A structure-preserving approximation of the discrete split rotating shallow water equations

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Motivation and Overview

Numerical schemes that lack structure preservation might lead to wrong statistics of long-term (climate) simulations. Structure-preserving schemes for geophysical fluid dynamics (GFD) can be consistently derived with finite element (FE) methods, e.g. [5], but they are often computationally expensive (large mass matrix) or non-local (weak form).

We suggest a novel split FE framework [2,3,4] for eqns. of GFD based on [1] with:

Key features

- discrete schemes consist of prognostic topological and diagnostic metric eqns.
- split schemes' properties: structure preservation results from topological eqns.; convergence, accuracy, dispersion relation from metric eqns.
  $\Rightarrow$ they preserve both Hamiltonian and split structures!
- structure-preservation is independent from realization of metric eqns.
- all differential operators are local (by avoiding weak form)
- split FEM results in efficient schemes in matrix-vector form

Conclusions and Outlook

- larger choice of FE spaces, compared to standard FE, permits to derive and study novel schemes
- systematic derivations of structure-preserving approximations is decoupled from modifications in metric equations (smoothing, adding noise, etc.)
- todo: study stability, higher dimensions, higher order

I. Split y-independent (slice) RSW equations

We introduce a y-independent model case as this provides insight towards developing schemes for the full 2D rotating shallow water (RSW) equations.

$$\frac{\partial u^{(1)}}{\partial t} - \tilde{\star}\widetilde{q}^{(0)}F_v^{(0)} + \mathrm{d}\,B^{(0)} = 0, \quad \frac{\partial \widetilde{v}^{(1)}}{\partial t} + \tilde{\star}\widetilde{q}^{(0)}\widetilde{F}_u^{(0)} = 0, \quad \frac{\partial \widetilde{h}^{(1)}}{\partial t} + \mathrm{d}\widetilde{F}_u^{(0)} = 0,$$
$$\widetilde{u}^{(0)} = \tilde{\star}u^{(1)}, \quad v^{(0)} = \tilde{\star}\widetilde{v}^{(1)}, \quad \widetilde{h}^{(1)} = \tilde{\star}h^{(0)},$$

using the definitions

$$\widetilde{F}_u^{(0)} := h^{(0)}\widetilde{u}^{(0)}, \quad F_v^{(0)} := h^{(0)}v^{(0)} \quad \text{(mass fluxes), and}$$
$$B^{(0)} := gh^{(0)} + \frac{1}{2}(\widetilde{u}^{(0)})^2 + \frac{1}{2}(v^{(0)})^2 \quad \text{(Bernoulli function), and}$$
$$\widetilde{q}^{(0)}\widetilde{h}^{(1)} = \mathrm{d}v^{(0)} + f dx \quad \text{(potential vorticity (PV))}$$

- 1d f-plane: f const. Coriolis param., g graviational constant, wave speed $c = \sqrt{gH}$, H mean height
- 0-forms (functions): fluid height $h^{(0)}(x,t)$, velocities $\widetilde{u}^{(0)}(x,t), v^{(0)}(x,t)$
- 1-forms: fluid height $\widetilde{h}^{(1)}(x,t)$, velocities $u^{(1)}(x,t) = u(x,t)dx$, $\widetilde{v}^{(1)}(x,t) = v(x,t)\widetilde{dx}$
- twisted Hodge-star $\tilde{\star} : \Lambda^k \to \widetilde{\Lambda}^{(1-k)}$ (resp. $\widetilde{\Lambda}^k \to \Lambda^{(1-k)}$), $\Lambda^k, \widetilde{\Lambda}^k$ space of all k-forms, $k = 0, 1$
- exterior derivative $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$ ($\mathrm{d} : \widetilde{\Lambda}^k \to \widetilde{\Lambda}^{k+1}$) is total deriv. $\mathrm{d}\,g^{(0)} = \partial_x g(x)dx \in \Lambda^1$ in 1d

The relations between operators and spaces is illustrated in diagram (1).

II. Suitable pairs of FE spaces

We introduce a double pair of compatible finite element spaces $\Lambda_h^0, \widetilde{\Lambda}_h^0 = CG_k$ (Continuous Galerkin) and $\Lambda_h^1, \widetilde{\Lambda}_h^1 = DG_{k-1}$ (Discontinuous Galerkin) with polynomial order k such that

$$\begin{array}{ccc} H^1 & \xrightarrow{\mathrm{d}} & L^2 \\ \downarrow \pi_0 & & \downarrow \pi_1 \\ \Lambda_h^0, \widetilde{\Lambda}_h^0 & \xrightarrow{\mathrm{d}} & \Lambda_h^1, \widetilde{\Lambda}_h^1 \end{array} \qquad \begin{array}{ccccccc} h_h^{(0)}, v_h^{(0)} & \in & \Lambda_h^0 & \xrightarrow{\mathrm{d}} & \Lambda_h^1 & \ni & u_h^{(1)} \\ & & \downarrow \tilde{\star}_h^0 & & \downarrow \tilde{\star}_h^1 & & \\ \widetilde{h}_h^{(1)}, \widetilde{v}_h^{(1)} & \in & \widetilde{\Lambda}_h^1 & \xleftarrow{\mathrm{d}} & \widetilde{\Lambda}_h^0 & \ni & \widetilde{u}_h^{(0)} \end{array} \qquad (1)$$

with commuting, bounded, surjective projections (π_0, π_1). The discrete Hodge stars $\tilde{\star}_h^0, \tilde{\star}_h^1$ map between straight and twisted spaces and are allowed to be non-invertible.

III. Split Hamiltonian FE discretization of split RSW

Let $\mathcal{H} : \Lambda^1 \times \widetilde{\Lambda}^1 \times \widetilde{\Lambda}^1 \to \mathbb{R}$ be the Hamiltonian functional of the split slice RSW model, defined by

$$\mathcal{H}[u_h^{(1)}, \widetilde{v}_h^{(1)}, \widetilde{h}_h^{(1)}] = \frac{1}{2}\langle u_h^{(1)}, \tilde{\star}h_h^{(0)}\widetilde{u}_h^{(0)}\rangle + \frac{1}{2}\langle \widetilde{v}_h^{(1)}, \tilde{\star}h_h^{(0)}v_h^{(0)}\rangle + \langle \widetilde{h}_h^{(1)}, \tilde{\star}gh_h^{(0)}\rangle,$$

with metric equations:

$$\widetilde{u}_h^{(0)} = \widetilde{u}_h^{(0)}[u_h^{(1)}] = \tilde{\star}_h u_h^{(1)}, \; v_h^{(0)} = v_h^{(0)}[\widetilde{v}_h^{(1)}] = \tilde{\star}_h \widetilde{v}_h^{(1)}, \; h_h^{(0)} = h_h^{(0)}[\widetilde{h}_h^{(1)}] = \tilde{\star}_h \widetilde{h}_h^{(1)} \quad (2)$$

in which [ ] indicates the dependency of a function from another one.

We define the (almost) Poisson bracket $\{,\}$ as

$$\{\mathcal{F},\mathcal{G}\} := -\langle \frac{\delta\mathcal{F}}{\delta\widetilde{h}_h^{(1)}}, \mathrm{d}\,\tilde{\star}\frac{\delta\mathcal{G}}{\delta u_h^{(1)}}\rangle - \langle \frac{\delta\mathcal{F}}{\delta u_h^{(1)}}, \mathrm{d}\,\tilde{\star}\frac{\delta\mathcal{G}}{\delta\widetilde{h}_h^{(1)}}\rangle + \langle \frac{\delta\mathcal{F}}{\delta u_h^{(1)}}, \tilde{\star}\widetilde{q}_h^{(0)}\tilde{\star}\frac{\delta\mathcal{G}}{\delta\widetilde{v}_h^{(1)}}\rangle - \langle \frac{\delta\mathcal{F}}{\delta\widetilde{v}_h^{(1)}}, \tilde{\star}\widetilde{q}_h^{(0)}\tilde{\star}\frac{\delta\mathcal{G}}{\delta u_h^{(1)}}\rangle$$

with PV $q_h^{(1)}$ defined implicitly by

$$\langle \tilde{\star}\widetilde{\phi}_h^{(0)}, \widetilde{q}_h^{(0)}\widetilde{h}_h^{(1)}\rangle + \langle \mathrm{d}\,\widetilde{\phi}_h^{(0)}, \widetilde{v}_h^{(1)}\rangle - \langle \tilde{\star}\widetilde{\phi}_h^{(0)}, f dx\rangle = 0, \qquad \forall\widetilde{\phi}_h^{(0)} \in \widetilde{\Lambda}_h^0.$$

Then, the dynamics for any functional $\mathcal{F} : \Lambda^1 \times \widetilde{\Lambda}^1 \times \widetilde{\Lambda}^1 \to \mathbb{R}$ is given by

$$\frac{d}{dt}\mathcal{F}[u_h^{(1)}, \widetilde{v}_h^{(1)}, \widetilde{h}_h^{(1)}] = \{\mathcal{F}, \mathcal{H}\}.$$

The discrete Hodge star operators $\tilde{\star}_h$ in (2) are realized by nontrivial Galerkin projections $(\mathbf{GP1}_h, \mathbf{GP0}_h, \mathbf{GP1}_u, \mathbf{GP0}_u)$ (see IV.).

IV. Family of split P0-P1 schemes

The introduction of double pairs of compatible FE spaces enriches the choice of potential schemes. For the low order P0-P1 double pairs, we find the following *family* of split low-order (P0-P1) FE schemes, consisting of one set of topological equations and 4 combinations of metric closure equations: $\mathbf{GP1}_u - \mathbf{GP1}_h$, $\mathbf{GP1}_u - \mathbf{GP0}_h/\mathbf{GP0}_u - \mathbf{GP1}_h$, or $\mathbf{GP0}_u - \mathbf{GP0}_h$, cf. [2]:

topol. moment. eqn.: $\frac{\partial}{\partial t}\mathbf{u}_e^1 + \mathbf{D}^{en}\mathbf{B}_n^0 - \widetilde{\mathbf{M}^{en}}(\tilde{\mathbf{q}}_n^0 \circ \mathbf{F}_n^v) = 0, \; \frac{\partial}{\partial t}\tilde{\mathbf{v}}_e^1 + \widetilde{\mathbf{M}^{en}}(\tilde{\mathbf{q}}_n^0 \circ \tilde{\mathbf{F}}_n^u) = 0$

$$\begin{array}{ccccccc} \mathbf{h}_n^0 & \in & \Lambda_h^0 \subset P1 & \xrightarrow{\mathbf{D}^{en}} & \Lambda_h^1 \subset P0 & \ni & \mathbf{u}_e^1, \tilde{\mathbf{v}}_e^1 \\ & \mathbf{GP1}_h\!: \mathbf{M}^{nn}\mathbf{h}_n^0 = \mathbf{P}^{ne}\tilde{\mathbf{h}}_e^1 \downarrow & & & \downarrow \mathbf{GP1}_u\!: \mathbf{M}^{nn}\tilde{\mathbf{u}}_n^0 = \mathbf{P}^{ne}\mathbf{u}_e^1 \;\&\; \mathbf{M}^{nn}\mathbf{v}_n^0 = \mathbf{P}^{ne}\tilde{\mathbf{v}}_e^1 & & \\ & \mathbf{GP0}_h\!: \mathbf{M}^{en}\mathbf{h}_n^0 = \tilde{\mathbf{h}}_e^1 \downarrow & & & \downarrow \mathbf{GP0}_u\!: \mathbf{M}^{en}\tilde{\mathbf{u}}_n^0 = \mathbf{u}_e^1 \;\&\; \mathbf{M}^{en}\mathbf{v}_n^0 = \tilde{\mathbf{v}}_e^1 & & \\ \tilde{\mathbf{h}}_e^1 & \in & \widetilde{\Lambda}_h^1 \subset P0 & \xleftarrow{\mathbf{D}^{en}} & \widetilde{\Lambda}_h^0 \subset P1 & \ni & \tilde{\mathbf{u}}_n^0, \mathbf{v}_n^0 \end{array}$$

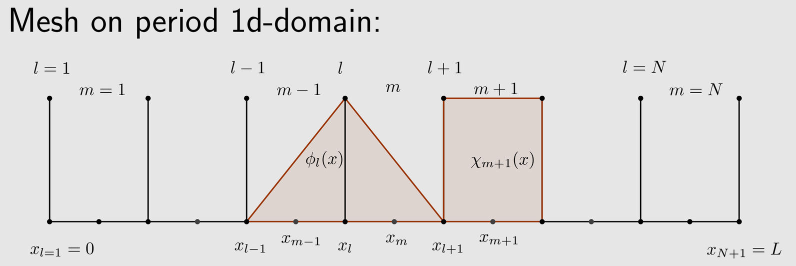
topol. continuity eqn.: $\frac{\partial}{\partial t}\tilde{\mathbf{h}}_e^1 + \mathbf{D}^{en}\tilde{\mathbf{F}}_n^u = 0.$

We use a second order Crank-Nicolson time integrator.

Definitions:

- For $\Lambda_h^0, \widetilde{\Lambda}_h^0$ we use a piecewise linear basis $\{\phi_l(x)\}_{l=1}^N$ to approximate functions, for $\Lambda_h^1, \widetilde{\Lambda}_h^1$ a piecewise constant basis $\{\chi_m(x)\}_{m=1}^N$ to approximate 1-forms, e.g.
$u_h^{(1)}(x,t) = \sum_{m=1}^N u_m(t)\chi_m(x)$,
$\widetilde{u}_h^{(0)} = \sum_{l=1}^N \widetilde{u}_l(t)\phi_l(x)$

Mesh on period 1d-domain:



Vector arrays:
-discrete 1-form $\mathbf{u}_e^1 = \mathbf{M}^{ee}\mathbf{u}_e$ with
$\mathbf{u}_e = \{u_m(t)|m = 1, ...N\}$, similar for $\tilde{\mathbf{v}}_e^1$ and $\tilde{\mathbf{h}}_e^1$
-height average $\tilde{\mathbf{h}}_n^1 := \mathbf{A}^{ne}\tilde{\mathbf{h}}_e^1$ with average op. $\mathbf{A}^{ne}$

Mass and stiffness matrices:
- $\mathbf{M}^{nn}, \mathbf{M}^{ee}, \mathbf{M}^{ne}$ are metric-depend $(N \times N)$ mass-matrices:
$(\mathbf{M}^{nn})_{ll'} = \int_L \phi_l(x)\phi_{l'}(x)dx$,
$(\mathbf{M}^{ee})_{mm'} = \int_L \chi_m(x)\chi_{m'}(x)dx$,
$(\mathbf{M}^{ne})_{ml'} = \int_L \chi_m(x)\phi_{l'}(x)dx$ with $\mathbf{M}^{en} = (\mathbf{M}^{ne})^T$.
- $\mathbf{M}^{ne} = \mathbf{P}^{ne}(\mathbf{\Delta x}_e)^T$ with metric-dependent part
$\mathbf{\Delta x}_e = (\Delta x_1, \ldots \Delta x_m, \ldots \Delta x_N)$ and metric-free part $\mathbf{P}^{ne}$.
- $\mathbf{D}^{ne}$ is $(N \times N)$ stiffness matrix with metric-independent coefficients:
$(\mathbf{D}^{ne})_{ml'} = \int_L \chi_m(x)\frac{d\phi_{l'}(x)}{dx}dx$ with $\mathbf{D}^{en} = (\mathbf{D}^{ne})^T$.

V. Approximation of full split schemes

We introduce a new, computationally more efficient split RSW scheme. We exploit the splitting of the topological and metric properties within the split FE framework to introduce structure-preserving approximations of the mass matrices used for (2). Instead of using the full nontrivial Galerkin projections $\mathbf{GP1}_h, \mathbf{GP0}_h$ for height or $\mathbf{GP1}_u, \mathbf{GP0}_u$ for velocity u, v, we use the **averaged** versions:

$$\mathbf{AVG}_h : \mathbf{h}_n^0 = \mathbf{P}^{ne}\tilde{\mathbf{h}}_e^1, \quad \mathbf{AVG}_u : \tilde{\mathbf{u}}_n^0 = \mathbf{P}^{ne}\mathbf{u}_e^1,$$

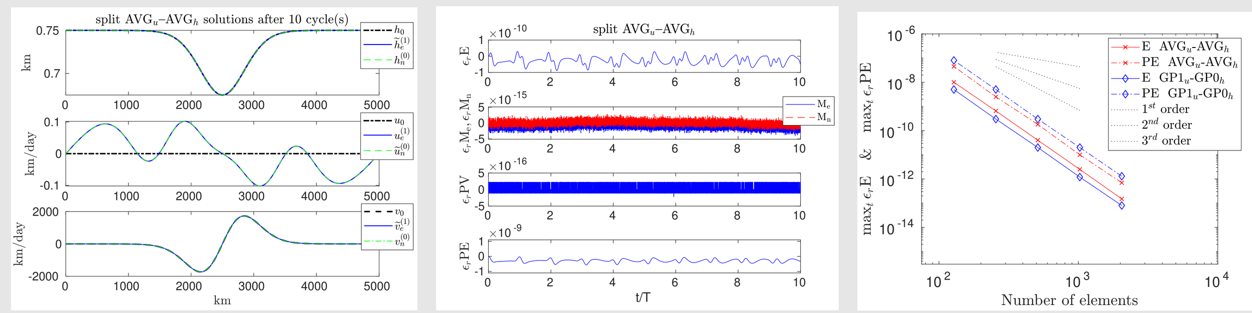
with averaging operator $\mathbf{P}^{ne}$ and denote the resulting scheme with $\mathbf{AVG}_u$- $\mathbf{AVG}_h$. Rather then solving linear systems, we obtain values for $\mathbf{h}_n^0, \tilde{\mathbf{u}}_n^0, \mathbf{v}_n^0$ simply by averaging.

VI. Results

- $\mathbf{AVG}_u$- $\mathbf{AVG}_h$ is computationally more efficient then full schemes: here in 1D by factor of 2 (wall clock time)
- all schemes preserve both Hamiltonian and split structures:

Properties resulting from topological equations:

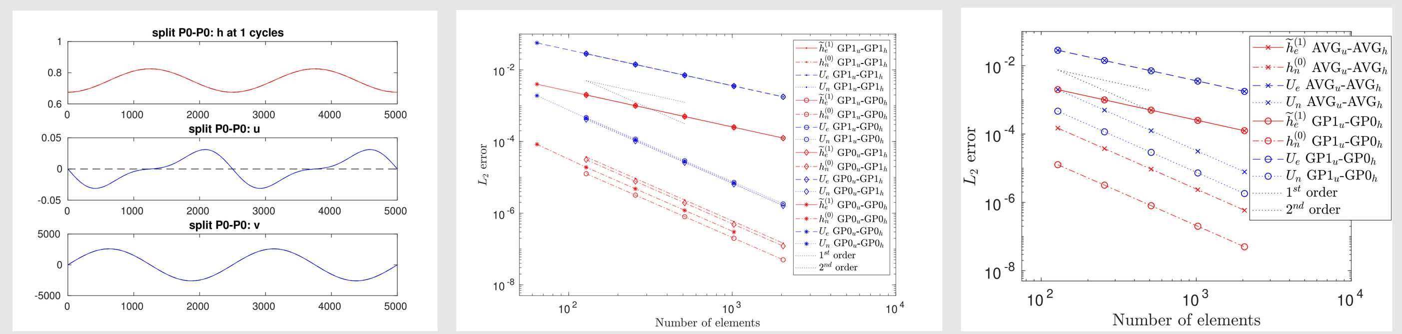
- Conservation of total energy E: $\frac{d}{dt}\mathcal{H}[u_h^{(1)}, \widetilde{v}_h^{(1)}, \widetilde{h}_h^{(1)}] = \{\mathcal{H}, \mathcal{H}\} = 0$
- Conservation of the Casimirs C (mass (M), pot. vorticity (PV) and enstrophy (PE)): $\frac{d}{dt}\mathcal{C} = \{\mathcal{C}, \mathcal{G}\} = 0$ for any $\mathcal{G}$: $\mathcal{C}$ in nullspace of $\{,\}$ (M,PV at 10^{-15})



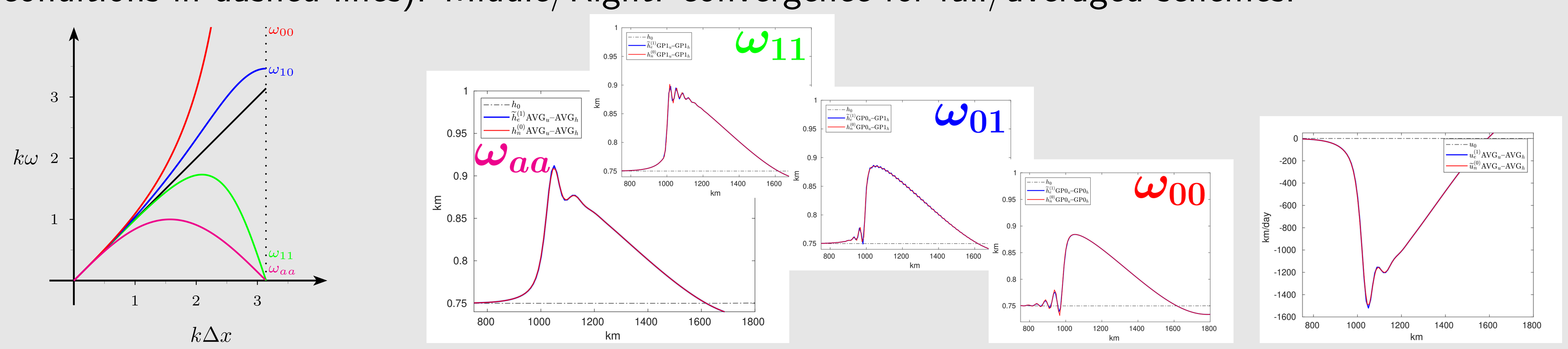
Solutions for the split $\mathbf{AVG}_u$- $\mathbf{AVG}_h$ scheme for a mesh with 512 elements. Initial fields are shown as dashed-dotted lines. Left: flow in geostrophic balance after 10 cycles. Middle: time series of the quantities of interest. Right: convergence rates of E and PE.

Properties resulting from metric equations:

- Convergence, accuracy, dispersion relation:



Convergence behavior of all schemes. Left: stationary RSW solutions after 1 cycles (initial conditions in dashed lines). Middle/Right: convergence for full/averaged schemes.



Left: dispersion relations: analytic (black) for $c = \sqrt{gH} = 1$, ω_{11} for $\mathbf{GP1}_u$-$\mathbf{GP1}_h$, ω_{10} for $\mathbf{GP1}_u$-$\mathbf{GP0}_h$ and $\mathbf{GP0}_u$-$\mathbf{GP1}_h$, ω_{00} for $\mathbf{GP0}_u$-$\mathbf{GP0}_h$, and ω_{aa} for $\mathbf{AVG}_u$-$\mathbf{AVG}_h$. Right: fields with oscillations at the wave fronts in dependency of the wave dispersion relations.

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