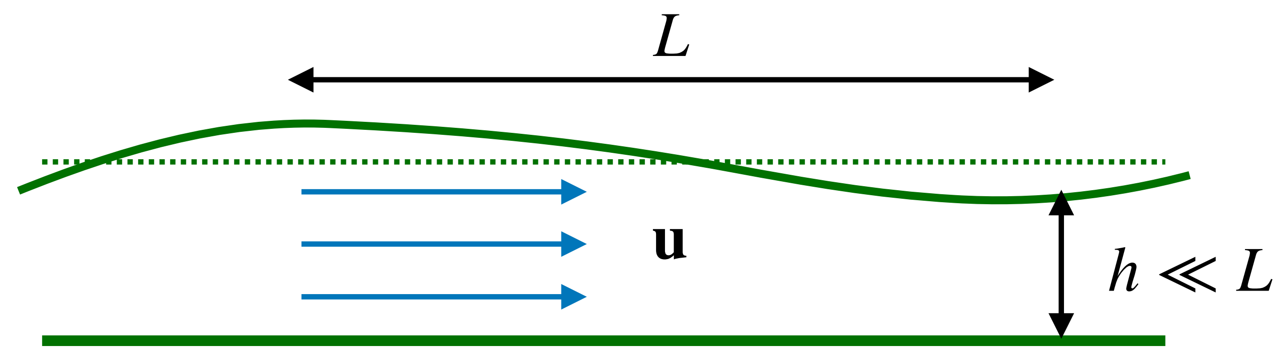


Equatorial Yanai Kelvin waves

I. Equatorial Waves

Shallow water model





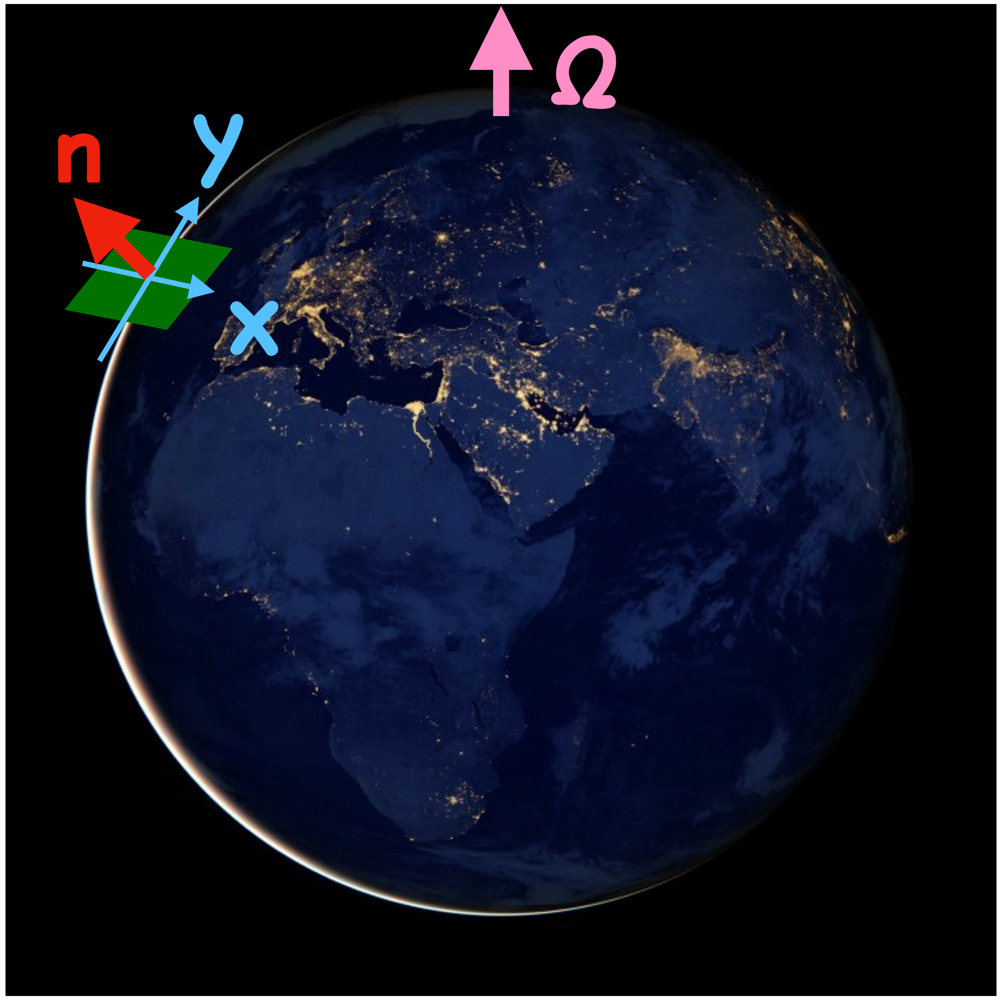
$$\begin{aligned} \partial_t h + \nabla \cdot (h\mathbf{u}) &= 0, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -g\nabla h - f\mathbf{n}\times\mathbf{u}. \end{aligned}$$

Coriolis parameter $f = 2\boldsymbol{\Omega}\cdot\mathbf{n}$

2D flow model, broken time reversal symmetry

Linear dynamics

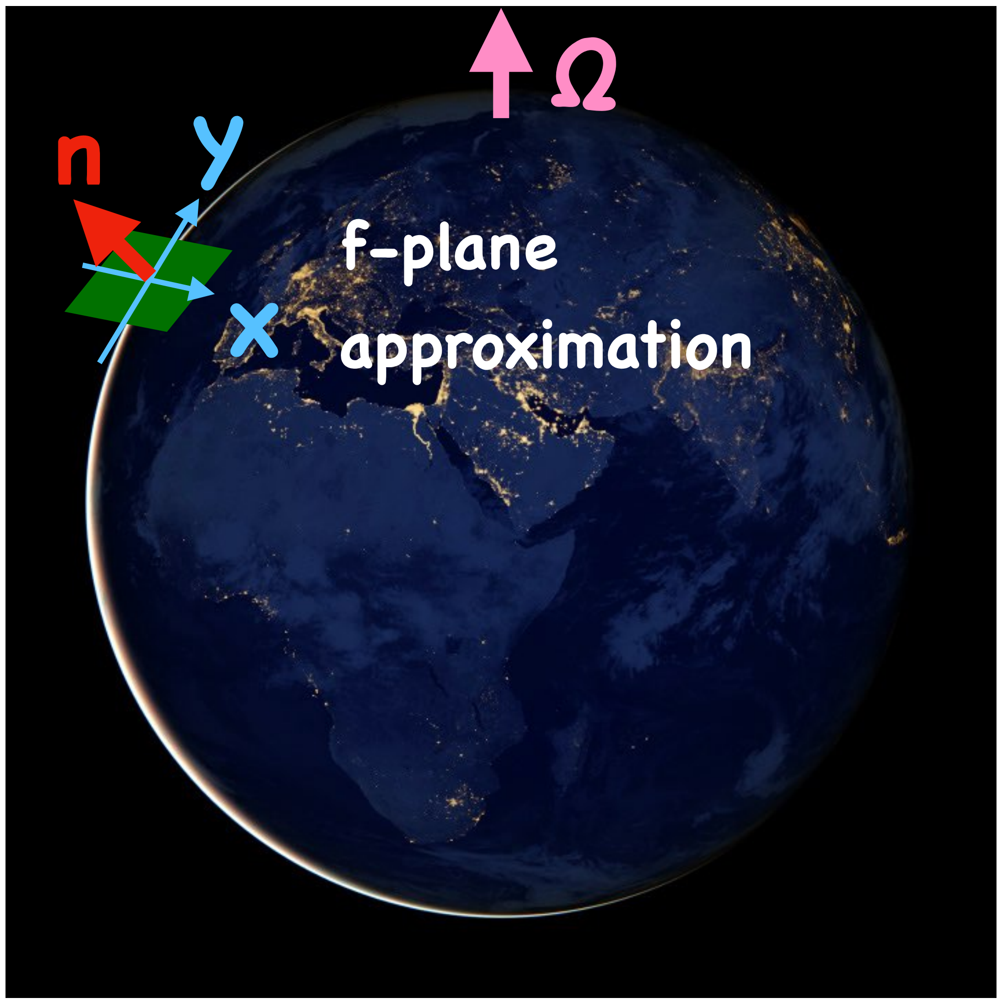
Linear dynamics



Free modes of
Laplace Tidal Equations

$$\partial_t u = -g\partial_x \eta + f v$$
$$\partial_t v = -g\partial_y \eta - f u$$
$$\partial_t \eta = -H\partial_x u - H\partial_y v$$

Linear dynamics



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f-plane waves

Kelvin 1879

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$$\Psi = (u, v, \eta), \quad \Psi = \hat{\Psi} e^{i\omega t - ik_x x - ik_y y}$$

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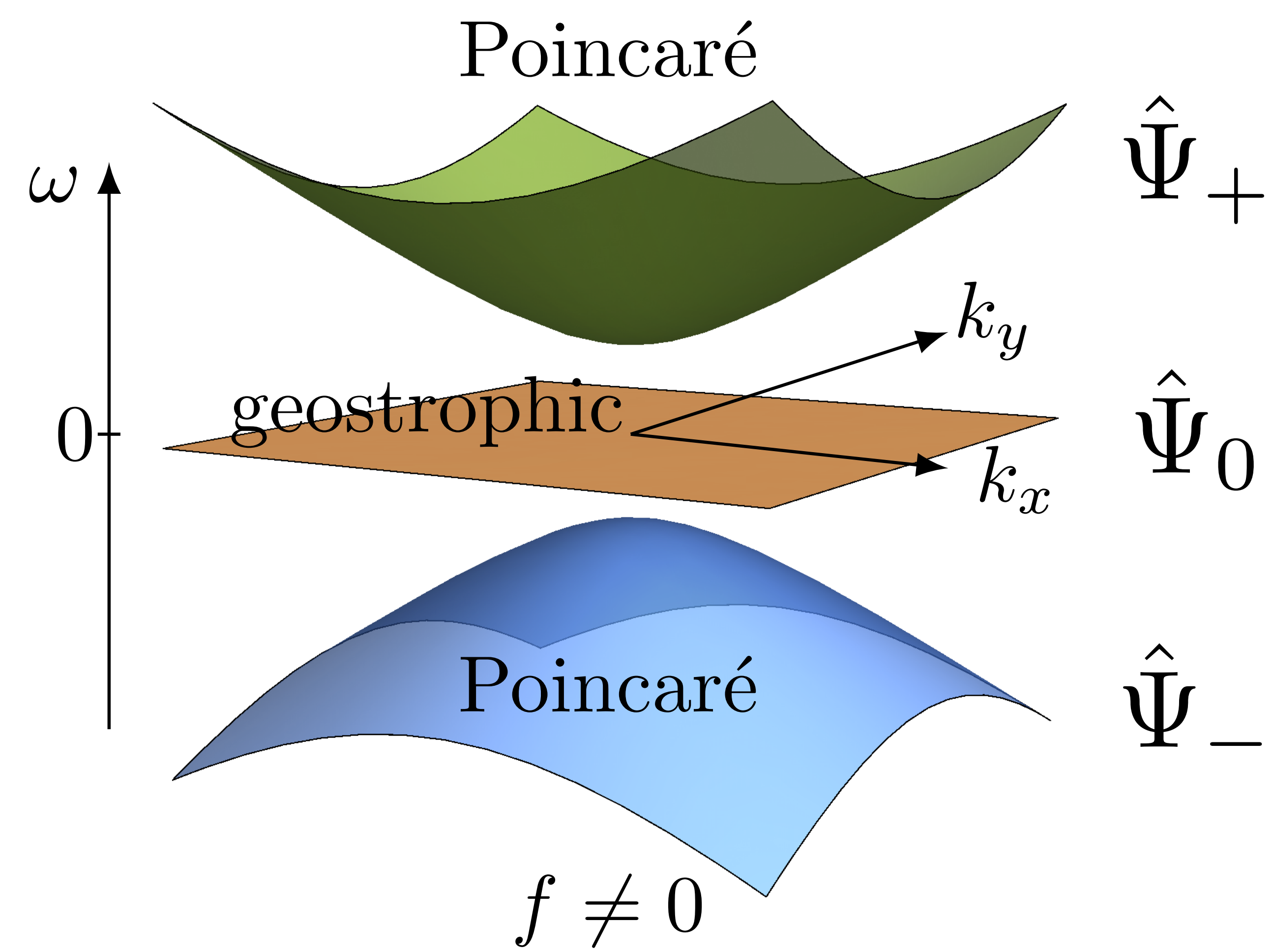

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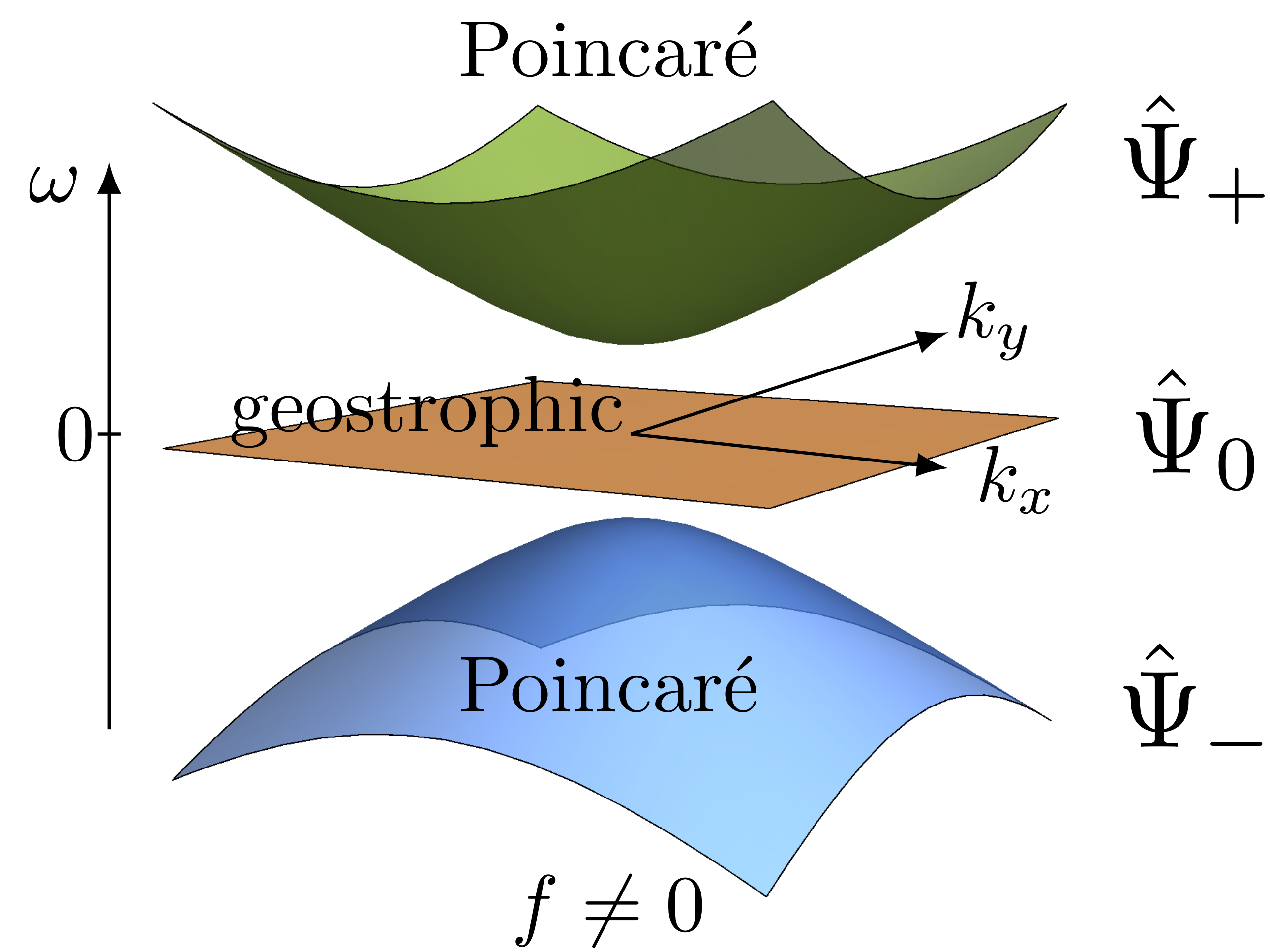

- Rotation opens a frequency gap

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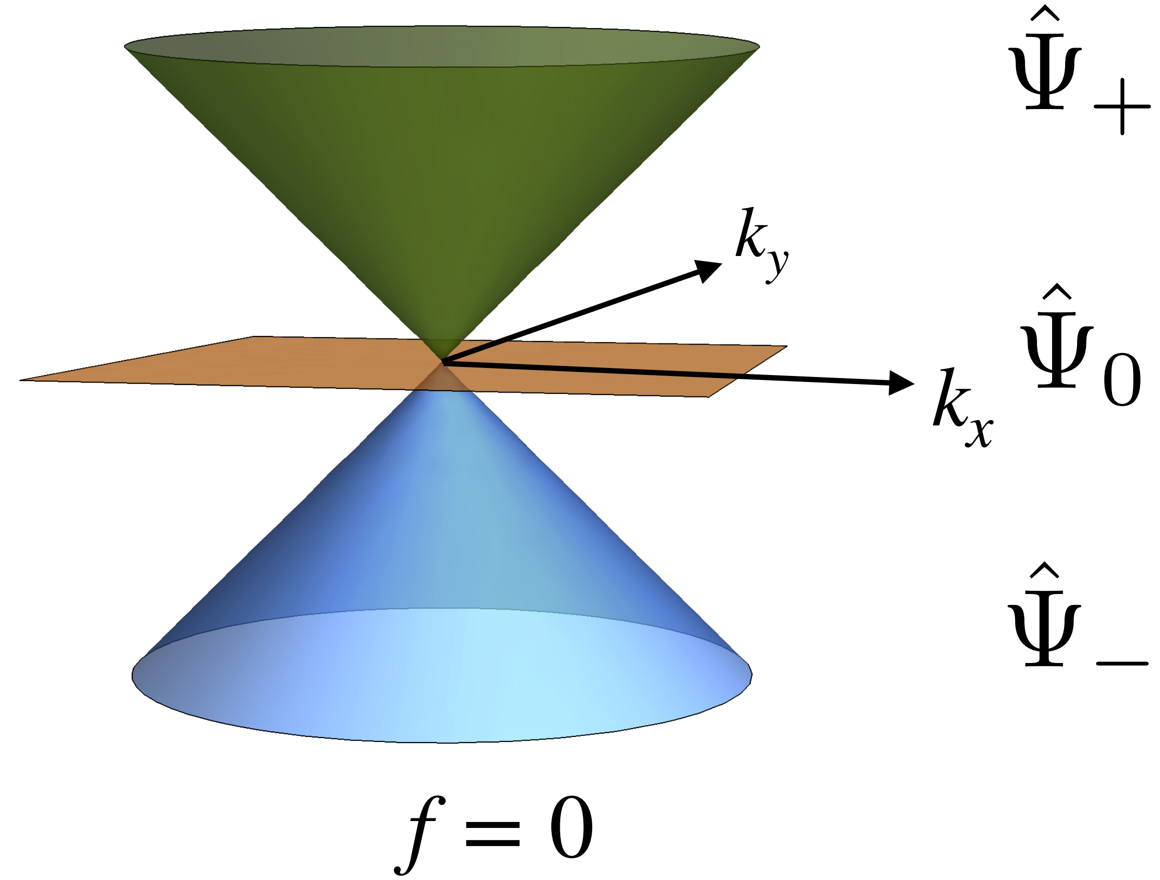
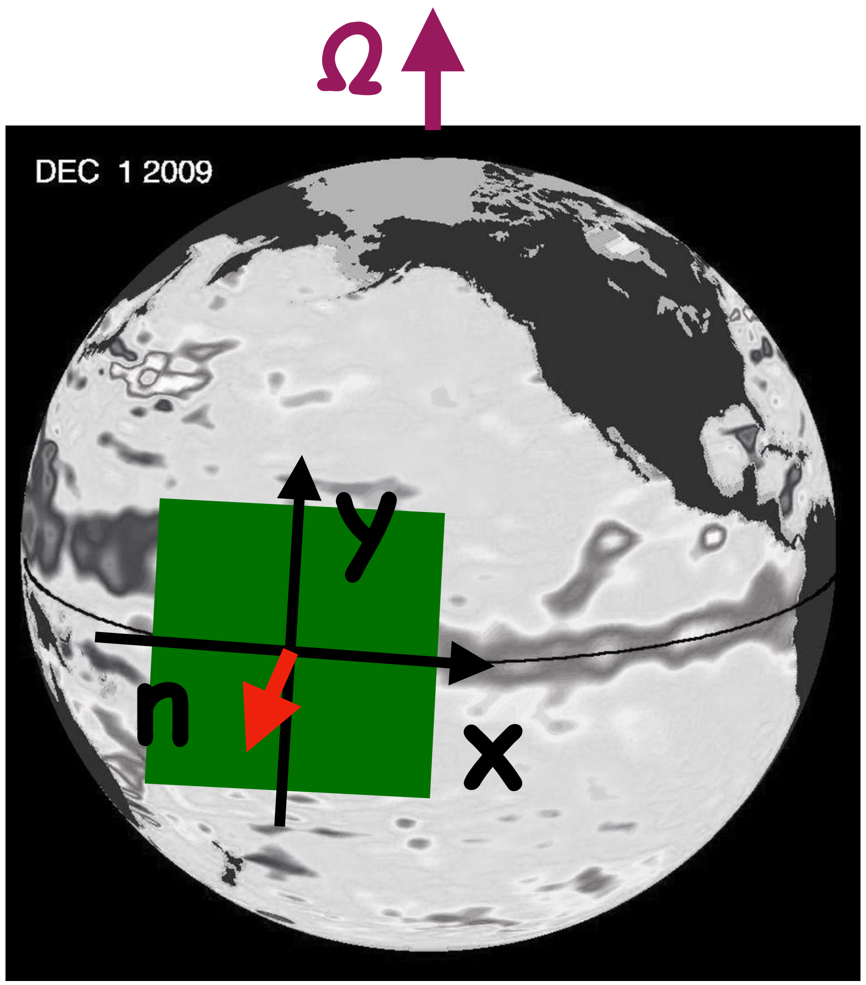


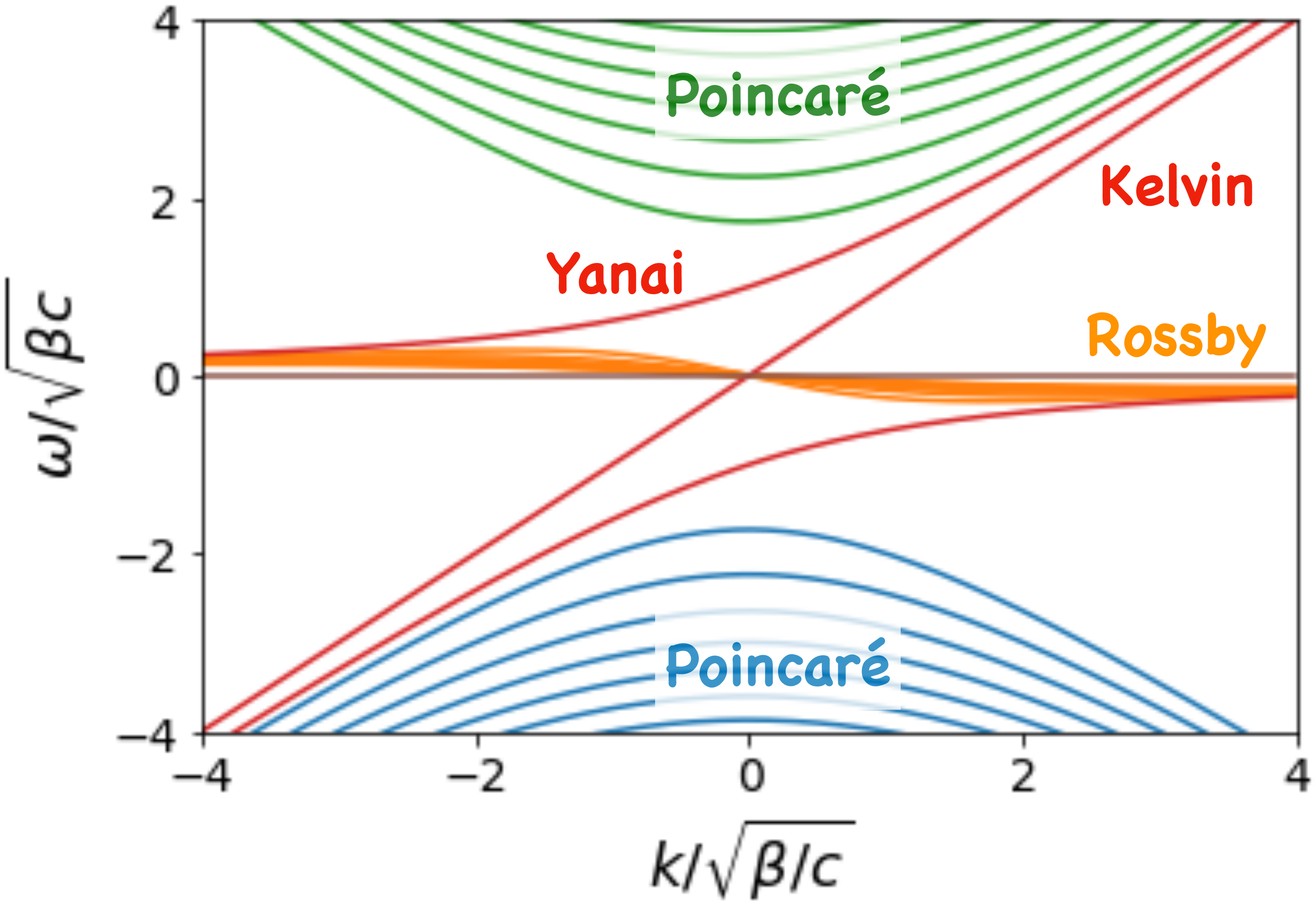
- Rotation opens a frequency gap

Equatorial Beta Plane



$$f = \beta y$$

Matsuno 1966



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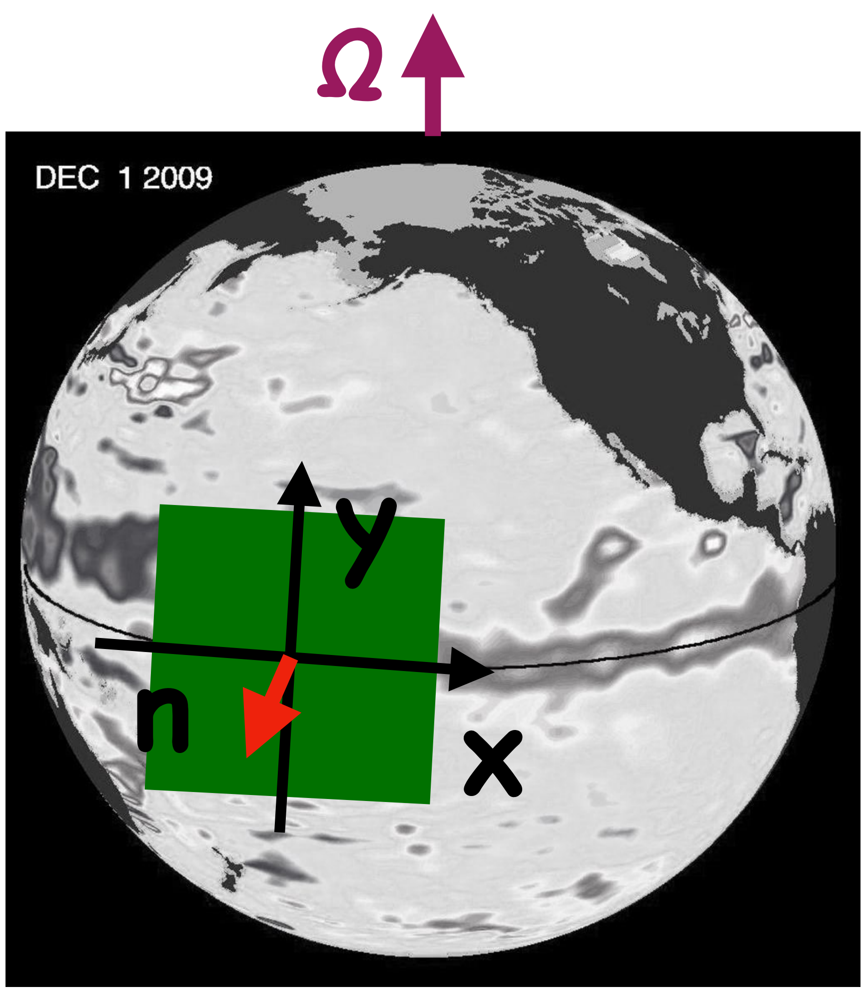

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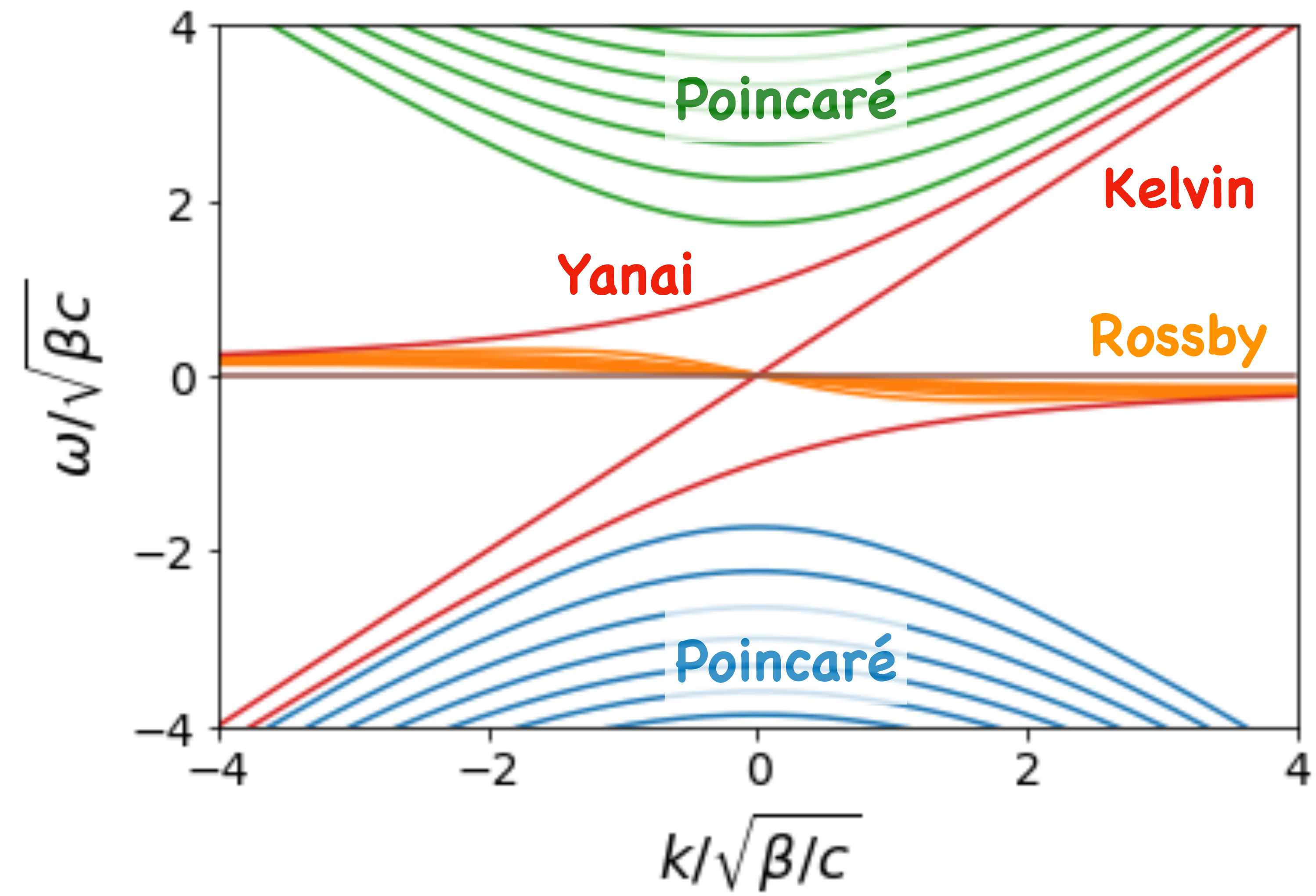
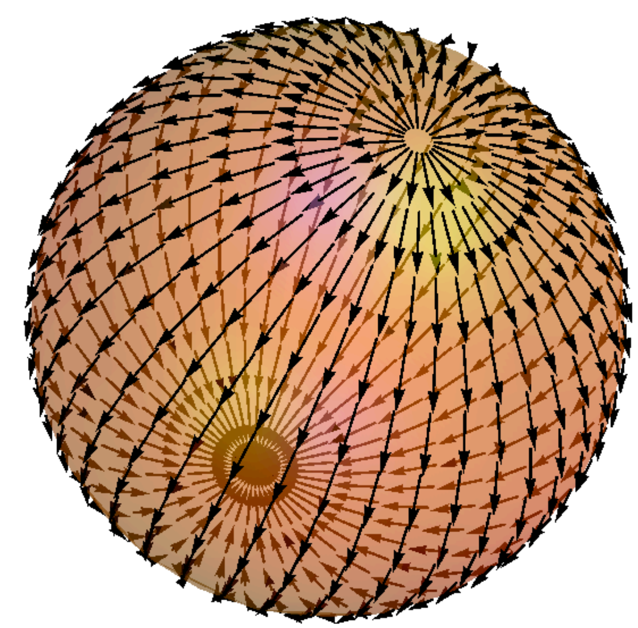


- Two modes with eastward group velocity filling the frequency gap.
- How to explain the global shape of the spectrum ?

Topology

It classifies objects according to global properties.



e.g. number of holes singularities in vector bundles

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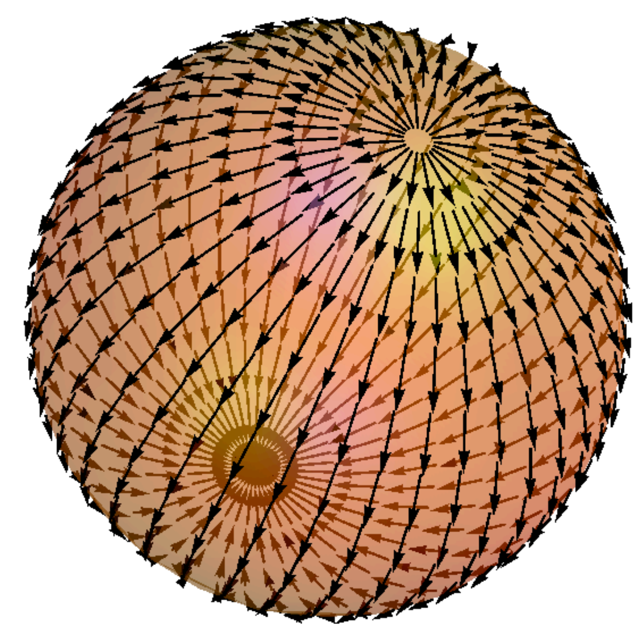
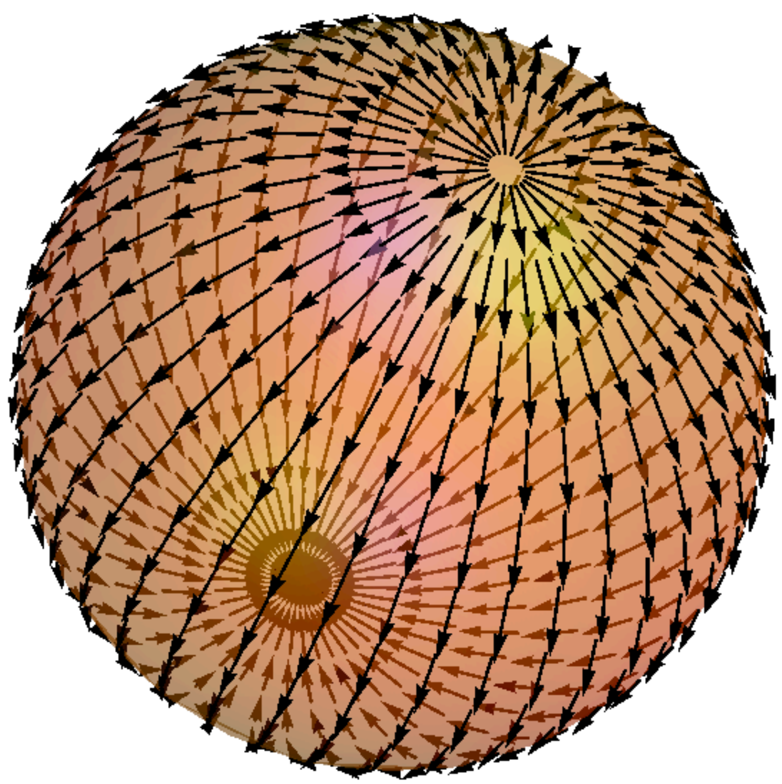
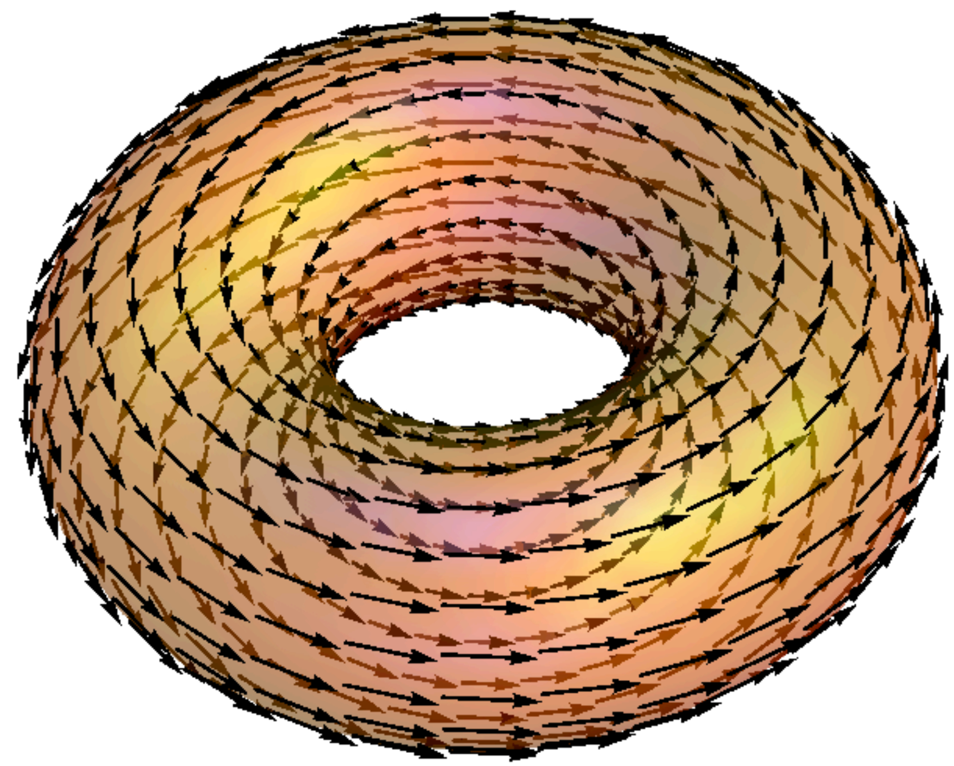


e.g. number of holes singularities in vector bundles

The first Chern number



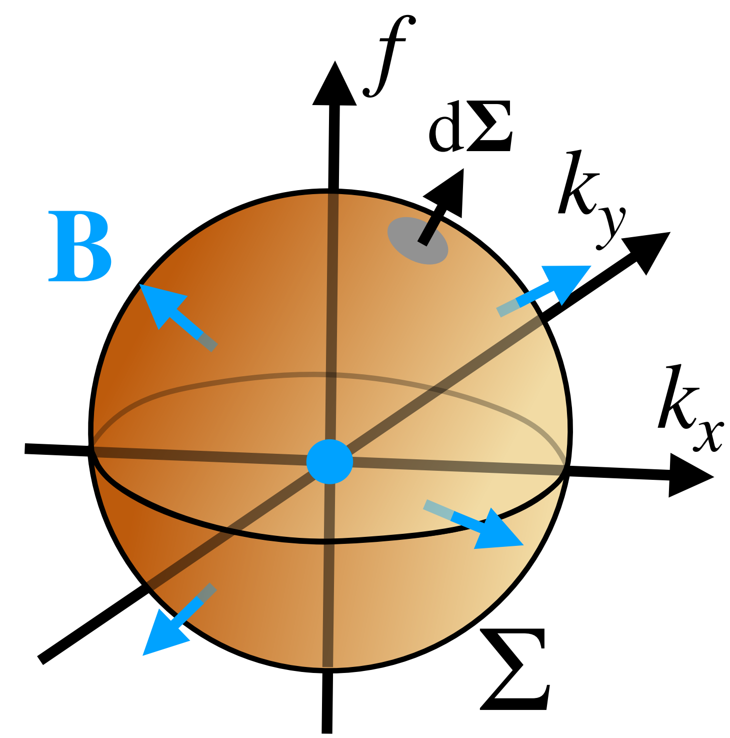


It counts the number of singularities in a bundle of vectors parameterized on a closed surface.

Winding of eigenmodes

f-plane shallow water eigenmode

$$\Psi = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix}$$



A (local) geometrical quantity

A (global) topological number

Berry curvature

$$\mathbf{B} = i\nabla_p \times \left(\Psi^\dagger \nabla_p \Psi\right)$$

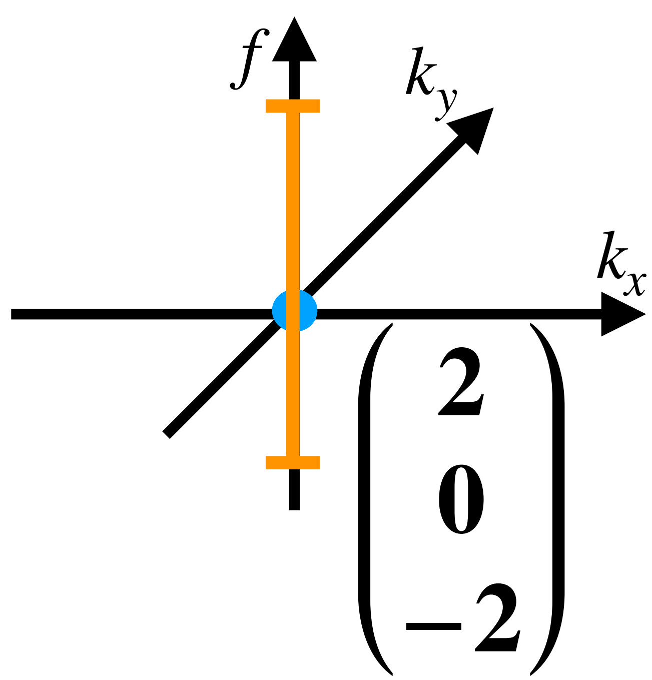
First Chern number

$$\chi = \frac{1}{2\pi}\int_\Sigma \mathbf{B}\cdot \mathrm{d}\boldsymbol{\Sigma}$$

Bulk-edge correspondence

Bulk : f constant

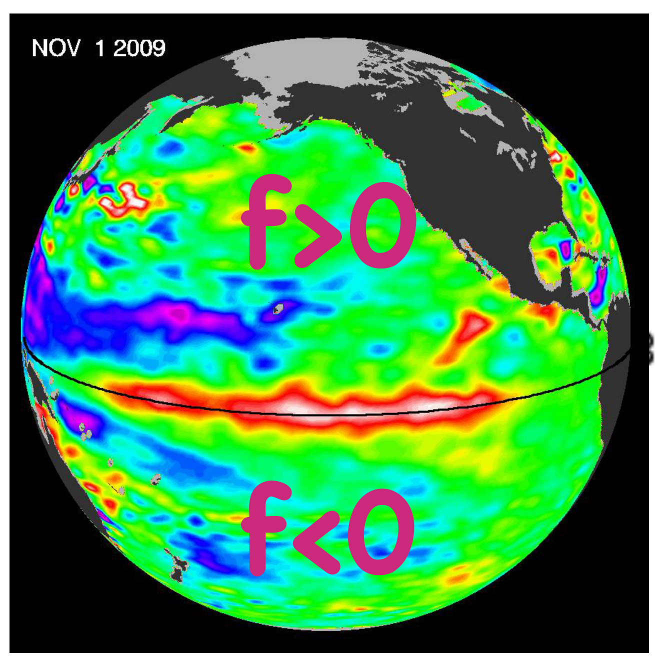
Kelvin 1879

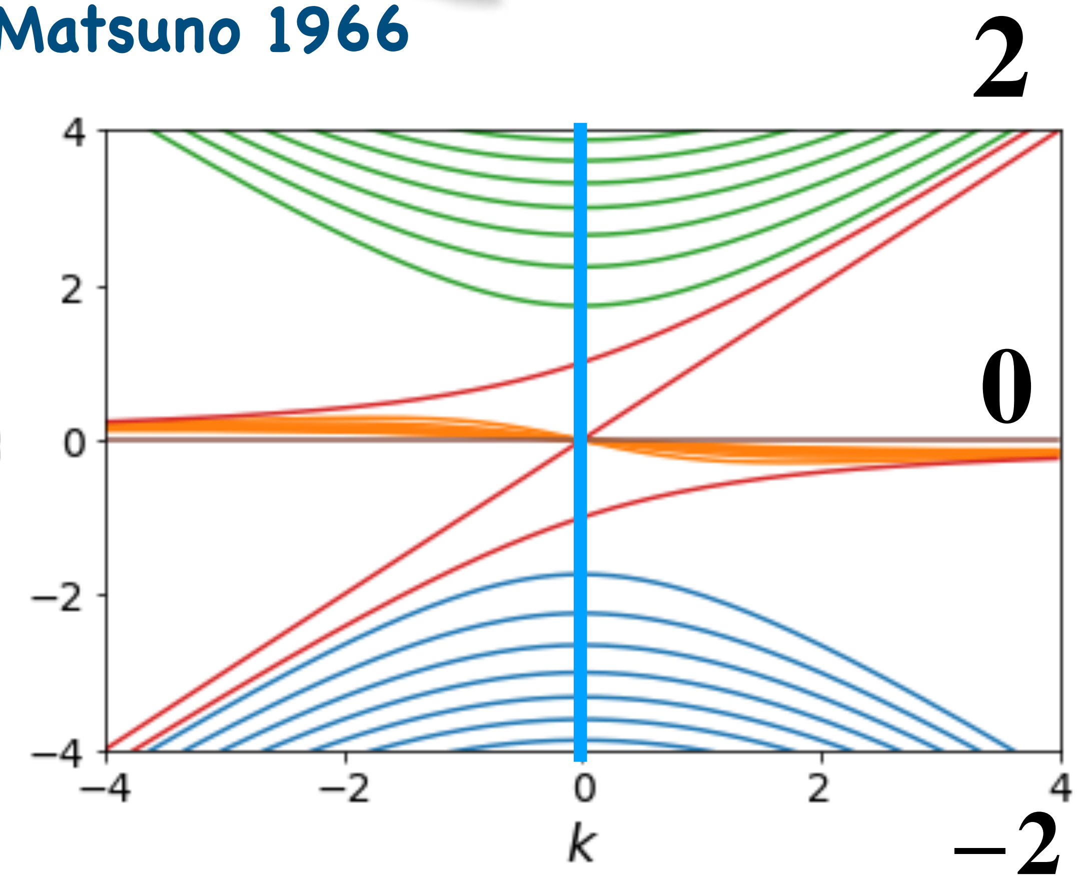


Topological index

Interface : f(y)

Matsuno 1966





Spectral index

The correspondence is guaranteed by the Atiyah-Singer index theorem Faure 2018

Bulk-edge correspondence

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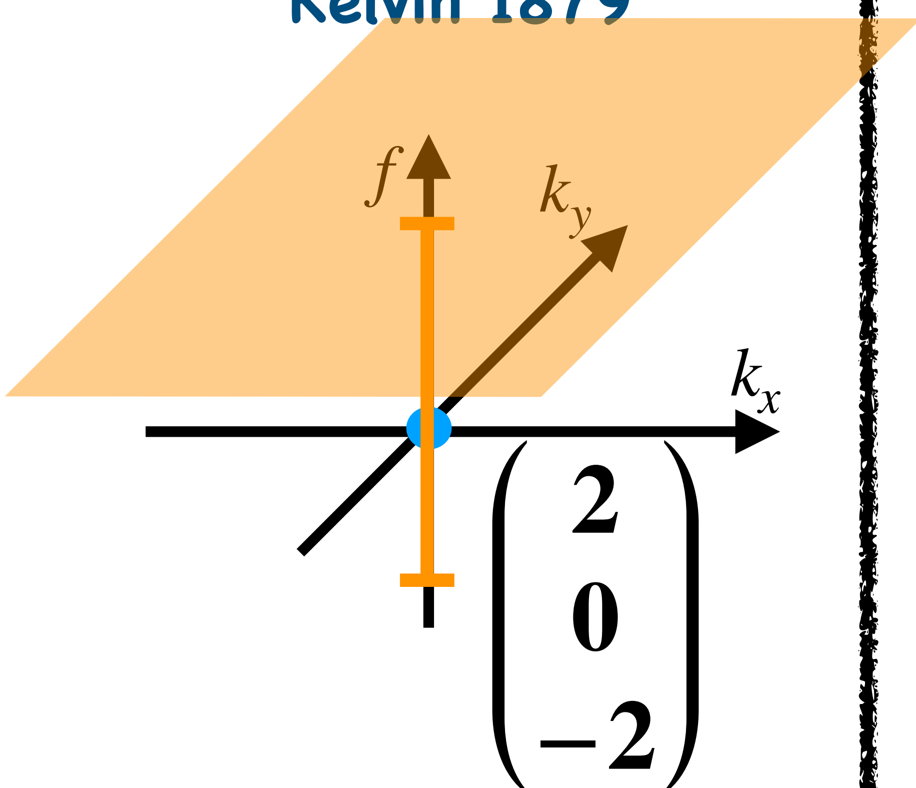

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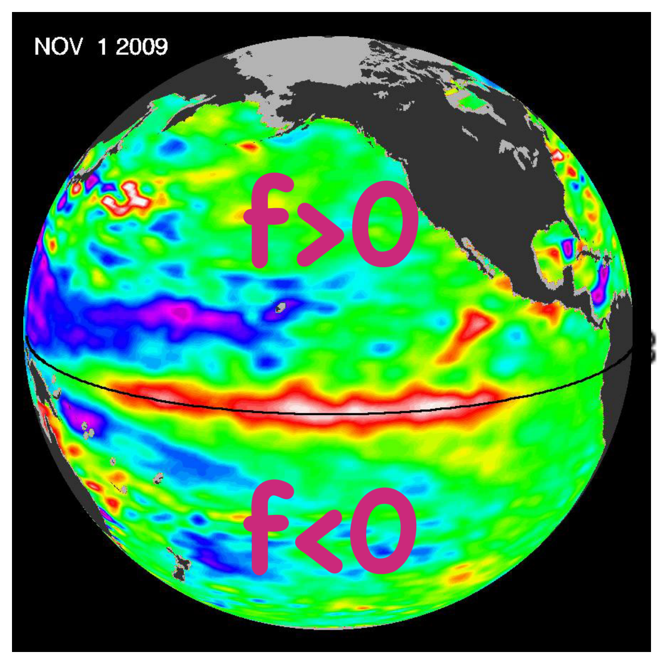

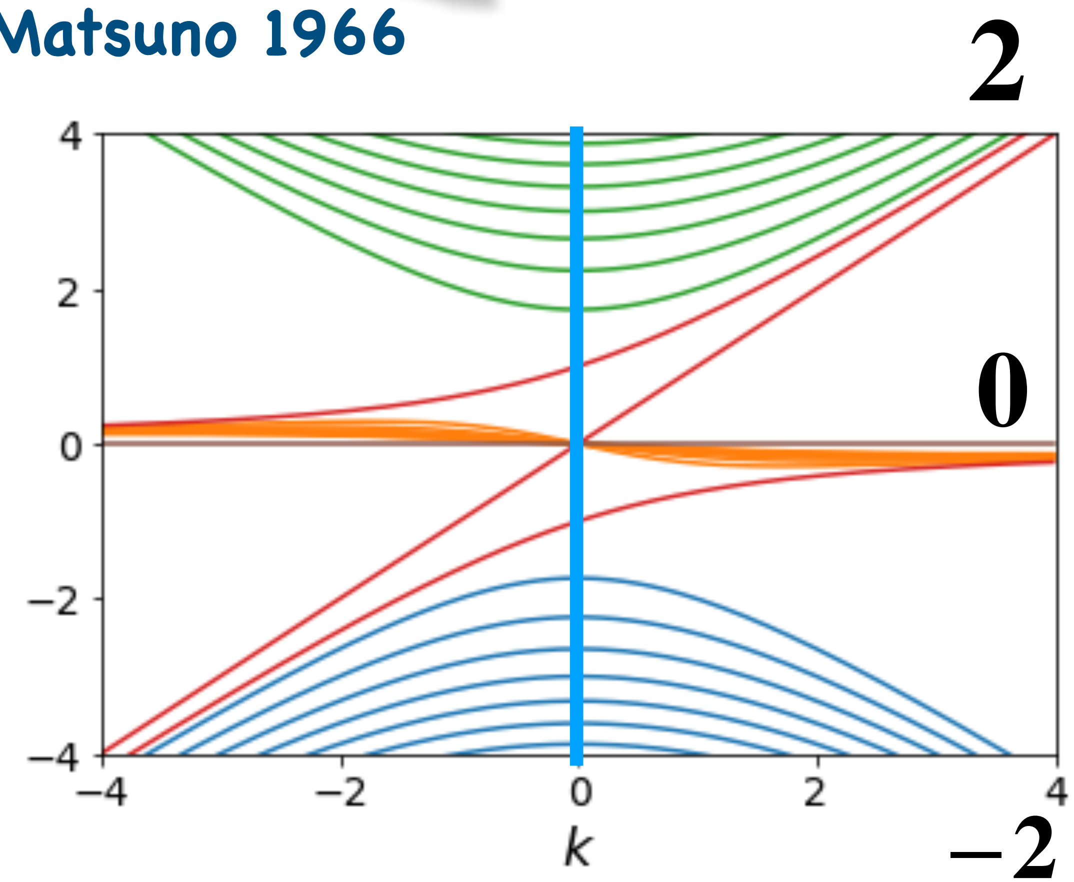

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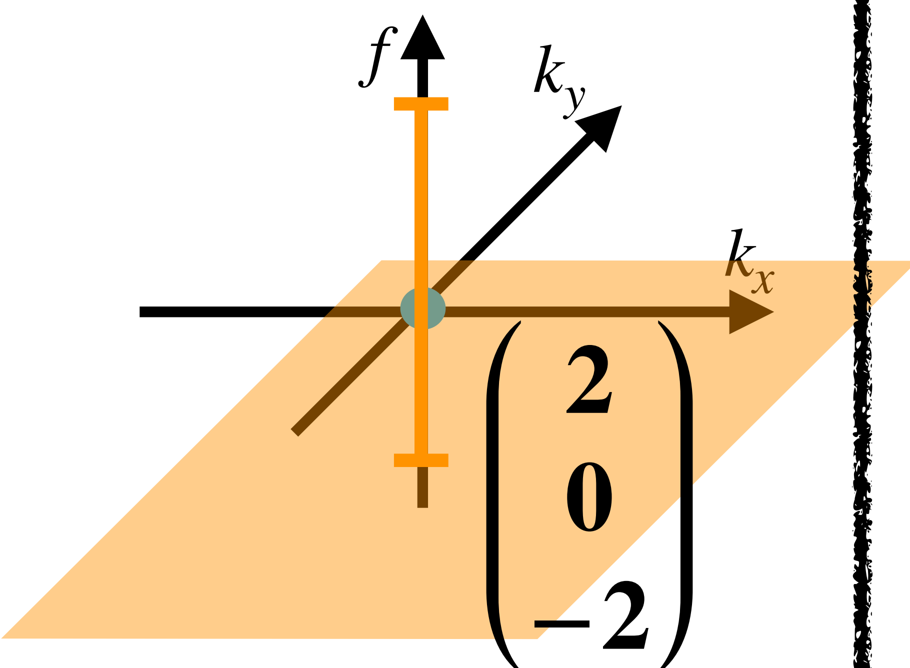

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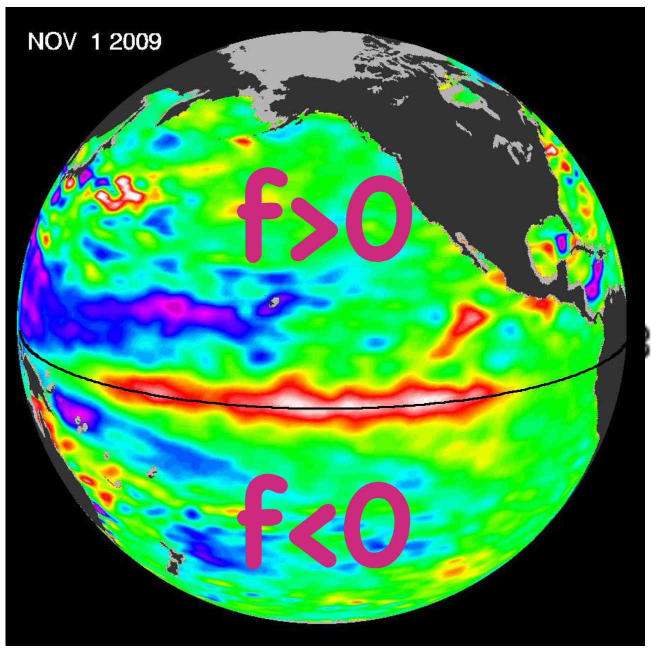

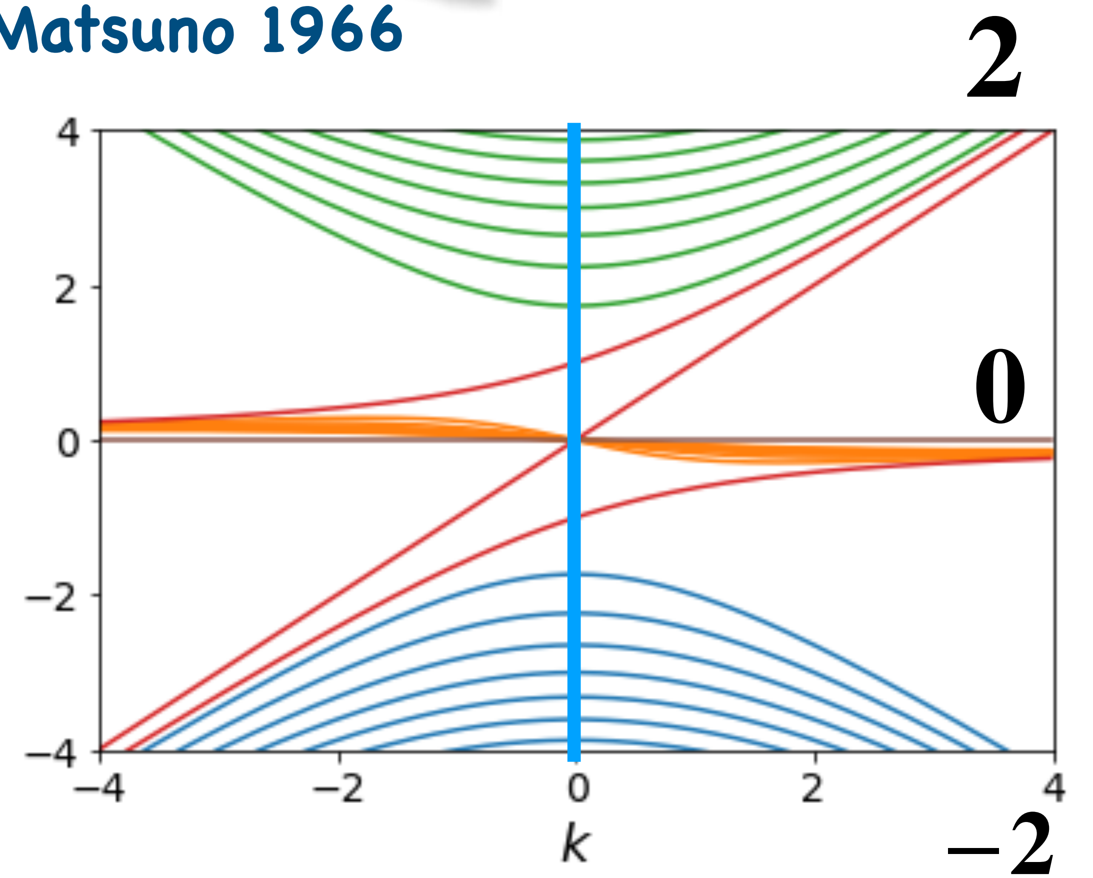

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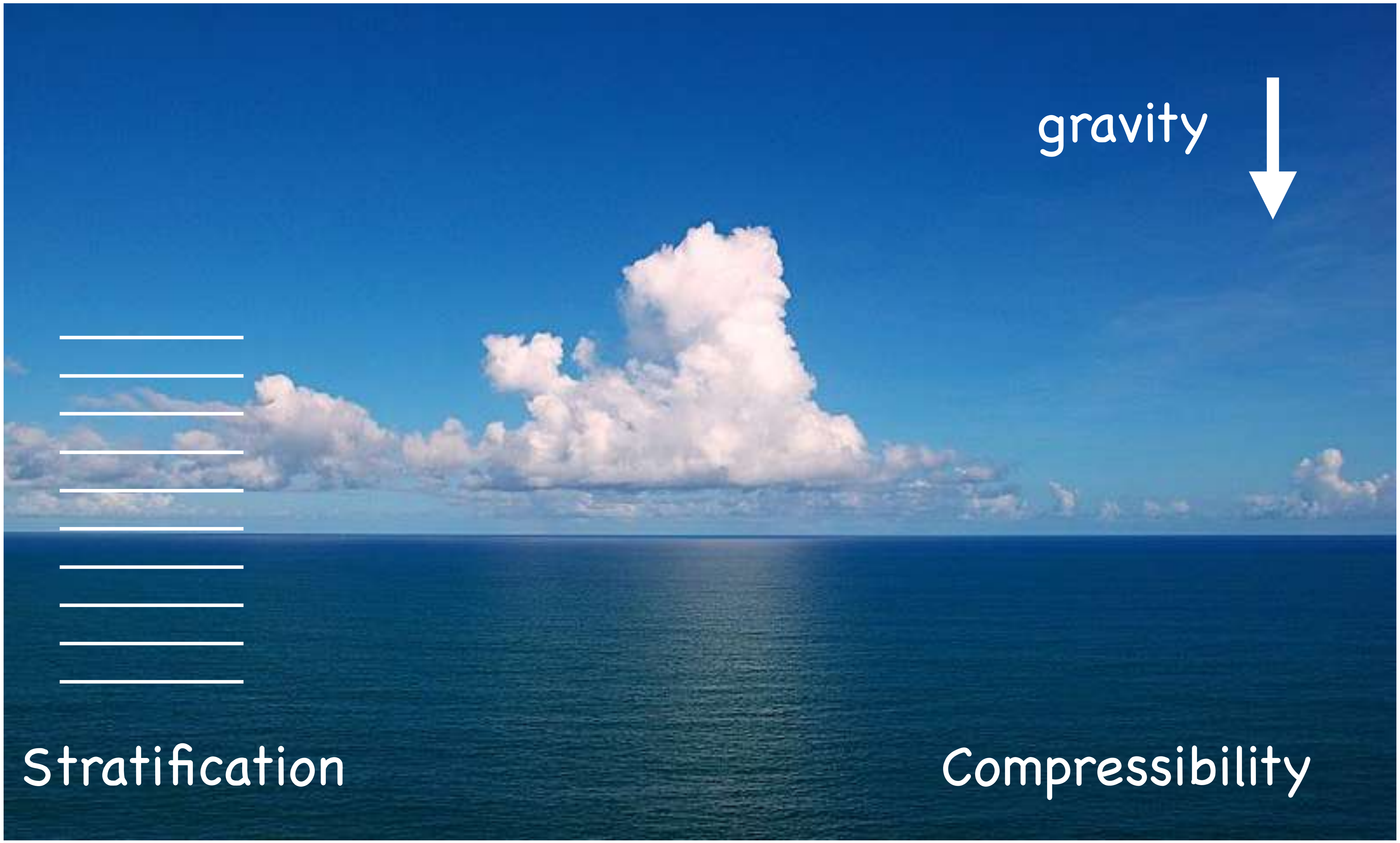
II. Acoustic-Gravity Waves

Breaking mirror symmetry



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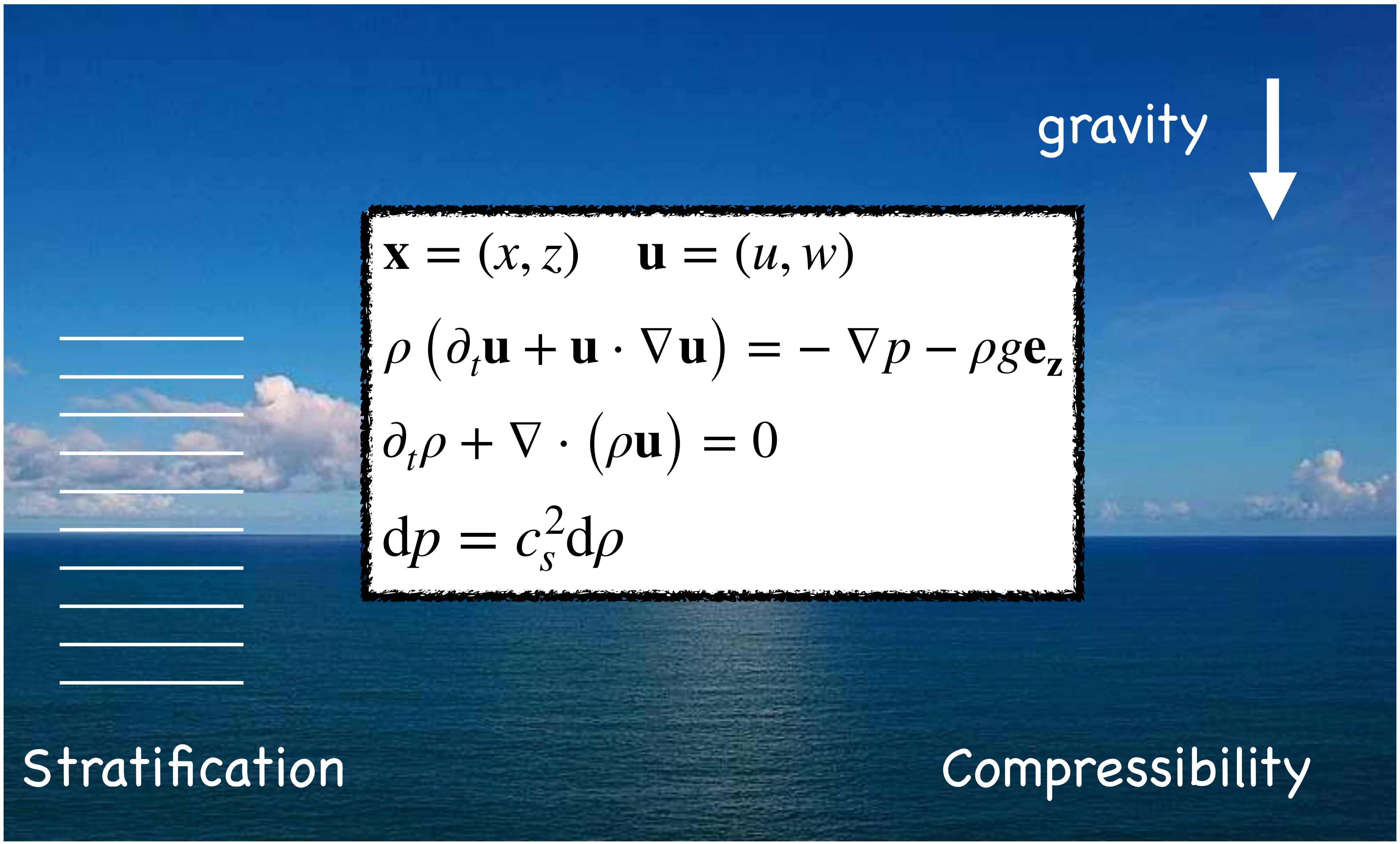

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$$\rho\left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p - \rho g \mathbf{e_z}$$

$$\partial_t \rho + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\mathrm{d}p = c_s^2 \mathrm{d}\rho$$

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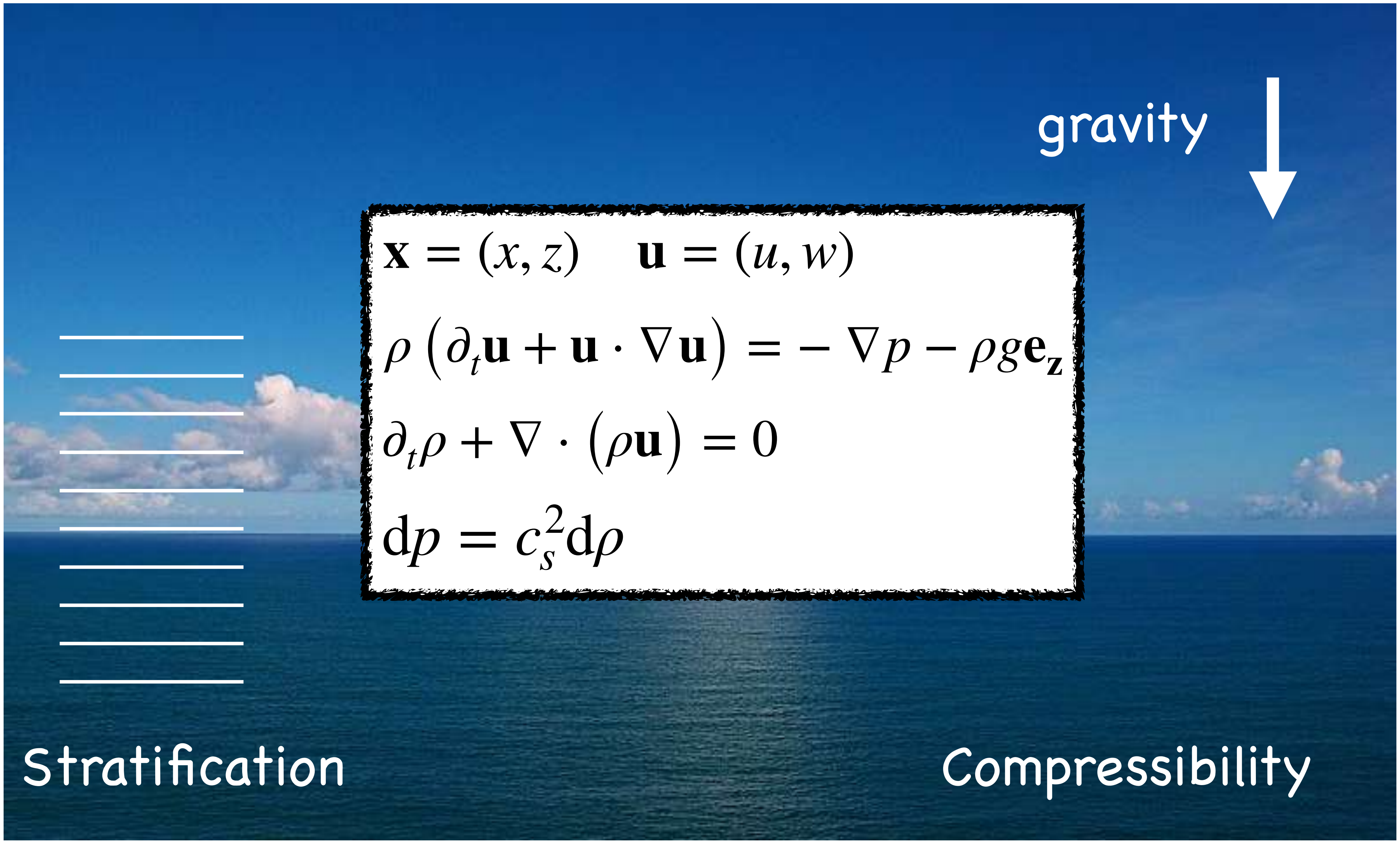

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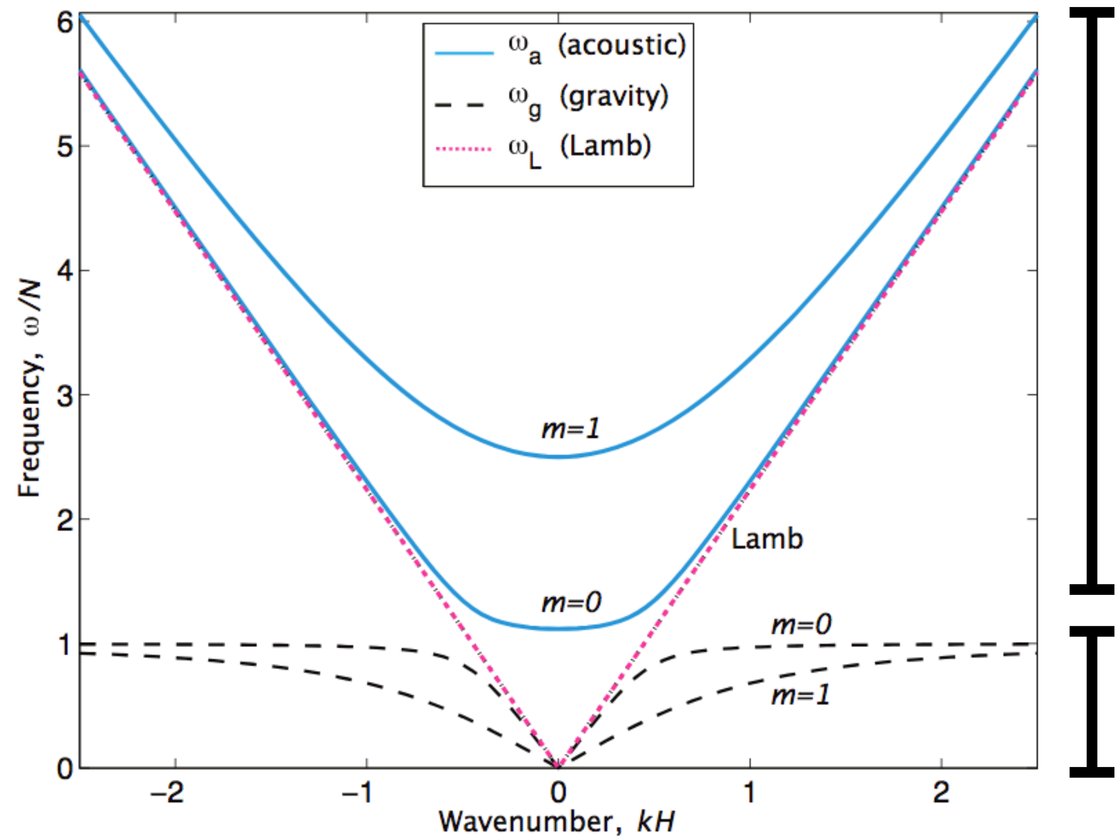
$$\mathrm{d}p = c_s^2 \mathrm{d}\rho$$

Linearize around a state of rest with density profile $\rho_o(z)$

Acoustic-gravity waves



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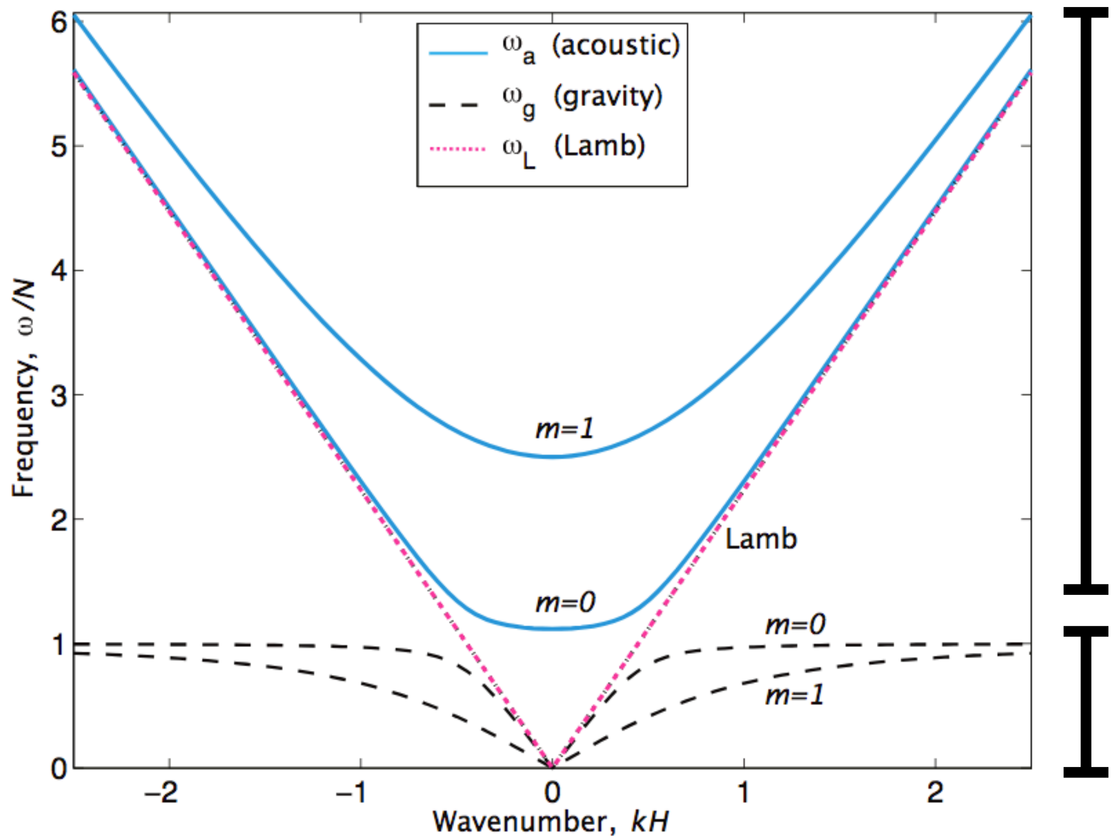
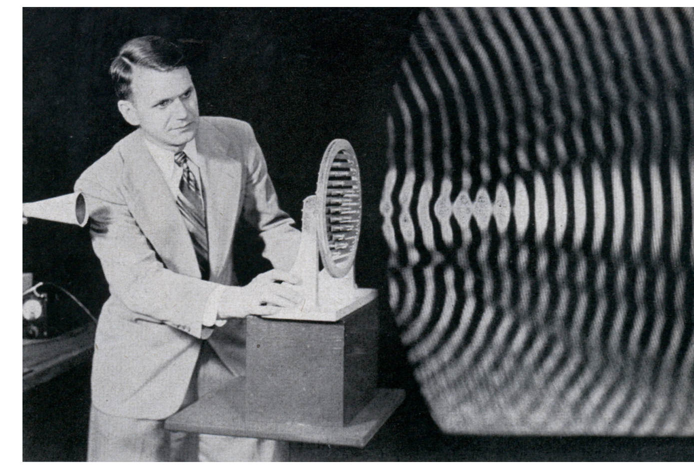
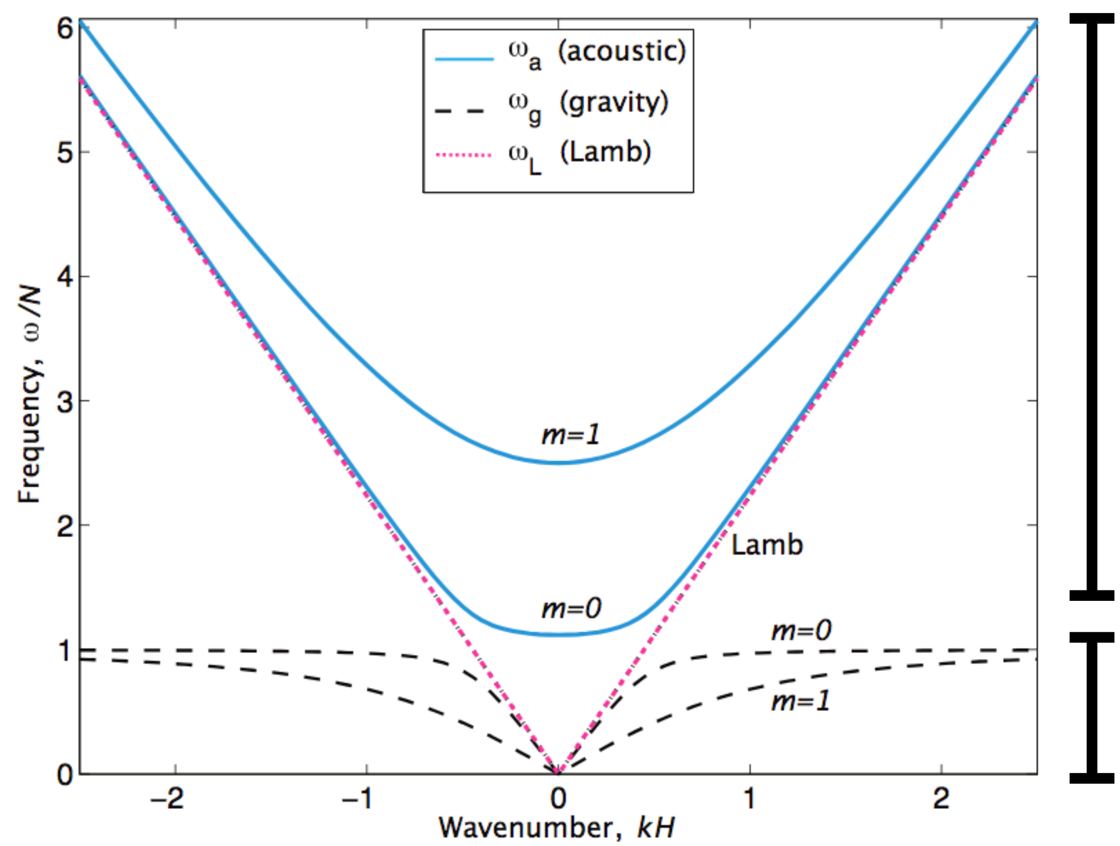
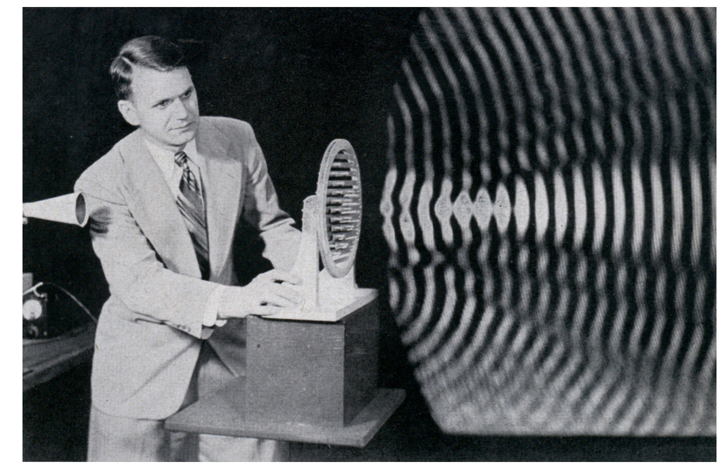




Compressibility: sound waves

Acoustic-gravity waves

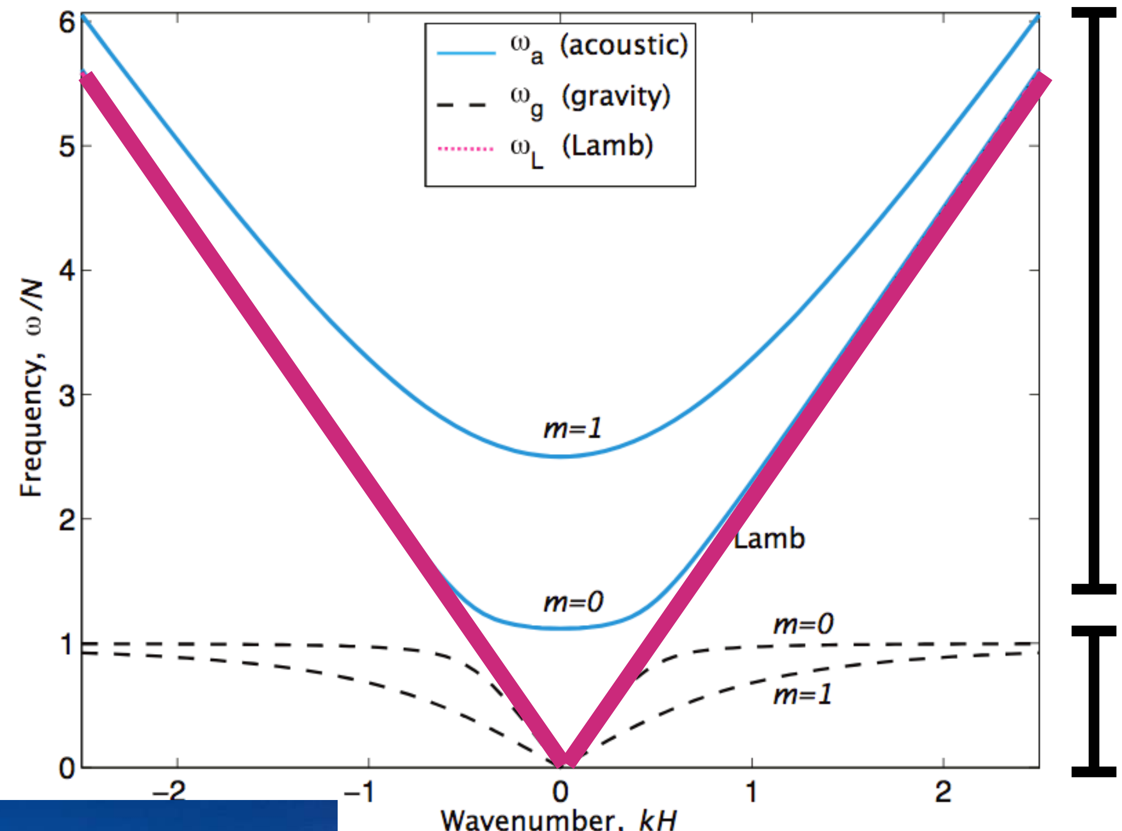


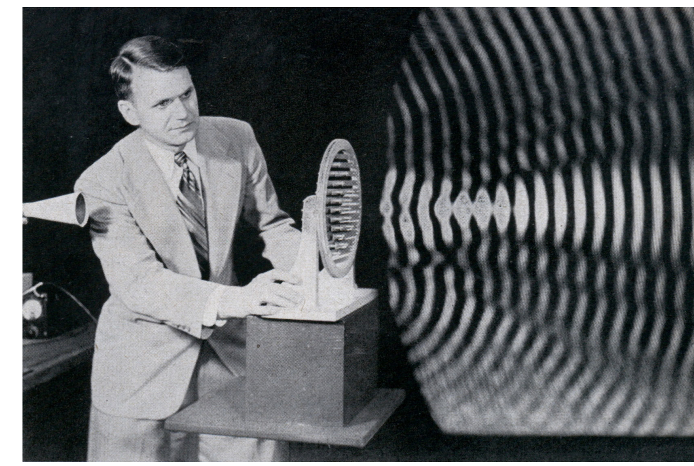


Compressibility: sound waves

Buoyancy: Gravity Waves

Acoustic-gravity waves





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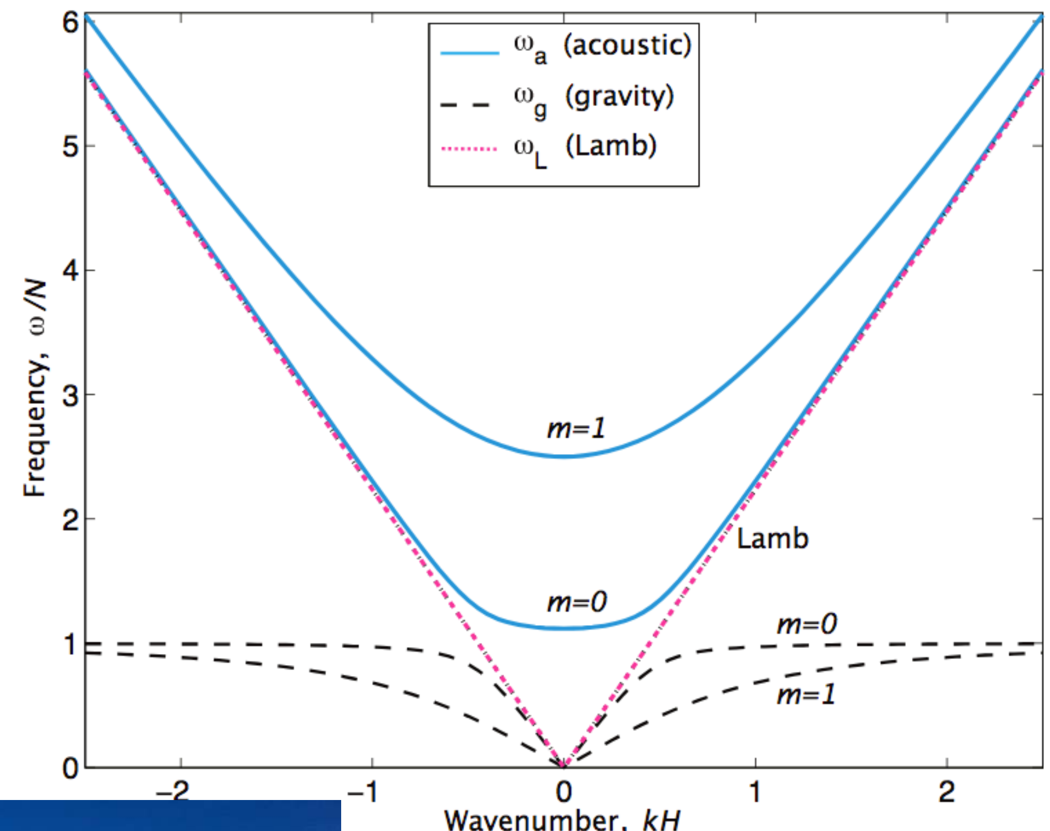
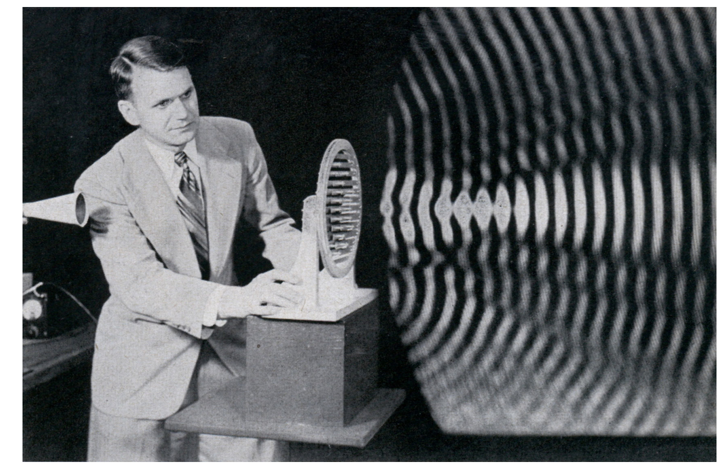




Compressibility: sound waves

Buoyancy: Gravity Waves

Lamb Waves

What is the origin of this edge mode ?

Does it always exist ?

Linearized equation

$$\partial_t \begin{pmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{\theta} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -c_s\partial_x \\ 0 & 0 & -N & S - c_s\partial_z \\ 0 & N & 0 & 0 \\ -c_s\partial_x & -S - c_s\partial_z & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{\theta} \\ \tilde{p} \end{pmatrix}$$

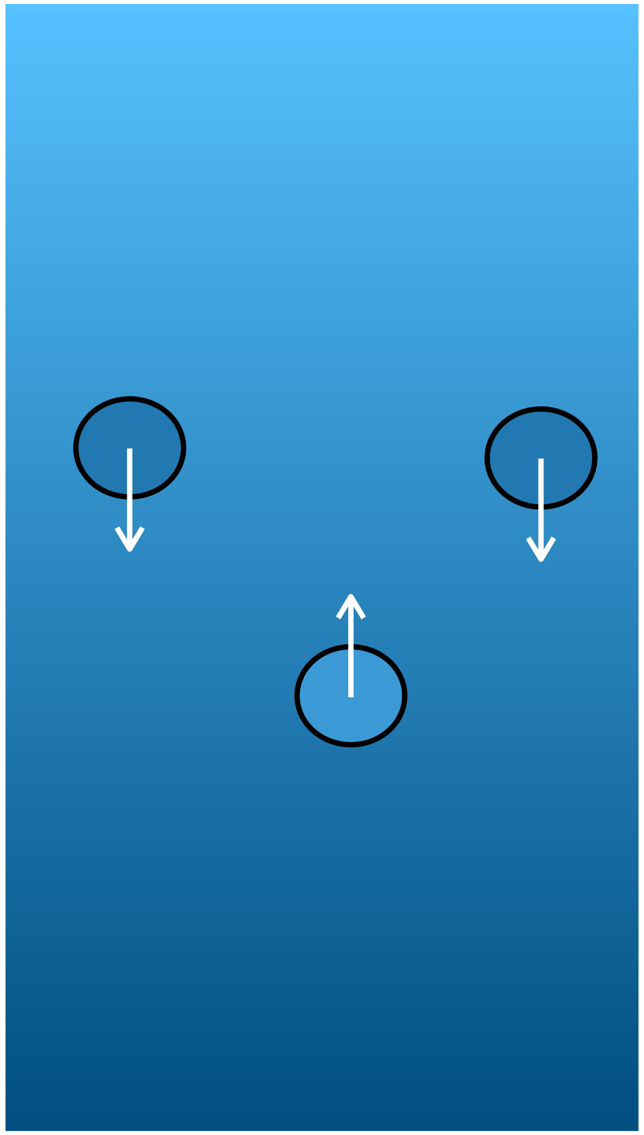
Buoyancy frequency

$$N^2 = -g\frac{\partial_z \rho_0}{\rho_0} - \frac{g^2}{c_s^2}$$

Stratification parameter

$$S = \frac{1}{2}\left(\frac{N^2 c_s}{g} - \frac{g}{c_s}\right)$$

Mirror symmetry recovered when S=0 !

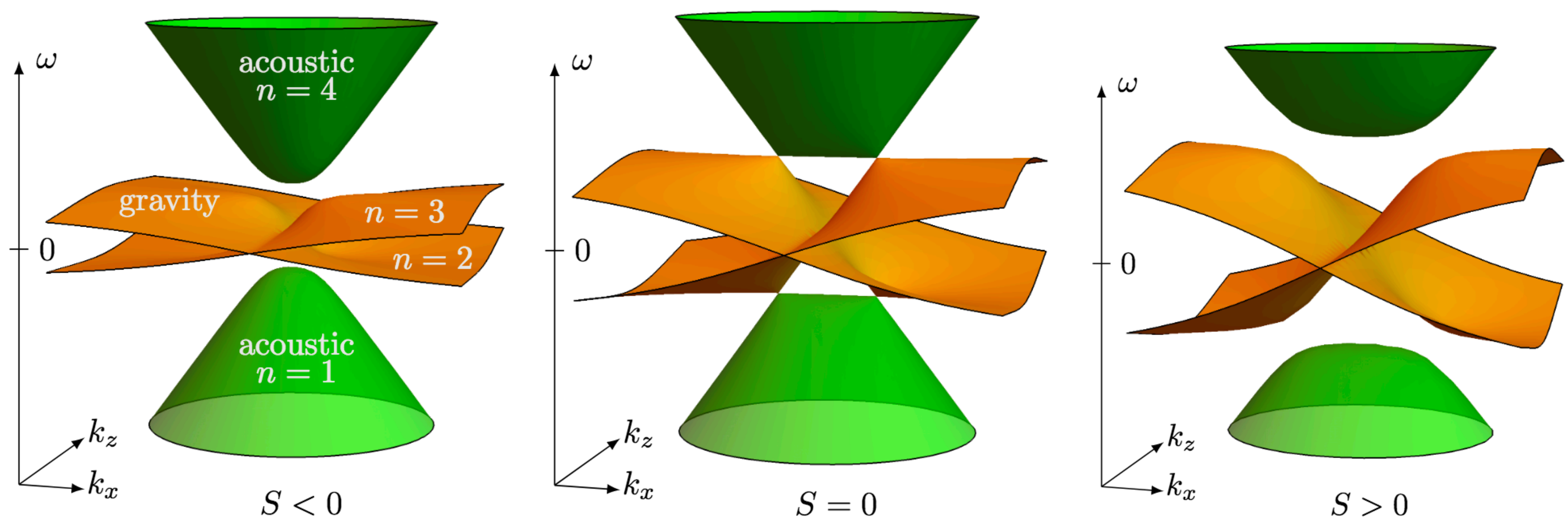


Bulk spectrum

$g = c_s = 1$

$$S = \frac{1}{2}\left(N^2 - 1\right)$$

Project on $e^{i\omega t - ik_x x - ik_z z}$



Manolis Perrot

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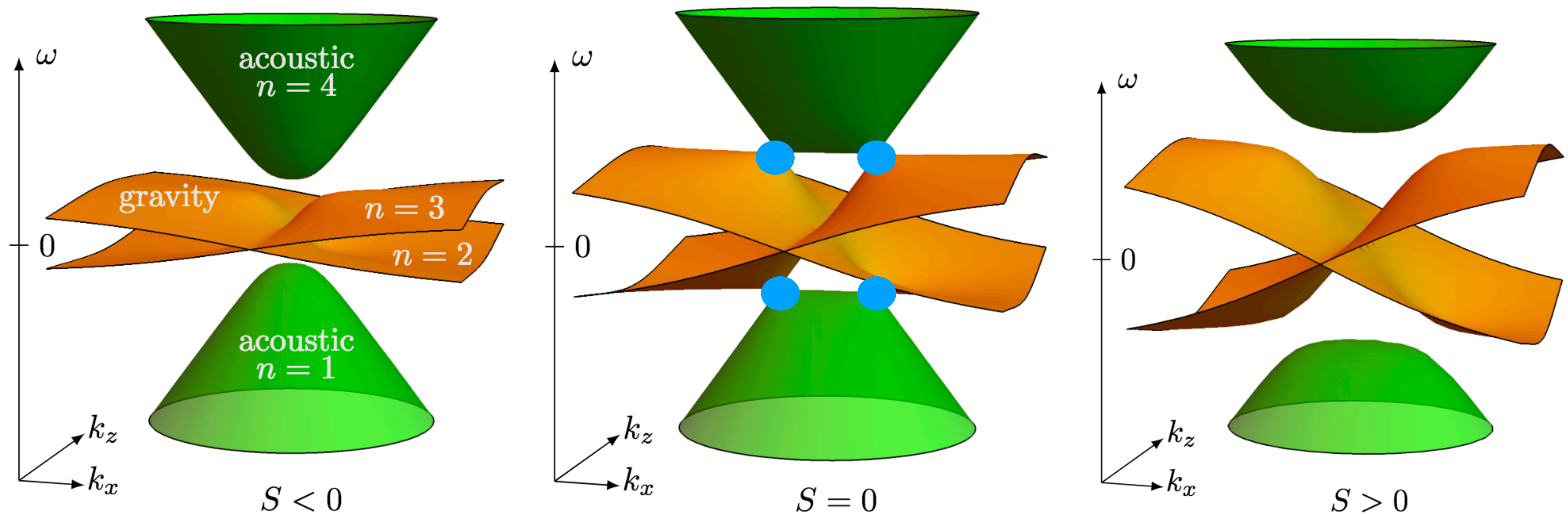
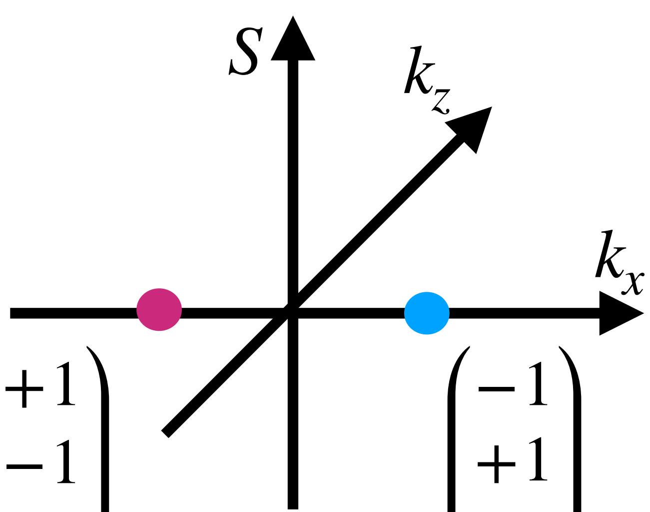


Four degeneracy points at S=0

Manolis Perrot

Spectral flow

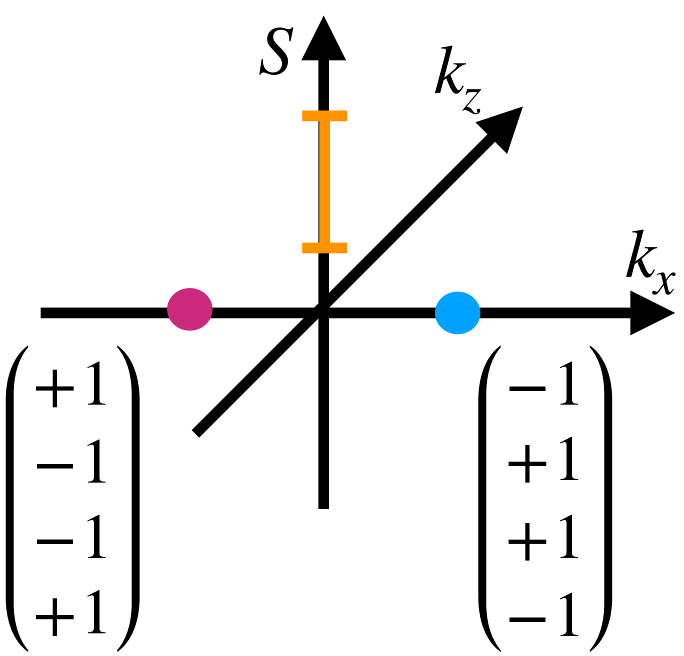
Bulk : S constant



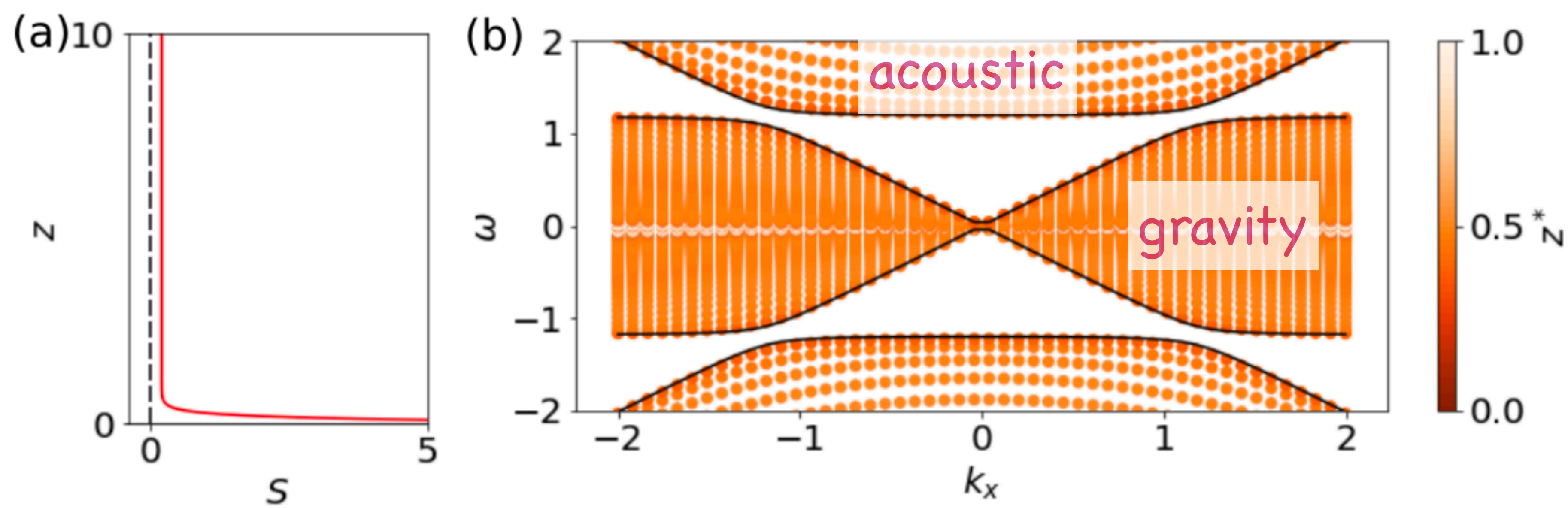
$$\begin{pmatrix} +1 \\ -1 \\ -1 \\ +1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ +1 \\ +1 \\ -1 \end{pmatrix}$$

Spectral flow

Bulk : S constant



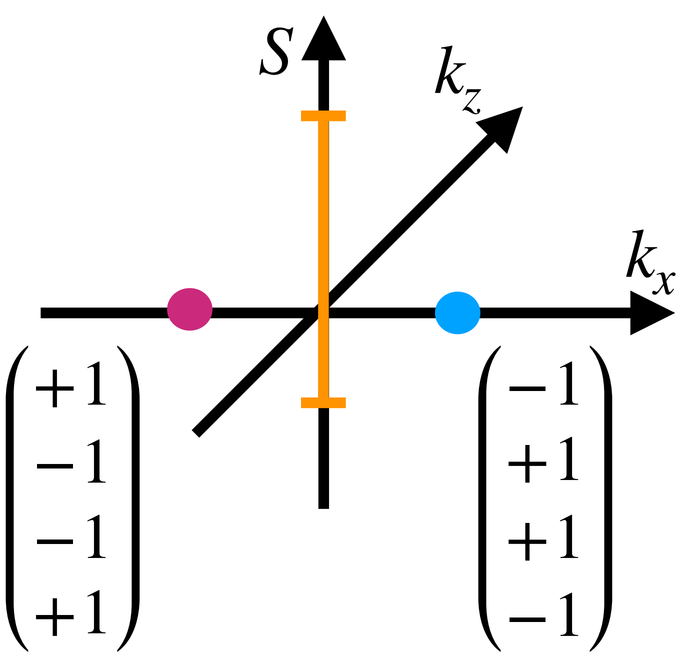
Interface : S(z)

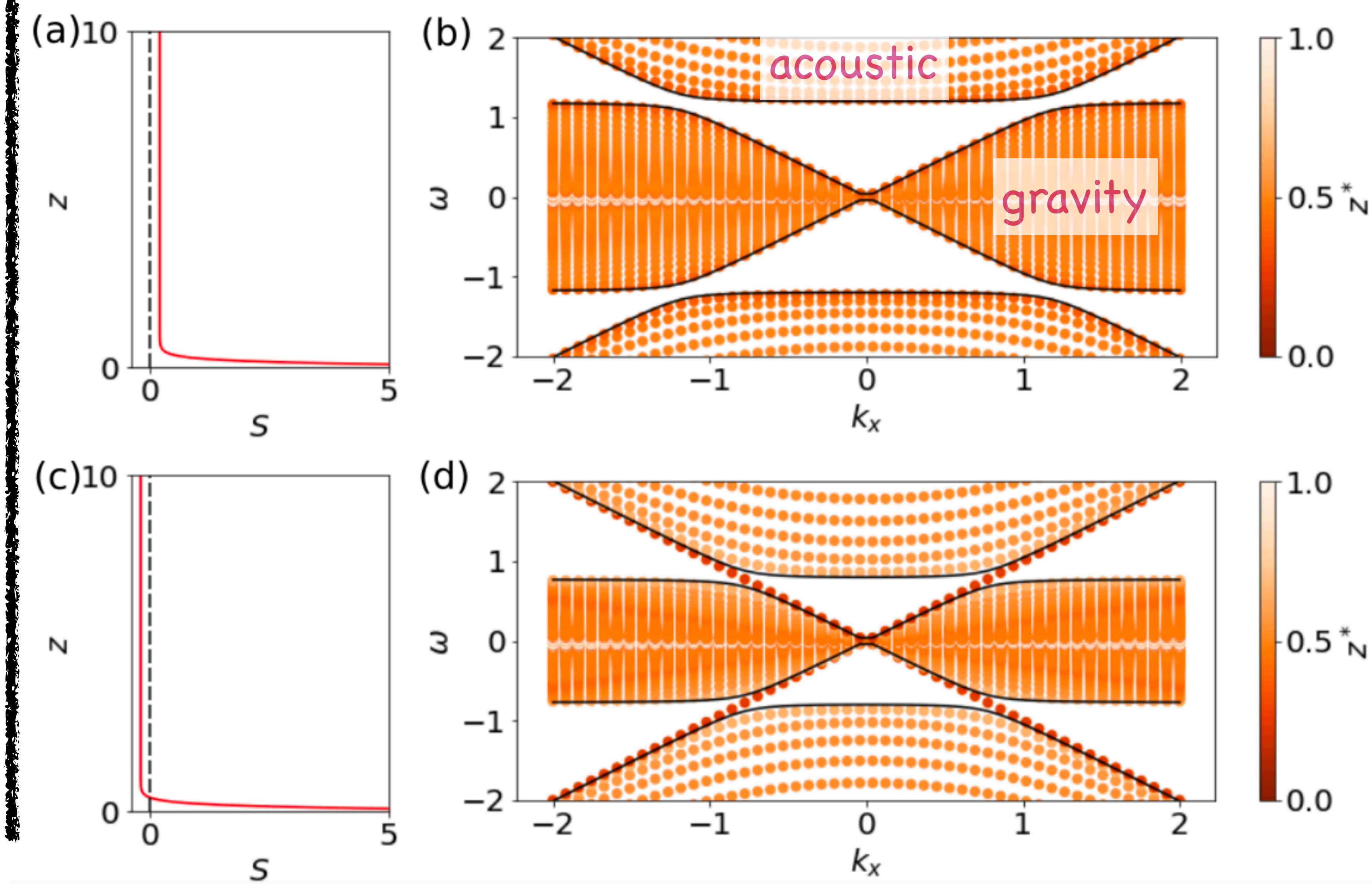


Spectral flow

Bulk : S constant

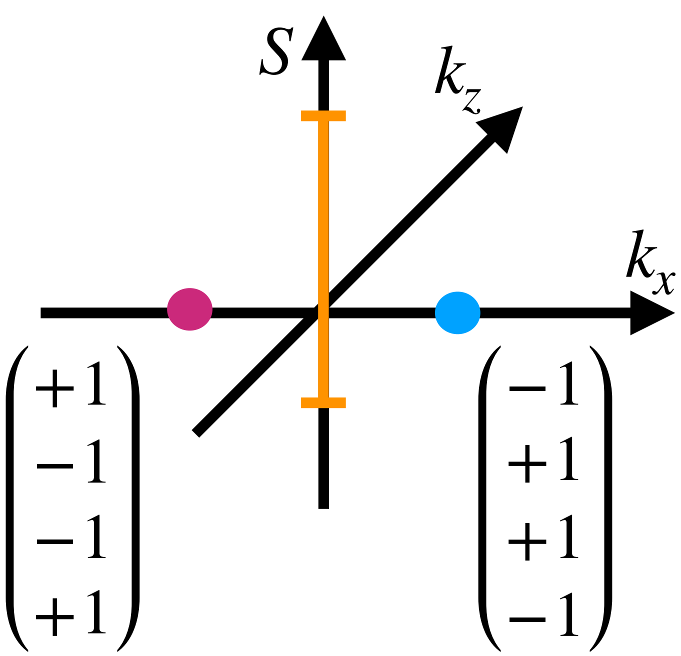
Interface : S(z)



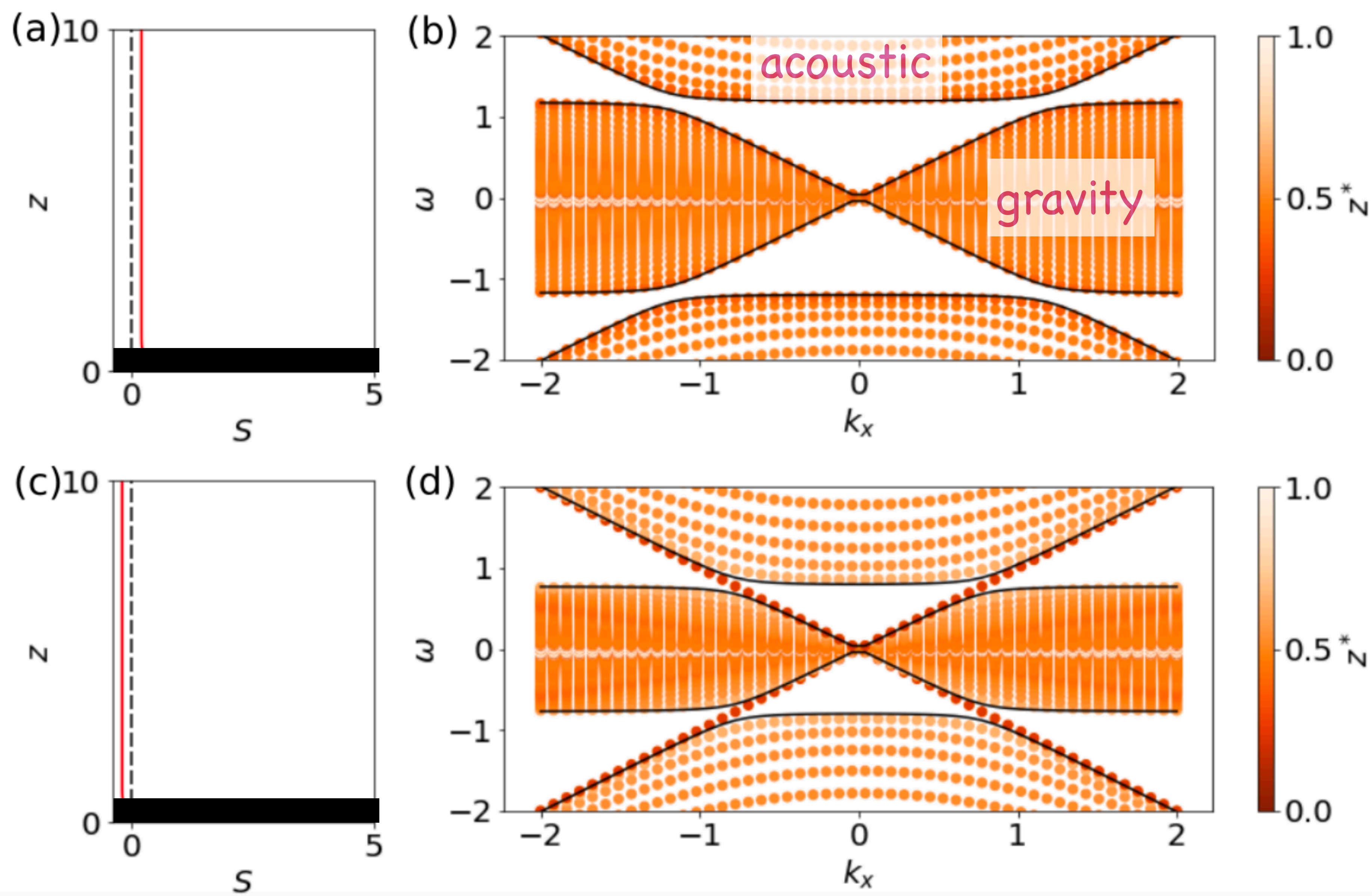


Spectral flow

Bulk : S constant



Interface : S(z)



Solid boundary

Prospect : observation of Lamb-like waves?

Existence of a point in the interior where

$$N = \frac{g}{c_s}$$

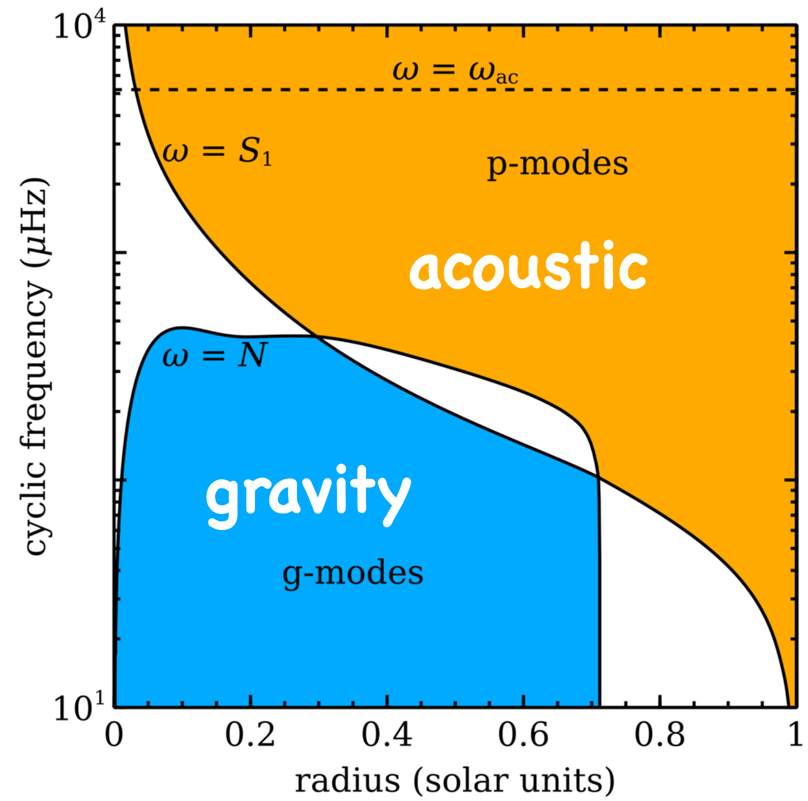
Finite size effects avoided if

$$H \gg \frac{c_s^2}{g}$$

In the ocean

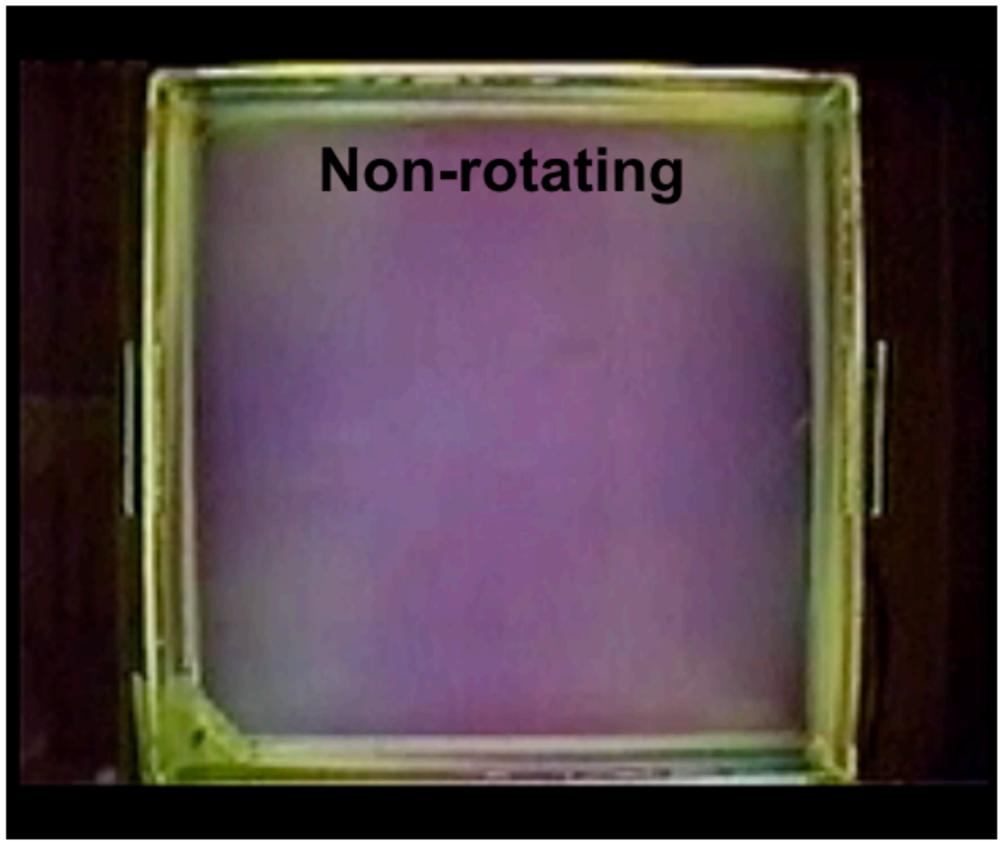
Iga 2001 predicts absence of Lamb like due to finite size effect

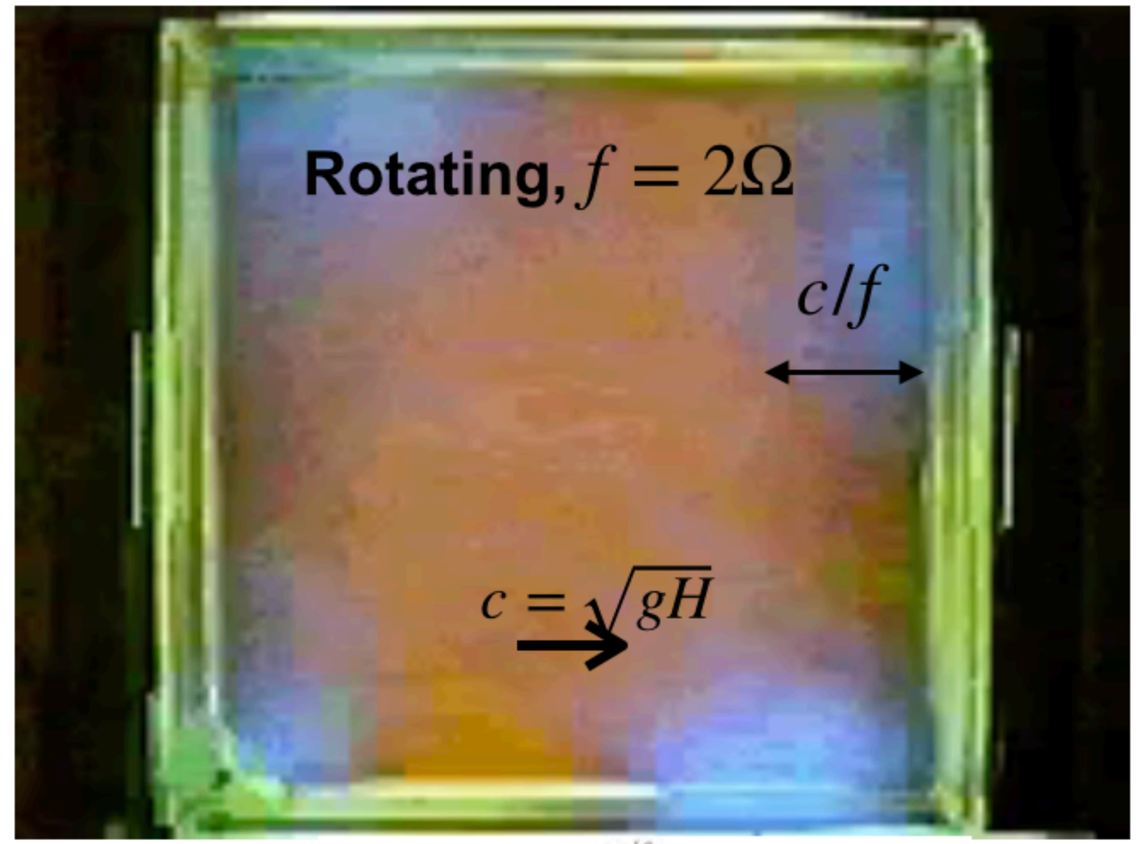
In stars

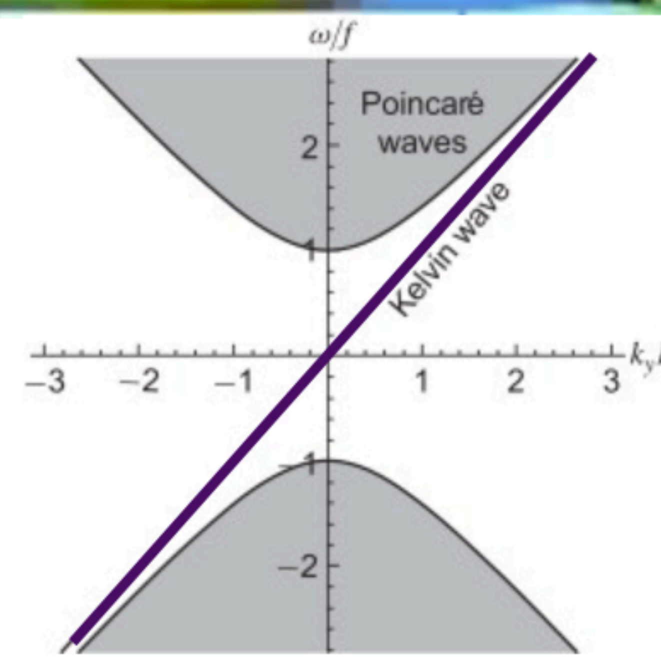


III. Other manifestations of topology in geophysical waves

III.1 Coastal Kelvin waves

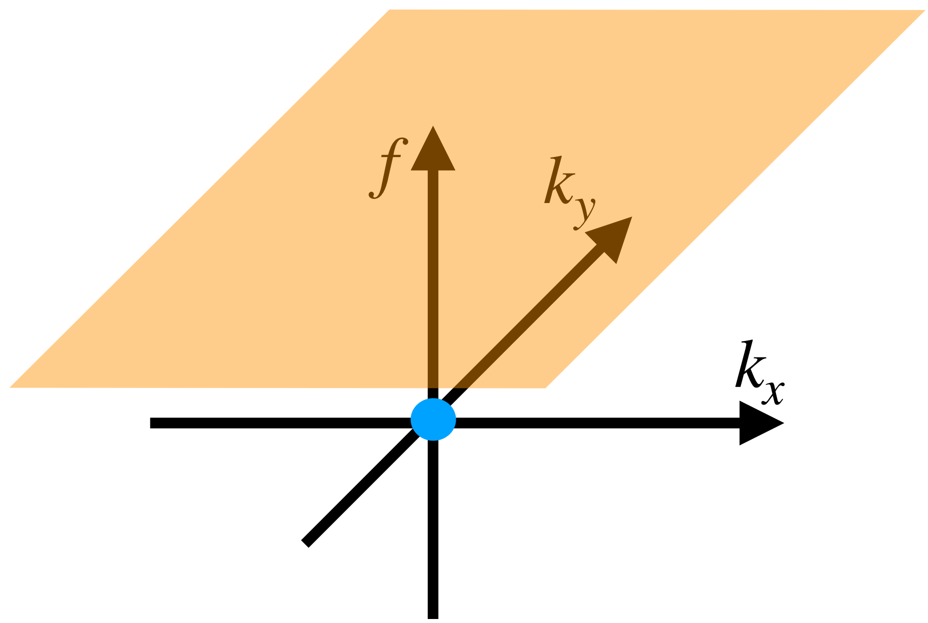




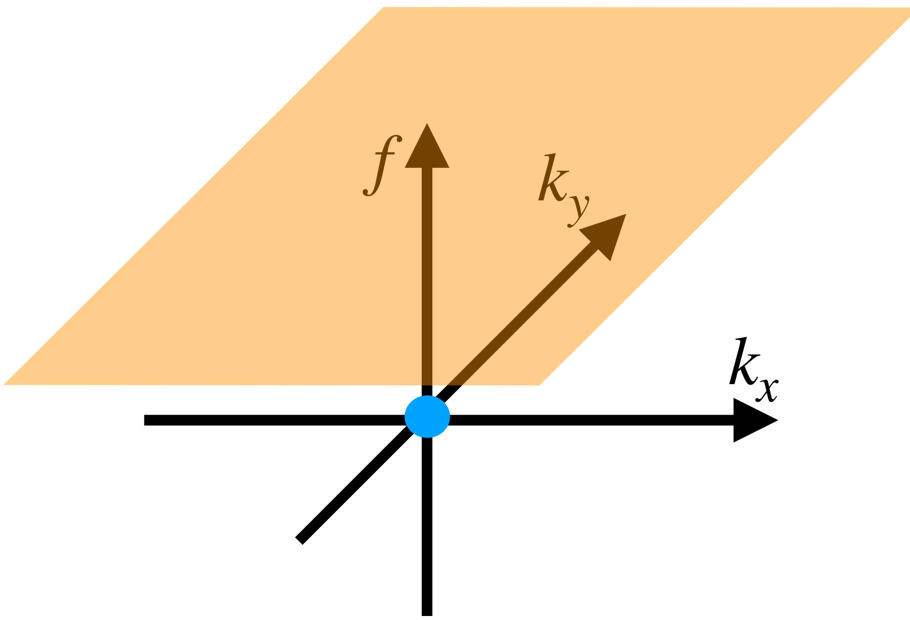


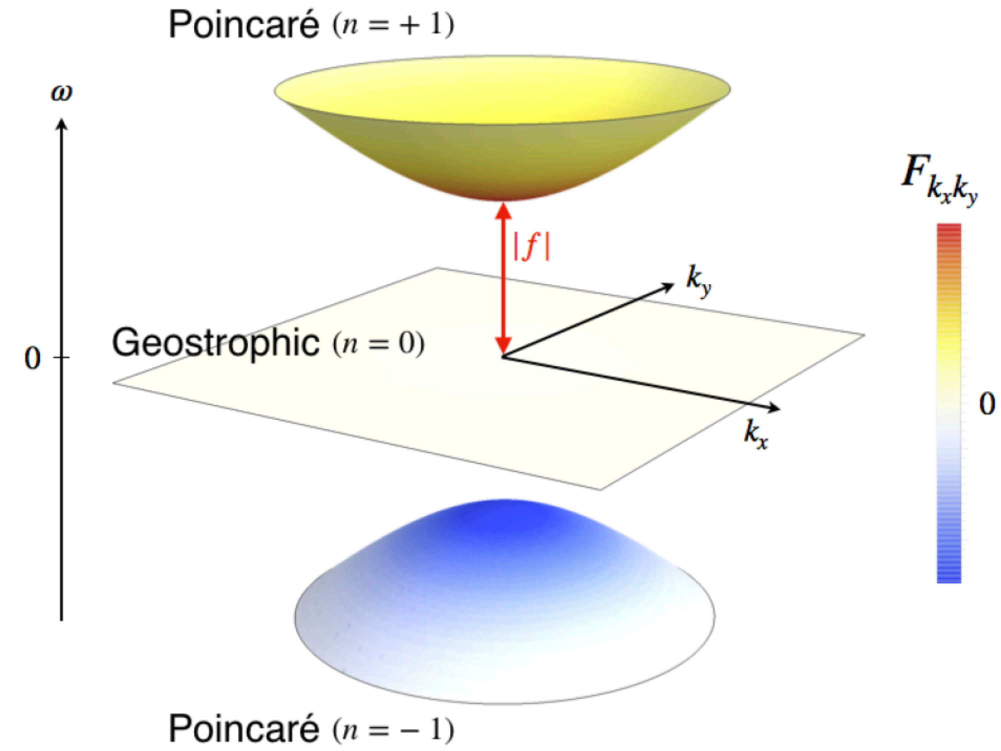
Is it topological? short answer is probably not. But…

A bulk topological invariant for f-plane waves ?

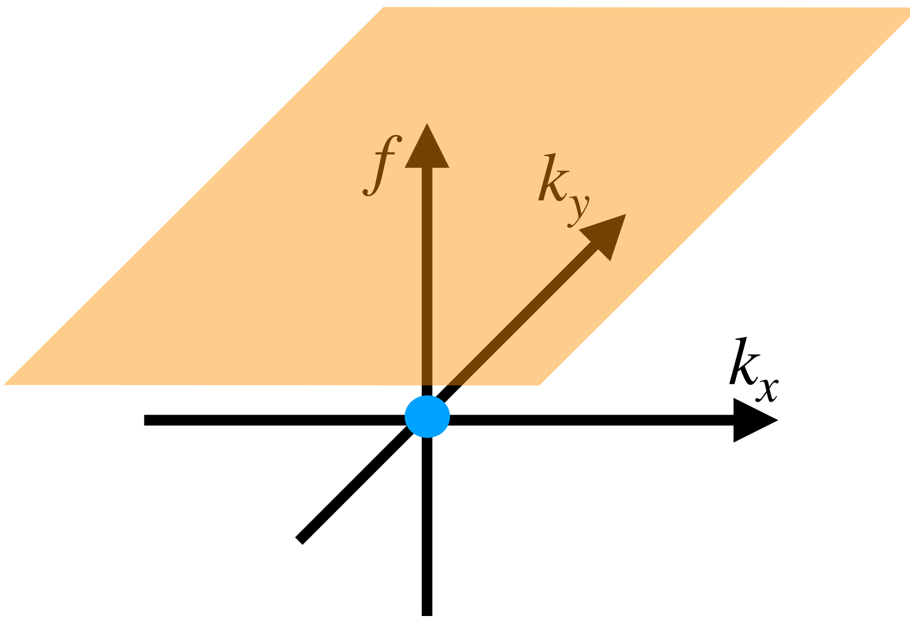


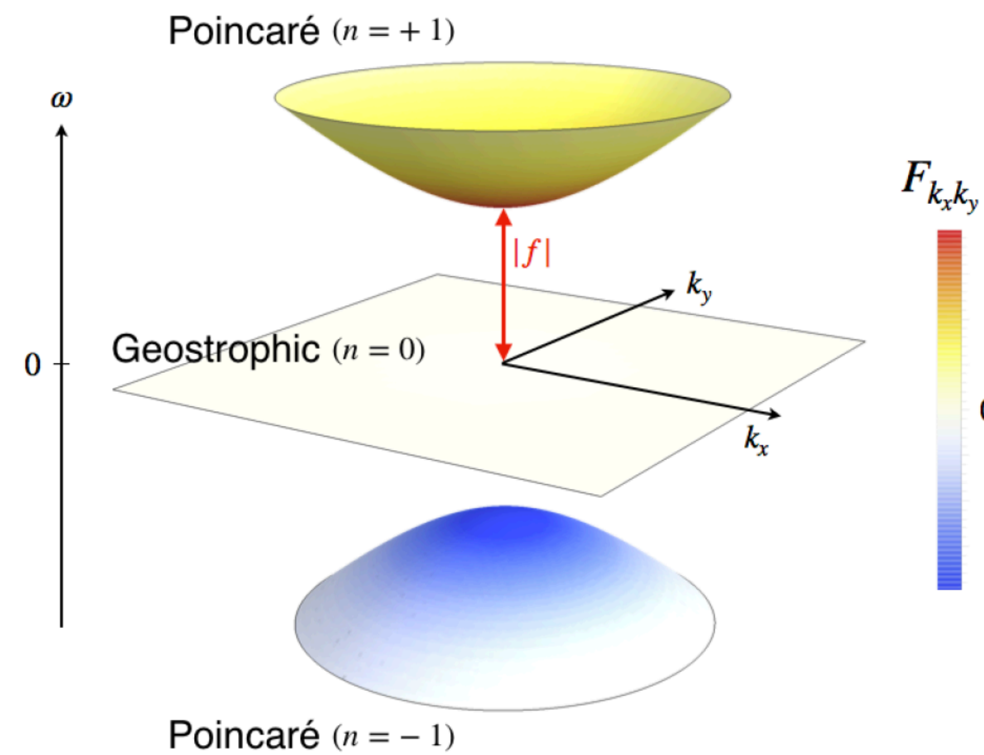
A bulk topological invariant for f-plane waves ?





A bulk topological invariant for f-plane waves ?



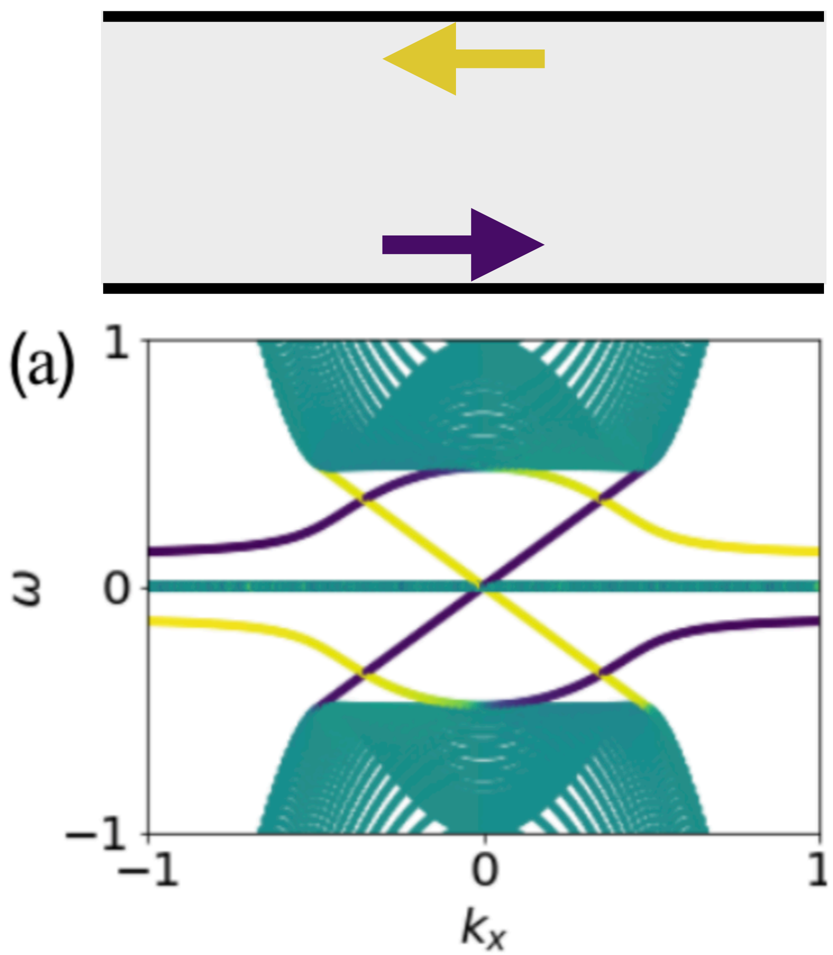


Clément Tauber

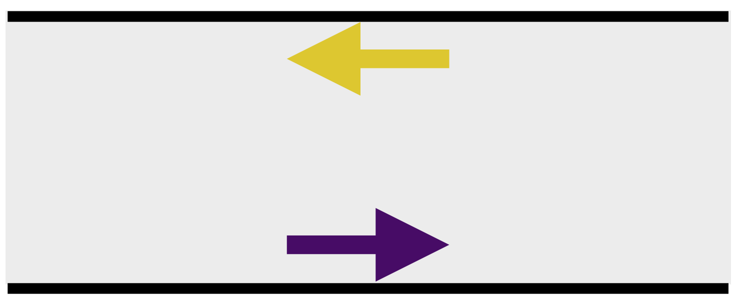
Need a regularization, e.g. with odd viscosity

$$\partial_t u = -g\partial_x \eta + \left(f + \nu_o \Delta\right) v$$
$$\partial_t v = -g\partial_y \eta - \left(f + \nu_o \Delta\right) u$$
$$\partial_t \eta = -H\partial_x u - H\partial_y v$$

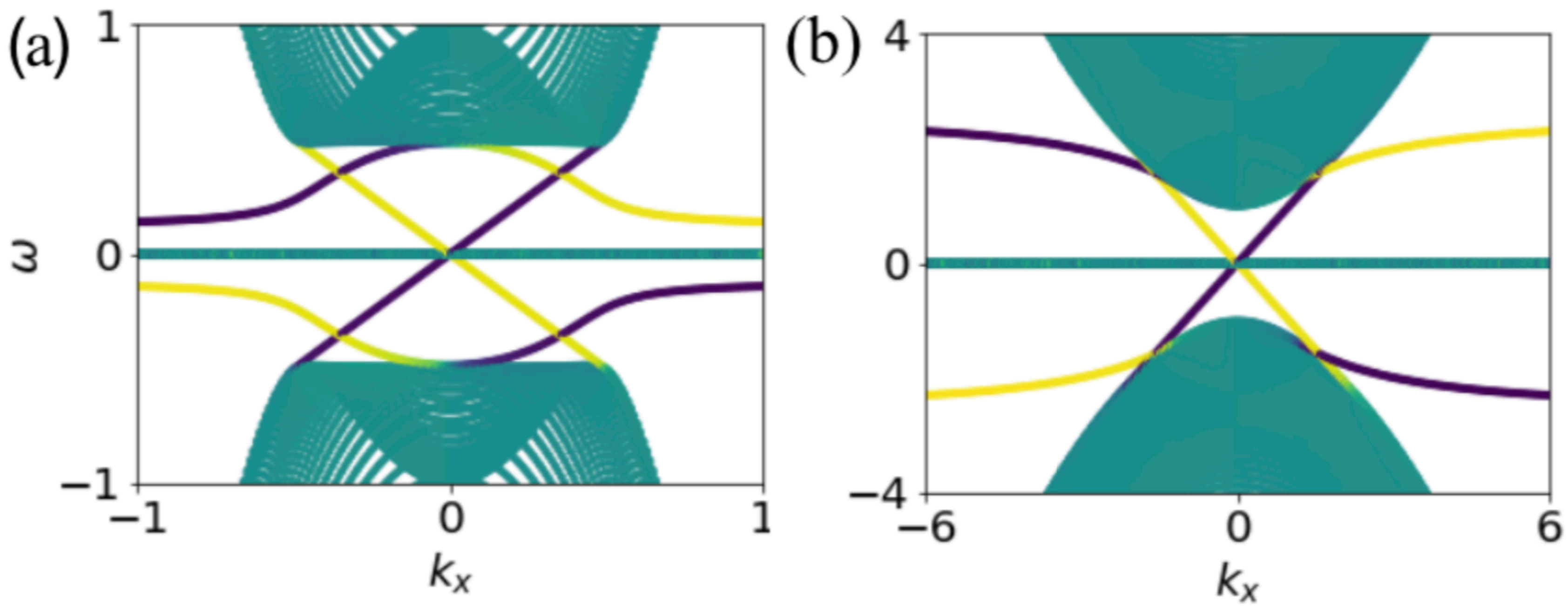
Apparent violation of bulk-boundary correspondence



Apparent violation of bulk-boundary correspondence

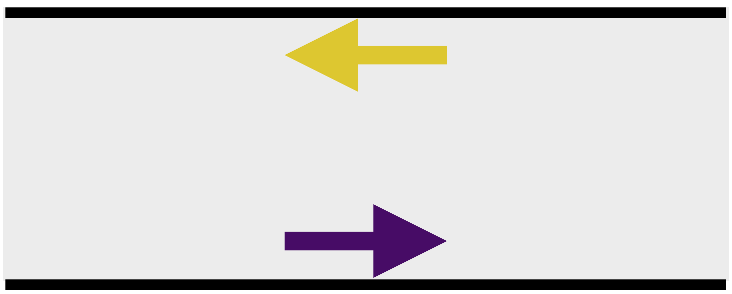


Change value of odd viscosity



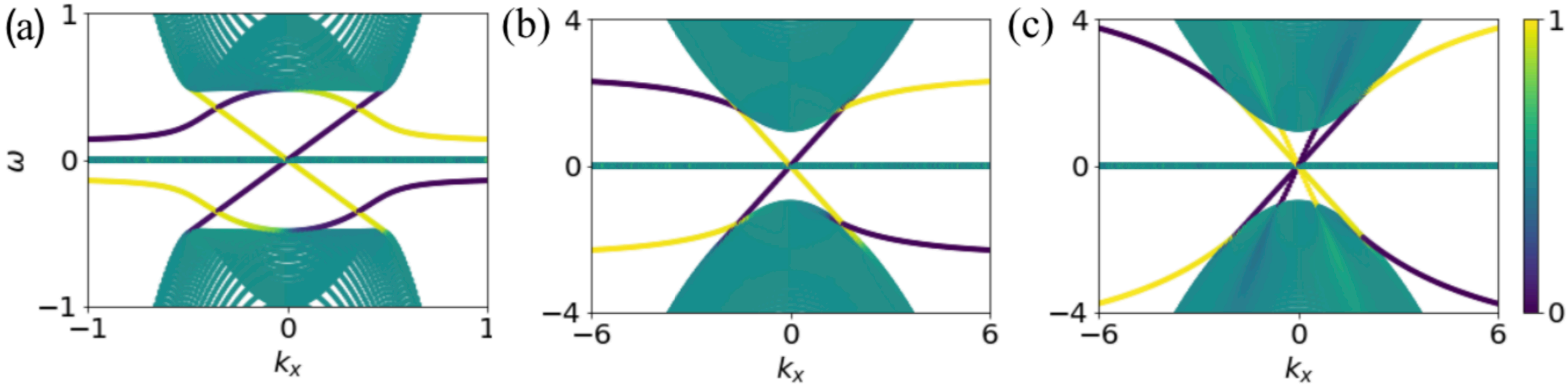
« The number of unidirectional states filling the frequency gap is topologically protected » **NO!**

Apparent violation of bulk-boundary correspondence



Change value of odd viscosity

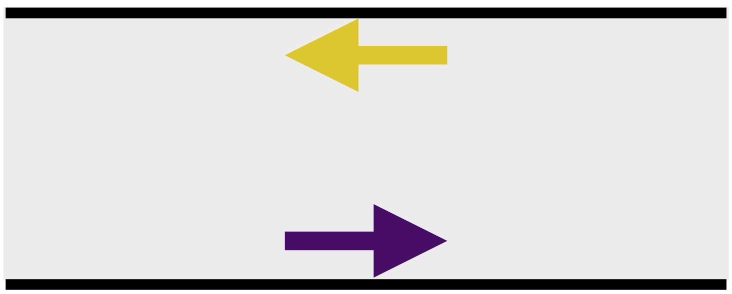
Change boundary condition



« The number of unidirectional states filling the frequency gap is topologically protected » **NO!**

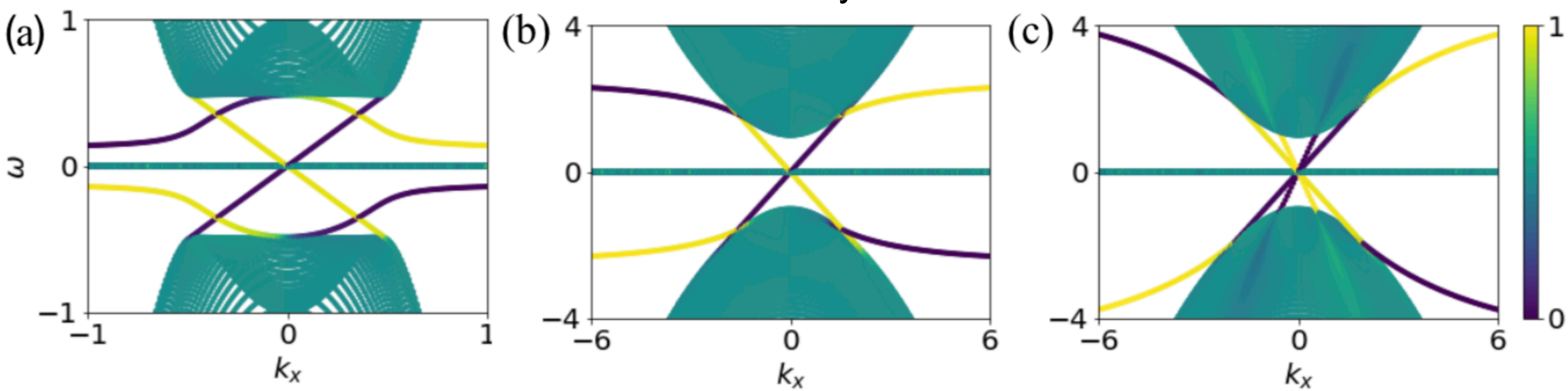
« The number of states gained by a given band when varying parameter kx is topologically protected » **NO!**

Apparent violation of bulk-boundary correspondence



Change value of odd viscosity

Change boundary condition

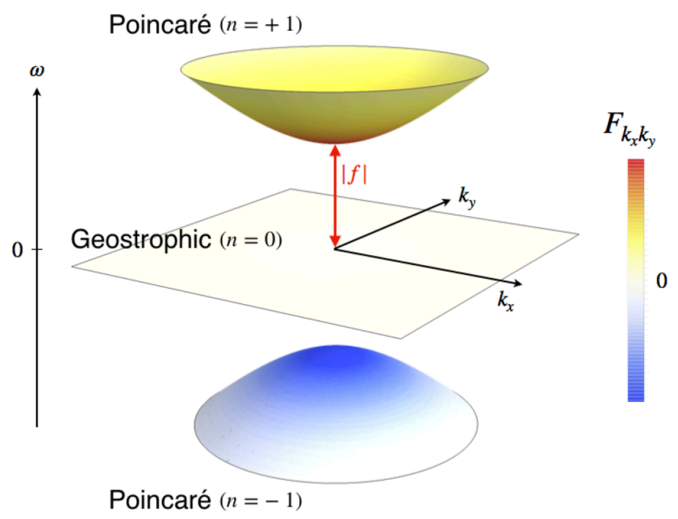


« The number of unidirectional states filling the frequency gap is topologically protected » **NO!**

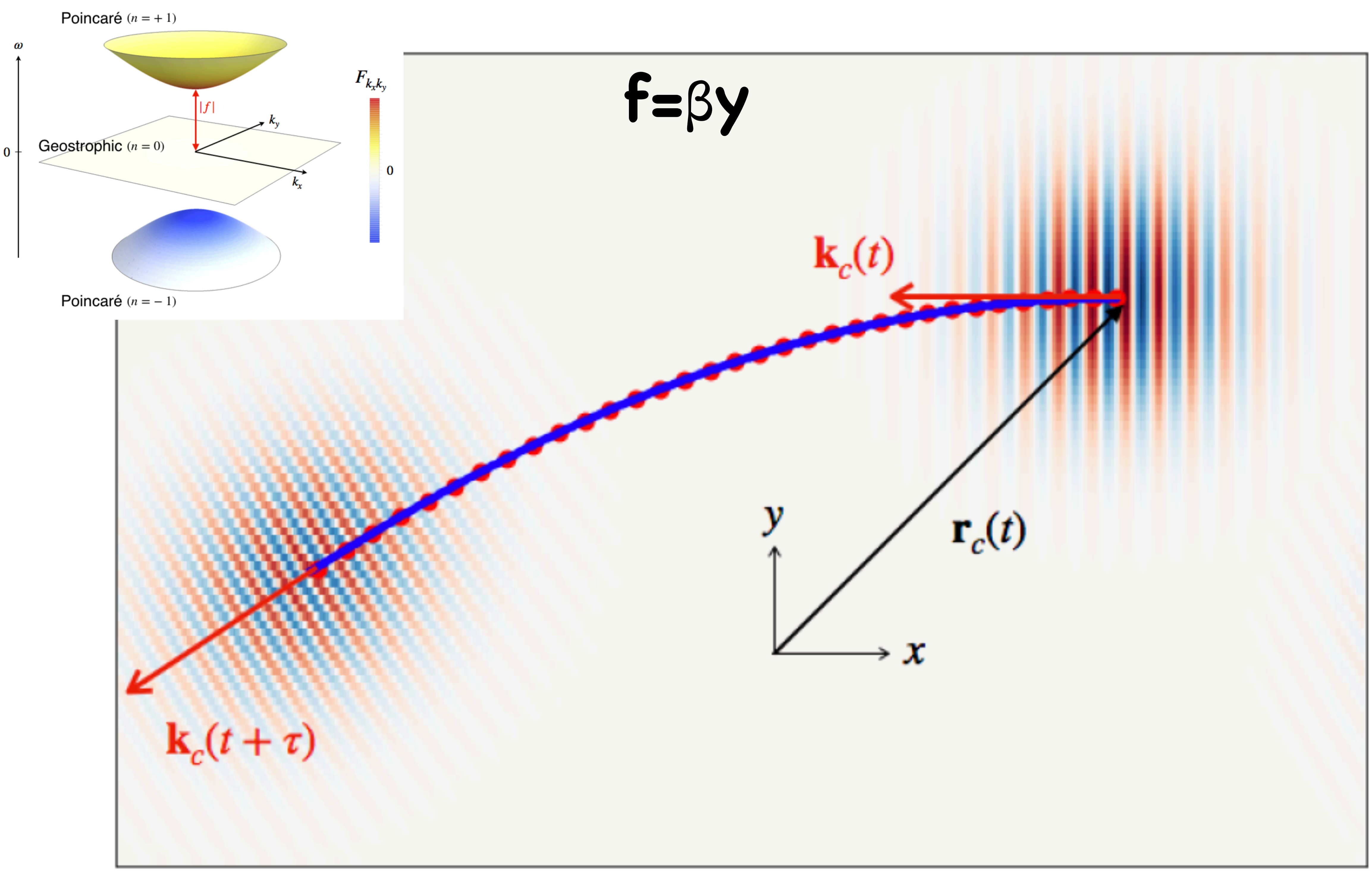
« The number of states gained by a given band when varying parameter kx is topologically protected » **NO!**

A weaker form of bulk-boundary correspondence still holds (thanks to C. TAUBER)

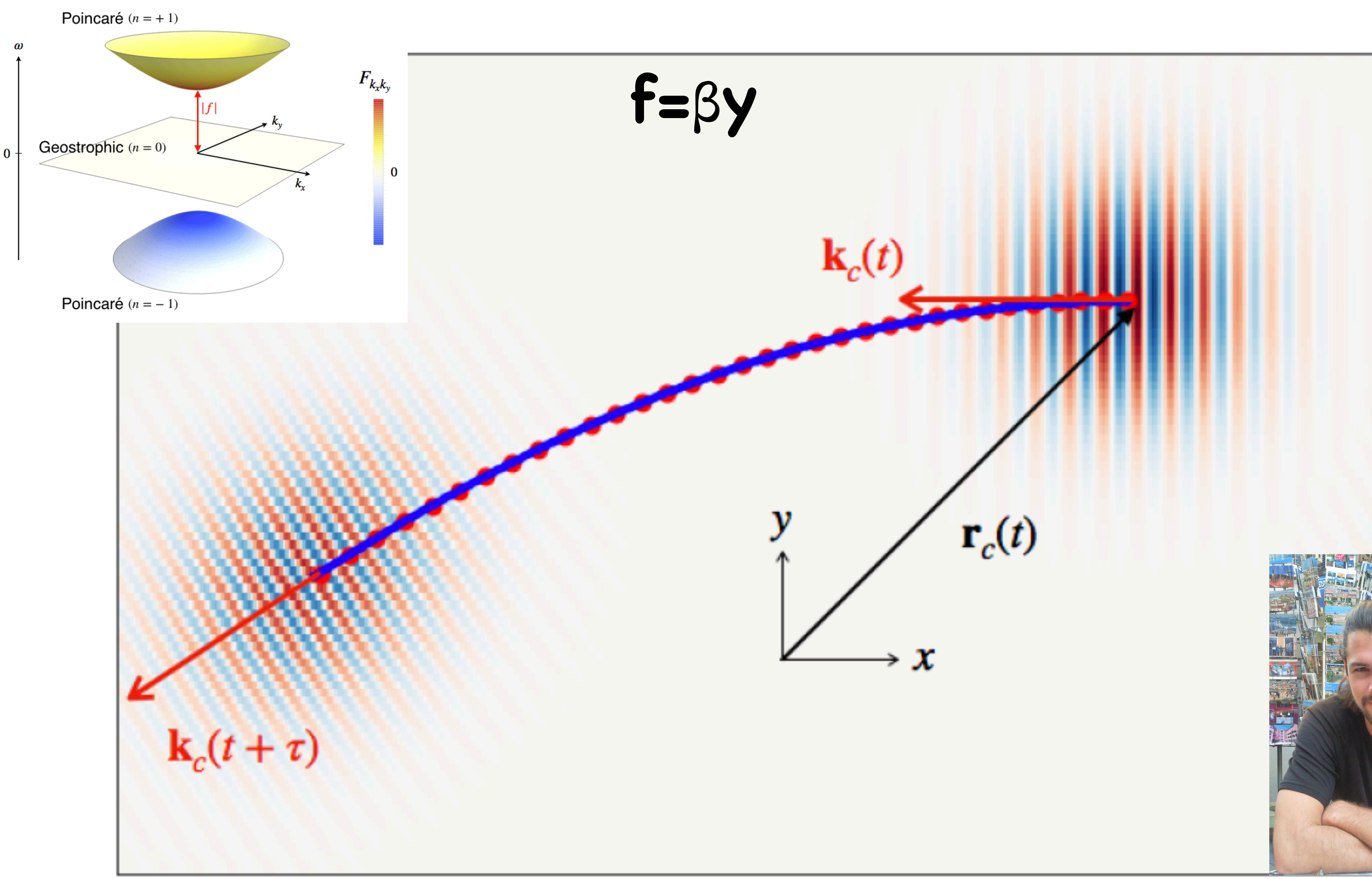
III.2 Footprint of Berry curvature on geophysical ray tracing

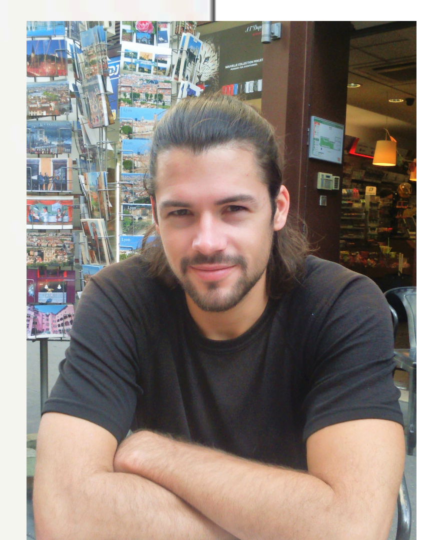


III.2 Footprint of Berry curvature on geophysical ray tracing

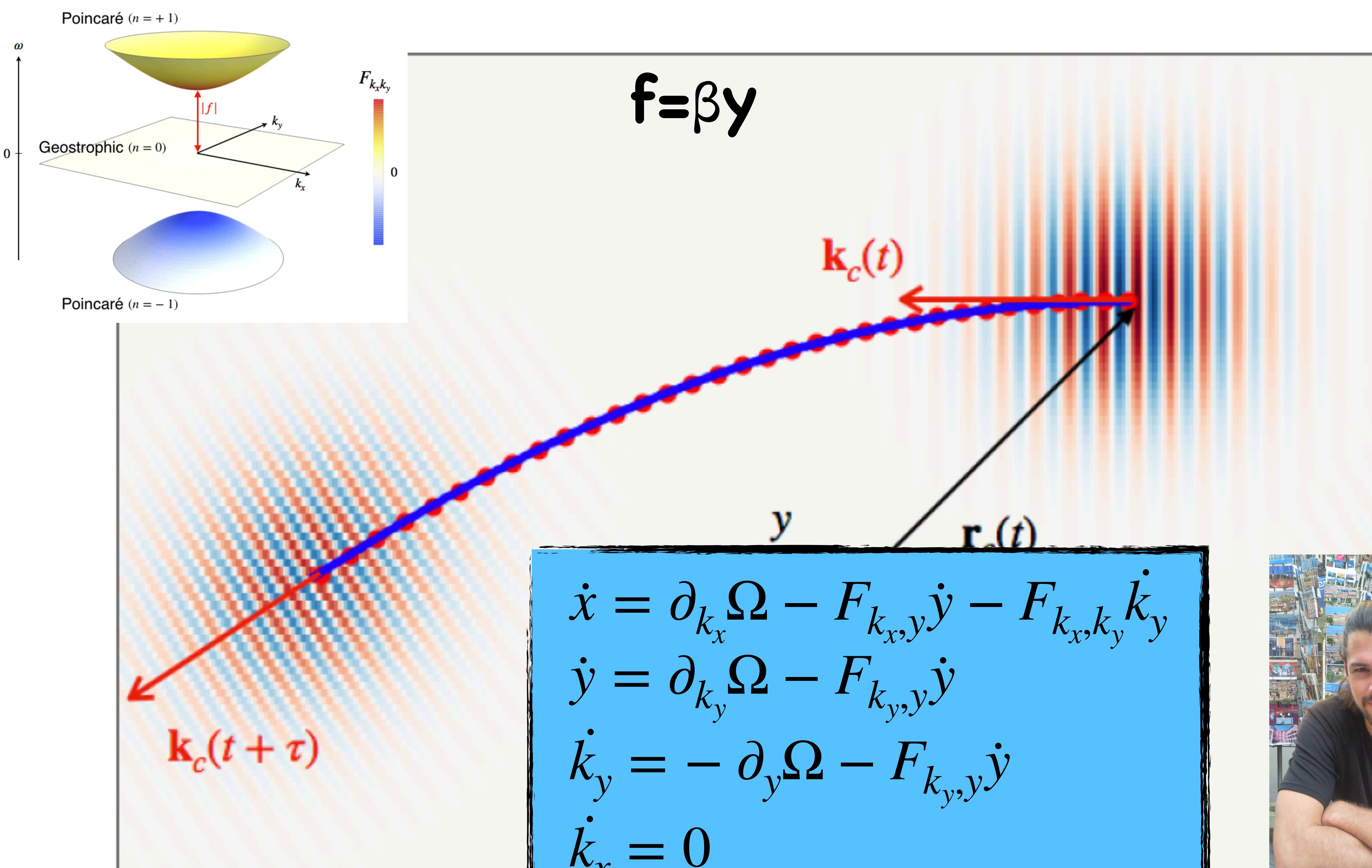


III.2 Footprint of Berry curvature on geophysical ray tracing

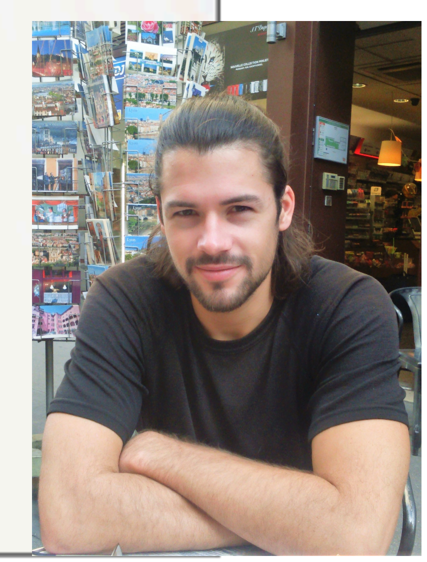



Nicolas Pérez

III.2 Footprint of Berry curvature on geophysical ray tracing



$$\dot{x} = \partial_{k_x}\Omega - F_{k_x,y}\dot{y} - F_{k_x,k_y}\dot{k}_y$$
$$\dot{y} = \partial_{k_y}\Omega - F_{k_y,y}\dot{y}$$
$$\dot{k}_y = -\partial_y\Omega - F_{k_y,y}\dot{y}$$
$$\dot{k}_x = 0$$



Nicolas Pérez

Conclusion

Breaking time reversal symmetry with rotation

- Topological invariant related to emergence of Yanai/Kelvin waves
- Delplace Marston Venaille 2017

Breaking mirror symmetry with gravity

- Topology predicts the emergence of Lamb-like waves. Observation ?
- Perrot Delplace Venaille 2019

Coastal Kelvin waves are also topological, but in a weaker sense

- Apparent breaking of «Bulk-boundary correspondence»
- Tauber Delplace Venaille 2019, 2020

Manifestation of Berry curvature in geophysical ray tracing

- Formal analogy between extraordinary Hall effect and equatorial drift
- Perez Delplace Venaille, in prep.

Collaborators

Pierre
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Brad
Marston (Brown)

Clément
Tauber (Paris)

Nicolas
Perez (Lyon)

Manolis
Perrot (Patagonia?)