Response and Sensitivity Using Markov Chains

Manuel Santos Gutiérrez
work with: V. Lucarini
Department of Mathematics and Statistics, University of Reading

Introduction

Dynamical systems are often subject to forcing or changes in their governing parameters and it is of interest to study how this affects their statistical properties. A prominent real-life example of this class of problems is the investigation of **climate response** to perturbations. In this work we address the problem of:

- Calculating the **linear response** of a system by analysing the unforced scenario
- Using the **transfer operator** approach to asses ergodic properties
- Extending the perturbation theory of **Markov chains** to continuous systems
- Applying such an approach in **non-equilibirum** and dissipative models

The Transfer Operator

Let $\{\phi^t\}_{t\in\mathbb{R}}$ be a dynamical system on $\mathcal{X}$. The transfer operator is defined as:

$$\mathcal{L}^t\rho(\mathbf{x}) = \rho\left(\phi^{-t}(\mathbf{x})\right)|\det D\phi^{-t}(\mathbf{x})|.$$

for $\rho \in L^1_\eta(\mathcal{X})$. The transfer operator describes the natural **pushforward** on densities. The finite representation of the transfer operator at time dt is given by:

$$\mathcal{M}^{dt}_{i,j} := \frac{1}{\eta(B_j)}\int_{B_j}\mathcal{L}^{dt}\mathbf{1}_{B_i}\eta(d\mathbf{x}),$$

where $\{B_i\}_{i=1}^N$ is a colection of **boxes** covering phase-space. $\mathcal{M}^{dt}$ defines a finite **Markov chain**.

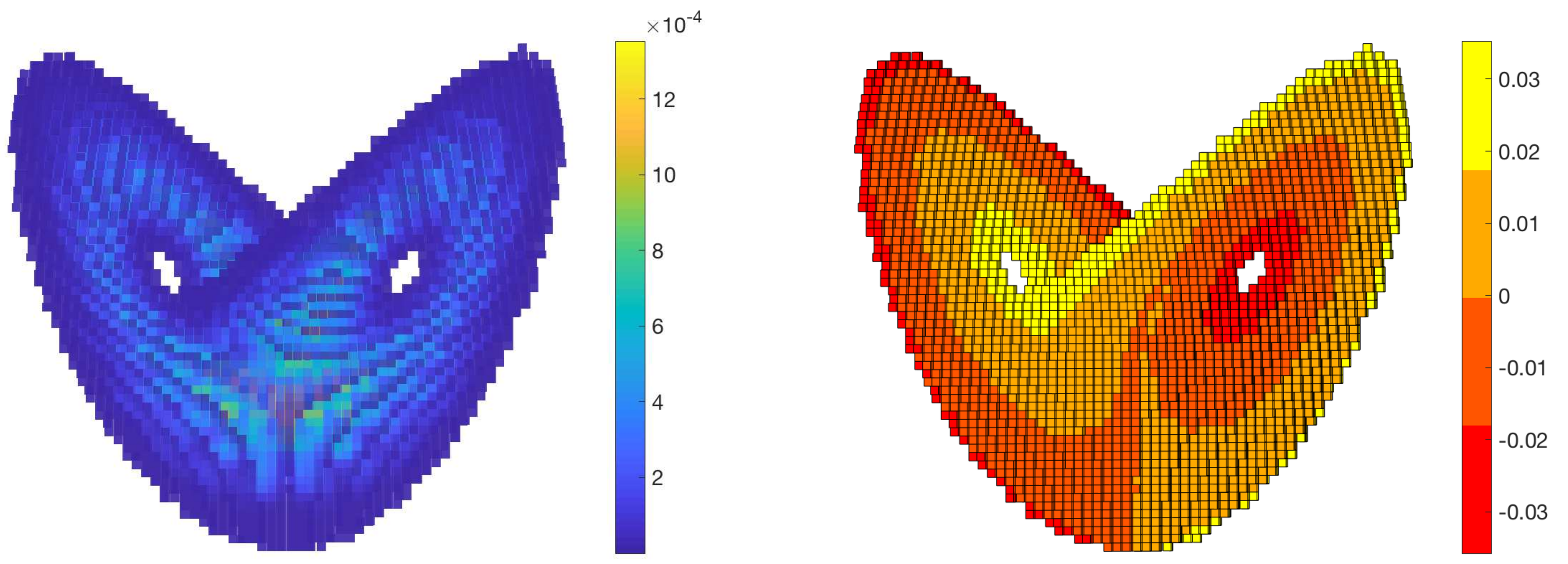


Figure 1: Example of box covering using the **Lorenz 63** convection model. The eigenvector associated with the largest eigenvalue of $\mathcal{M}^{dt}$ gives us an estimate of the **invariant measure** of the system (left). The eigenvector paired with the second largest eigenvalue helps us to detect **almost-invariant** sets on phase-space (right).

Response Formulas for Finite Markov Chains

Consider multiparametric perturbations of Markov matrices:

$$\mathcal{M} \longrightarrow \mathcal{M} + \epsilon_1 m_1 + \ldots + \epsilon_n m_n, \tag{1}$$

with invariant measures $\mathbf{u}$ and $\mathbf{v}$ solving

$$\mathcal{M}\mathbf{u} = \mathbf{u} \;\&\; \left(\mathcal{M} + \epsilon_1 m_1 + \ldots + \epsilon_n m_n\right)\mathbf{v} = \mathbf{v}.$$

We can show that

$$\boxed{\mathbf{v}(\epsilon_1,\ldots,\epsilon_n) = \mathbf{u} + \sum_{k=1}^{\infty}\left(\epsilon_1\Psi_1 + \ldots + \epsilon_n\Psi_n\right)^k\mathbf{u}} \tag{2}$$

Where $\Psi_k = (1-\mathcal{M})^{-1}m_k$ is the **linear response operator**. This expression allows us to predict the perturbed invariant measure and isolate the components of the response.

Remark. The validity of this formula relies on the rate of **mixing** of the Markov chain $\mathcal{M}$, determined by its **spectral** properties.

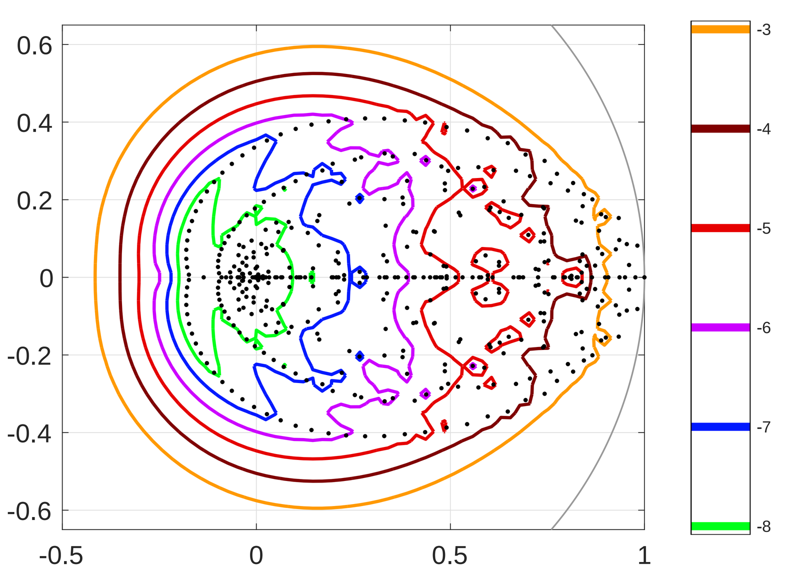


Figure 2: Spectrum $\sigma(\mathcal{M})$ (dots) and ϵ-psedospectra $\sigma_\epsilon(\mathcal{M})$ (coloured lines for $\epsilon = 10^{-3}, \ldots, 10^{-8}$) of the Lorenz 63 model. Where,

$$\sigma_\epsilon(\mathcal{M}) = \{\sigma(\mathcal{M}+E) : \|E\| \leq \epsilon\}.$$

- The **spectral gap**, $1-|\lambda_2|$, determines the rate of mixing.
- Large ϵ-pseudospectra indicate high **sensitivity** to perturbations.

A Case Study: the Lorenz 63 system

We consider the perturbed Lorenz 63 system:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}) + \epsilon\mathbf{G}(\mathbf{x}) = \begin{cases} s(y-x) \\ x(r+\epsilon-z)-y \\ xy-bz \end{cases},$$

with $s=10$, $b=8/3$, $r=28$ and $\epsilon\in\mathbb{R}$. **Question**:

Can we predict the statistics for different ϵ by only integrating the system when $\epsilon = 0$?

The associated **Liouville** equation describes the infinitesimal "pushforward" of measures ρ:

$$\partial_t\mathcal{L}^t\rho_0(\mathbf{x}) = \partial_t\rho(\mathbf{x},t) = \underbrace{-\nabla\cdot(\mathbf{F}\rho(\mathbf{x},t))}_{\text{Unperturbed component}}\underbrace{-\epsilon\nabla\cdot(\mathbf{G}\rho(\mathbf{x},t))}_{\text{Perturbation operator}}.$$

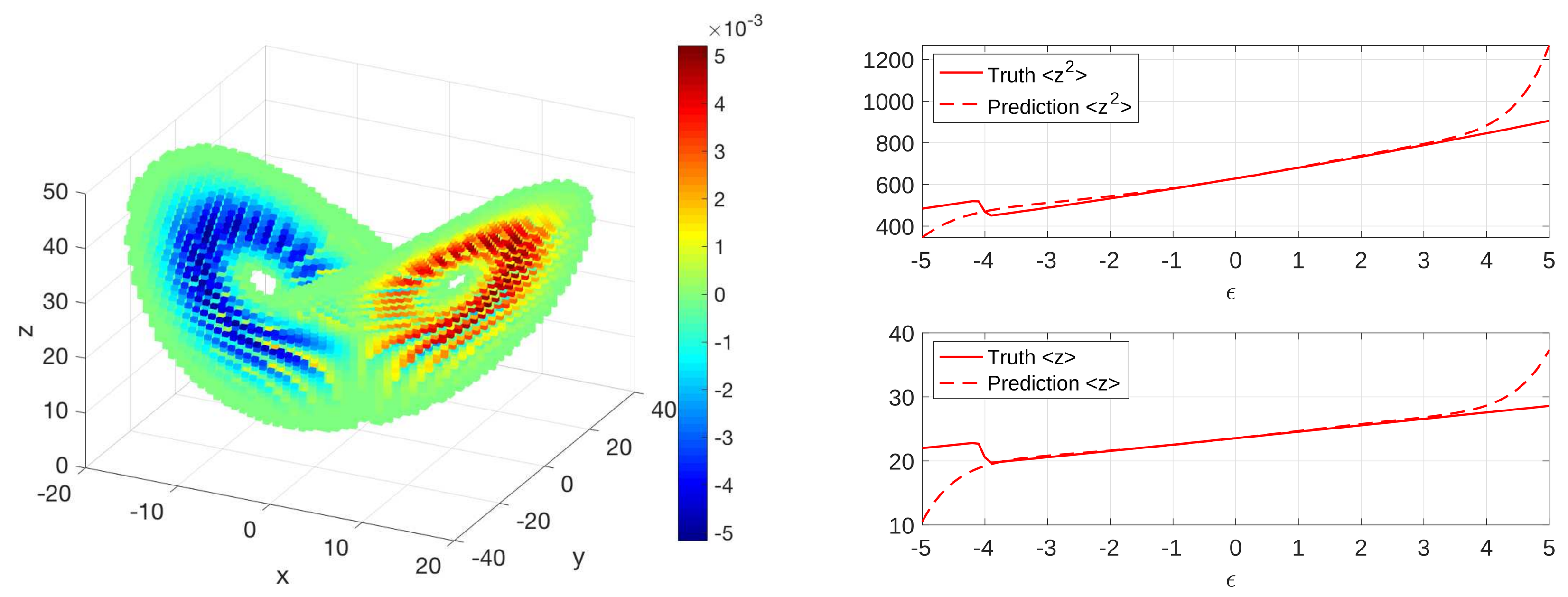
Differencing the left-hand side with time-step $dt>0$, we can approximate to **first-order**:

- Unperturbed component $\approx \mathcal{M}^{dt}$, using **time-series** $\{\mathbf{x}_k\}_{k=1}^T$ for $\epsilon = 0$
- Perturbation operators $\approx m$, using **finite-volume** methods

$$\boxed{\mathcal{M}^{dt}_{i,j} = \frac{\#\{(\mathbf{x}_k\in B_j)\wedge(\mathbf{x}_{k+1}\in B_i)\}}{\#\{\mathbf{x}_k\in B_j\}}} \;\&\; \boxed{m \approx -dt\nabla\cdot(\mathbf{G}\circ)}$$

$\Longrightarrow$ We will apply the pertubation problem as in Eq. (1).

Numerical Experiments



$\epsilon = 0.1$	$\langle z\rangle$	$\langle z^2\rangle$	$\langle z\rangle_\epsilon$	$\langle z^2\rangle_\epsilon$	$\Psi[z]$	$\Psi[z^2]$
L63	23.54	628.78	23.65	633.90	1.01	50.31
$N = 2^{12}$	23.55	629.75	23.62	633.98	0.89	42.39
$N = 2^{15}$	23.55	629.55	23.66	634.32	1.08	50.27
$N = 2^{18}$	23.55	629.22	23.65	634.30	1.11	50.80

Figure 3: Discrete derivative $m\mathbf{u}$ on the attractor (top-left). Expectation value of the observable z calculated using the perturbative expansion Eq. (2) (top-right). The table shows statistical quantities computed using Eq. (2), including the linear response. N indicates the number of boxes employed in the experiment.

Comments and Future Work

- The transfer operator naturally links the response theory of finite Markov chains with **continuous time dynamical systems**:
  - the **spectral gap** is necessary to apply Eq. (2) in any sense
  - the response of a **fractal attractor** can be calculated by observing the unforced system
- Real-world models posses a large number of degrees of freedom. For such reason, phase-space is projected onto variables of interest, forcing a **loss of the Markov property**. To what extent can these techniques be applied?

References

☞ M. Chekroun, J. Neelin, D. Kondrashov, J. McWilliams, M. Ghil, Rough Parameter Dependence in Climate Models and the Role of Ruelle-Pollicot Resonances, *PNAS*, Vol: 111, 1684–1690, 2014.

☞ A. Tantet, V. Lucarini, H. A. Dijkstra, Resonances in a Chaotic Attractor Crisis of the Lorenz Flow, *Journal of Statistical Physics*, Vol: 170, 584–616, 2018.

☞ M. Santos Gutiérrez, V. Lucarini, Response and Sensitivity Using Markov Chains, *Journal of Statistical Physics*, 2020.

EPSRC
Engineering and Physical Sciences Research Council
Centre for Doctoral Training
Mathematics of Planet Earth
Imperial College London
University of Reading