Centrifugal Instability of a Geostrophic Jet

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Outline

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Types of Jets Considered

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Jets can experience to Baroclinic, Barotropic, Gravitational (GI) and Centrifugal instabilities (CI).
The Work of Carnevale et al.

- Examined unstratified Gaussian barotropic jets.
- Analytically approximated the nonlinear behaviour of the flow:
  - Onset and saturation of CI.
  - Onset and saturation of secondary barotropic instability.
- Provided approximations for the effect of CI.
- Supported these with numerical simulations.
The Work of Ribstein et al.

- Studied the instability of a stratified Bickley jet.

- Linear theory for barotropic jets:
  - Confirmed an ultraviolet catastrophe in the inviscid case.
  - Viscosity arrests the ultraviolet catastrophe.

- Nonlinear simulation for the baroclinic jet:
  - Used WRF, which tends to be diffusive.
  - First a CI instability and then a secondary barotropic instability.
Governing Equations and Nondimensional Parameters

- Governing system:
  \[
  \frac{D\ddot{\vec{u}}}{Dt} + \vec{f} \times \vec{u} = -\nabla \Phi + b\hat{z} + \nu \nabla^2 \vec{u} - \nu \nabla^2 \vec{u}, \tag{1}
  \]
  \[
  \nabla \cdot \vec{u} = 0, \tag{2}
  \]
  \[
  \frac{Db}{Dt} = \kappa \nabla^2 b - \kappa \nabla^2 \bar{b}. \tag{3}
  \]

- BCs: Periodic in \(x\) and free slip conditions in \(y\) and \(z\).

- Nondimensional parameters:
  \[
  \text{Ro} = \frac{U}{fL}, \quad \text{Re} = \frac{UL}{\nu}, \quad \text{Bu} = \left(\frac{NH}{fL}\right)^2, \quad \delta = \frac{H}{L}, \quad \text{and} \quad \text{Pr} = \frac{\nu}{\kappa}
  \]
Linear Instability Theory

- Consider a solution $\vec{B}$ to a nonlinear system $\partial_t \vec{B} = \mathcal{N}(\vec{B})$.

- Add a small perturbation and linearize the equations to yield

$$\partial_t \vec{b} = \mathcal{L}(\vec{B})\vec{b}.$$ 

- Use a Fourier decomposition of $\vec{b}$ in time and in $x$-direction

$$-i\omega \hat{b}(\vec{x}) = \mathcal{L}(\vec{k}; \vec{B})\hat{b}(\vec{x}).$$

- Solve to determine linear stability characteristics.
Can also be unstable to barotropic and baroclinic instabilities.
Jiao and Dewar found that CI can efficiently mix the flow. We examine mixing efficiency with flux Richardson number:

\[ R_i = \frac{B}{B + \epsilon}. \]

- \( B = -\overline{w'b'} \) is the transfer of energy from APE and KE to the BPE.
- \( \epsilon = 2\nu (e_{ij}e_{ij} - 1/3(e_{ii})^2 \), where \( e_{ij} = 0.5(\partial_{x_j}u_i + \partial_{x_i}u_j) \) is the viscous dissipation.
- Can be shown analytically that \( R_i \in [0, 1] \).
- Kelvin-Helmholtz has a typical efficiency of \([0.2, 0.3]\).
Eigenvalue Problems for the LSA Problems

We simplify by making the hydrostatic approximation.

EVP for barotropic jet

- 1D EVP of the form $\omega \begin{bmatrix} \Phi' & u' & iv' \end{bmatrix}^T = A \begin{bmatrix} \Phi' & u' & iv' \end{bmatrix}^T$. 
- $A$ depends on $k$ and $m$, parameters and $\overline{U}(y)$.

EVP for baroclinic jet

- 2D generalized EVP of the form $\omega B \begin{bmatrix} \Phi' & u' & iv' \end{bmatrix}^T = C \begin{bmatrix} \Phi' & u' & iv' \end{bmatrix}^T$. 
- $B$ and $C$ depend on $k$, parameters and $\overline{U}(y, z)$.
Eigenvalue Solvers for the LSA Problems

1D EVP for the barotropic jet
- Used a direct EVP solver with a Chebyshev grid.
- Domain must contain the most unstable mode.

2D EVP for the baroclinic jet
- Shift-and-Invert Arnoldi method with linear spacing.
  - Used the barotropic EVP to provide a guess.
  - Want eigenvalues with large growth rates.
- Pick domain to contain region of negative EPV.
Spectral Parallel Incompressible Navier-Stokes (SPINS)

Spins is Spectrally accurate and highly parallelizable.

For our purposes we:

- Use periodic BC in the direction of the jet and Free slip BCs in the orthogonal directions.
- Use a Fourier basis and FFTs that scale well using MPI.
- Add a force to balance the dissipation of the jet.
Case 1 - LSA Results I

Nondimensional parameters -
$(\text{Ro, Re, Bu, } \delta, \text{Pr}) = (2, 1.1 \times 10^8, 17.26, 0.03, \infty)$.

Barotropic vs baroclinic jets

2D EVP for Barotropic jet

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Centrifugal Instability
Case 1 - LSA Results II

Both instabilities have comparable vertical wavelengths.

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Centrifugal Instability
Case 1 - Verification of LSA Results I

Random initial perturbation

Initial perturbation given by fastest growing mode

Initial perturbation given by second fastest growing mode

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Case 1 - Nonlinear Saturation of Cl I

2D 32798 × 4096 results:

- $b$
- $w$
- EPV

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Centrifugal Instability
Case 1 - Nonlinear Saturation of CI II

3D $256 \times 512 \times 1024$ EVP field:
Case 1 - Mixing Efficiency
Case 2 - LSA Results I

Nondimensional parameters -

\((\text{Ro, Re, Bu, } \delta, \text{Pr}) = (2, 2.2 \times 10^5, 17.26, 0.03, \infty)\).

2D EVP for barotropic jet  

Barotropic vs baroclinic jets

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Centrifugal Instability
This simulation also had the wavelength predicted by linear theory.
Case 2 - Nonlinear Saturation of CI

3D $256 \times 256 \times 1024$ EVP field:
Case 2 - Mixing Efficiency
Case 3 - LSA Results I

Nondimensional parameters -
\[(Ro, Re, Bu, \delta, Pr) = (2, 2.2 \times 10^5, 17.26, 0.1, \infty).\]

2D EVP for barotropic jet

Barotropic vs baroclinic jets
Case 3 - Nonlinear Simulation

3D $256 \times 256 \times 1024$ results:

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Case 3 - Nonlinear Saturation of CI

3D $256 \times 256 \times 1024$ EVP field:
Case 3 - Mixing Efficiency

![Graph showing mixing efficiency over time](image)

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Centrifugal Instability
Case 1
Case 2
Case 3
Parameter Study

Parameter Study I

\( \delta: \) The growth rates agree except for small \( \delta. \)
Bu: BT growth rates are slightly larger than BC.
Ro: The growth rates agree for all Ro.
Parameter Study IV

Re: Strong dependency on dissipation

![Graph showing dependency on dissipation](image)
Conclusions

- The jets we examined are stable to CI when Ro < $9/(4\sqrt{3})$.
- For small Reynolds numbers the stability properties of baroclinic and barotropic jets can differ significantly.
- CI is generally efficient at mixing the water column.
- Depending on the flow parameters, CI may generate a secondary instability.
Future Work

- What are the effects of non-uniform stratification?
- Can we classify the different types of nonlinear saturation based on nondimensional parameters?
- What is the effect of changing the Prandtl number?
- How robust are the parameter study results?
Background and Governing Equations
Numerical methods
Results
Conclusions and Future work
References


