Approximate expansions for water infiltration into dual permeability soils

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1. Abstract and introduction

The understanding of hydrological processes requires the investigation of preferential flows. In particular, the infiltration compartment is strongly affected by preferential flows. Recently, Lassabatere et al. (2014) proposed an analytical model for infiltration in soils impacted by preferential flow. These authors extended the model developed by Haverkamp et al. (1994) for single permeability soils to the case of dual permeability soils. However, this model remains implicit, requiring an inversion procedure for the quantification of the bulk cumulative infiltration. Such an implicit feature prevents from direct computation and may annoy any fellow who wants a direct and simple computation procedure. In this paper, we develop two approximate expansions for both transient and steady states. For that, we use the approximate expansions proposed by Haverkamp et al. (1994) for single permeability systems. These expansions are written for each compartment of the dual permeability soils, i.e. the matrix and the fast-flow regions and are combined for the computation of the bulk infiltration. After formulation, these expansions are assessed in terms of their capability to accurately reproduce the complete implicit model. Their validity time intervals are also determined and discussed. The main limitation for the use of these expansions results from the fact that the time intervals that define the transient and steady states are contrasted between the matrix and the fast-flow regions. However, some domain of validity can be defined allowing the use of these approximate expansions.

2. Theory

Recently, Lassabatere et al. (2014) proposed an analytical model for the 3-D axisymmetric cumulative infiltration into 2K soils, assuming that the cumulative infiltration was the summation of the contribution of the matrix and the fast-flow regions:

$$I_{3D,2K}(t) = w_f \left(I_{1D,f} + \frac{\gamma_f \, S_f^2}{r_d \, \Delta\theta_f} t \right) + \left(1 - w_f\right) \left(I_{1D,m} + \frac{\gamma_m \, S_m^2}{r_d \, \Delta\theta_m} t \right) \qquad [1a]$$

$$\frac{2\Delta K_f^2}{S_f^2} t = \frac{1}{1-\beta_f} \left[\frac{2\Delta K_f}{S_f^2} \left(I_{1D,f} - K_{0,f} t \right) - \ ln \left(\frac{exp\left(\beta_f \frac{2\Delta K_f}{S_f^2} \left(I_{1D,f} - K_{0,f} t \right) \right) + \beta_f - 1}{\beta_f} \right) \right] \qquad [1b]$$

$$\frac{2\Delta K_m^2}{S_m^2} t = \frac{1}{1-\beta_m} \left[\frac{2\Delta K_m}{S_m^2} \left(I_{1D,m} - K_{0,m} t \right) - ln \left(\frac{exp\left(\beta_m \frac{2\Delta K_m}{S_m^2} \left(I_{1D,m} - K_{0,m} t \right) \right) + \beta_m - 1}{\beta_m} \right) \right] \qquad [1c]$$

where the subscripts 'm' and 'f' denote the parameters of the matrix and the fast-flow regions, respectively, the variables r_d, $\Delta\theta$ $[\theta_s - \theta_0]$, ΔK $[K_s - K_0]$, β, and γ are the disc radius, the difference in water content and hydraulic conductivity, and two infiltration constants set at 0.6 and 0.75, respectively; w_f is the void ratio occupied by the fast-flow region. This equation is referred to as QEI-2K since it corresponds to the extension of the quasi-exact implicit model (QEI) developed for single permeability soils by Haverkamp et al. (1994) to dual permeability (2K) systems.

The sorptivity, S, of each region can be computed considering the usual approximation proposed by Parlange et al. (1975):

$$S^2(\theta_0, \theta_s) = \int_{\theta_0}^{\theta_s} (\theta_s + \bar{\theta} - 2\theta_0)\, D(\bar{\theta}) d\bar{\theta} \quad [2a]$$

$$D(\theta) = K(\theta) \frac{dh}{d\theta} \quad [2b]$$

where $\bar{\theta}$ is a dummy variable and the diffusivity corresponds to the product of the hydraulic conductivity with the derivative of the water retention curve $D(\theta) = K(\theta) \frac{dh}{d\theta}$.

The QEI-2K model defined by equation [1] may be complicated to compute since the function is defined in an implicit way. For any time, t, the infiltrations into the matrix and the fast-flow regions, $I_{1D,m}$ and $I_{1D,f}$, must be computed by resolving the equations [1b] and [1c]. Then, in this study, we try to design approximate expansions for the direct explicit quantification of the bulk cumulative infiltrations at both transient and steady states. For that, we consider the approximate expansions already proposed by Haverkamp et al. (1994) for single permeability systems before combining them. These expansions read:

$$I_{O(1)}(t) = S\sqrt{t} \quad [3a]$$

$$I_{O(2)}(t) = S\sqrt{t} + \left(\frac{2-\beta}{3}\Delta K + K_0 + \frac{\gamma\, S^2}{r_d\, \Delta\theta}\right) t \quad [3b]$$

$$I_{+\infty}(t) = \ K_s t - \frac{ln(\beta)}{2(1-\beta)} \frac{S^2}{\Delta K} \quad [3c]$$

These equations can be applied to both the matrix and the fast-flow regions and combined using equation [1a], for the case of the dual permeability soils:

$$I_{O(1)\_2K}(t) = \left(1 - w_f\right) I_{O(1)\_m}(t) + w_f\, I_{O(1)\_f} \quad [4a]$$

$$I_{O(2)\_2K}(t) = \left(1 - w_f\right) I_{O(2)\_m}(t) + w_f\, I_{O(2)\_f} \quad [4b]$$

$$I_{+\infty\_2K}(t) \ = \left(1 - w_f\right) I_{+\infty\_m}(t) \ + w_f\, I_{+\infty\_f} \quad [4c]$$

After rearranging the terms, the following approximate expansions come out. For the fake of clarity, was assume similar values for the infiltration constants, β and γ, for the two regions:

$$I_{O(1)\_2K}(t) = \left(\left(1 - w_f\right) S_m + w_f\, S_f\right)\sqrt{t} \quad [5a]$$

$$I_{O(2)\_2K}(t) = \left(\left(1 - w_f\right) S_m + w_f\, S_f\right)\sqrt{t} + \left(\frac{2-\beta}{3}\Delta K_{2K} + K_{0,2K} + \frac{\gamma}{r_d}\left(\left(1 - w_f\right)\frac{S_m^2}{\Delta\theta_m} + w_f\,\frac{S_f^2}{\Delta\theta_f}\right)\right) t \quad [5b]$$

$$I_{+\infty\_2K}(t) = K_{s,2K}t - \frac{ln(\beta)}{2(1-\beta)}\left(\left(1 - w_f\right)\frac{S_m^2}{\Delta K_m} + w_f\,\frac{S_f^2}{\Delta K_f}\right) \quad [5c]$$

The equations [5] and their accuracy to reproduce the quasi-exact implicit model, QEI-2K model defined by equations [1] is studied below.

3. Material and methods

In this section, we illustrate the accuracy of equations [5] for the specific case of a loamy soil with 1-mm radius macropores. This synthetic dual permeability soil has already been studied by Lassabatere et al. (2019). The bulk water content and hydraulic conductivity functions combine the contributions of the matrix and fast-flow regions (Gerke and van Genuchten, 1993):

$$\theta_{2K}(h) = w_f \theta_f(h) + \left(1 - w_f\right) \theta_m(h) \quad [6a]$$

$$K_{2K}(\theta) = w_f K_f\left(\theta_f\right) + \left(1 - w_f\right) K_m(\theta_m) \quad [6b]$$

For each region, the water retention and hydraulic conductivity functions are defined with van Genuchten model (1980) with Burdine's conditions and Brooks and Corey model (1964):

$$\theta(h) = \theta_r + (\theta_s - \theta_r)\left(1 + \left|\frac{h}{h_g}\right|^n\right)^{-m} \qquad \text{with} \qquad m = 1 - \frac{2}{n} \qquad [7a]$$

$$K(S_e) = K_s\, S_e^{\frac{2}{mn}+3} \qquad [7b]$$

Where $S_e = (\theta - \theta_r)/(\theta_s - \theta_r)$ is the saturation degree.

For the matrix, we considered the shape and scale parameters related to loamy soils as defined by Carsel and Parrish (1988). The fast-flow region was assumed to occupy 10% of the bulk soil, i.e., $w_f = 0.1$. Its residual water content, $\theta_{r,f}$, was set at zero and its saturated water content, $\theta_{s,f}$, was set at a large value, i.e., 0.70. Its shape parameter, n_f, was set at 3.75, as typical of coarse soils (Schaap et al., 2001). The scale parameter for water pressure head of the fast-flow region, $h_{g,f}$, was derived from the pore radius, $r_{g,f}$, using the Young–Laplace equation (see Lassabatere et al., 2019, for more details). The value of hydraulic conductivity, $K_{s,f}$, was computed from that of the loamy matrix, $K_{s,m}$, assuming a linear increase with the square of the pore radius, as indicated by Poiseuille's law (Sutera and Skalak, 1993).

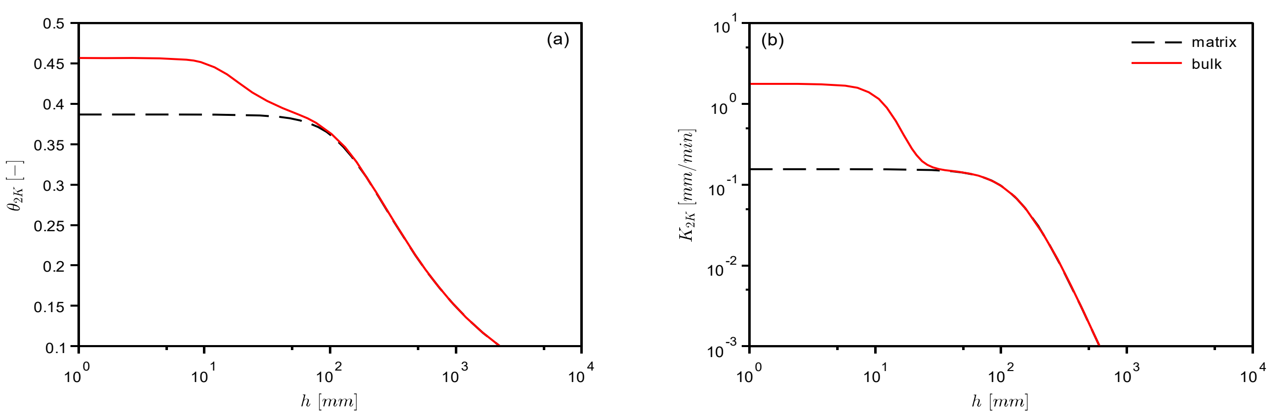

Figure 1: Hydraulic water retention (a) and hydraulic conductivity (b) functions for the synthetic dual permeability soil.

The water retention and hydraulic conductivity functions are detailed in Figure 1. The fast-flow region increases significantly the bulk water content and hydraulic conductivity close to saturation. It occupies 10% of the bulk soil increasing the soil porosity by an additional 15.3% (Figure 1a). Regarding hydraulic conductivity, the increase is much more important. The bulk hydraulic conductivity is increased tenfold (Figure 1b).

4 Results

4.1 Separate infiltrations into the matrix and the fast-flow regions

Cumulative infiltrations were computed for the case of a Beerkan run, i.e. zero water pressure head at surface. At first, we computed the cumulative infiltration for each region, considering the quasi-exact implicit (QEI) formulation, i.e., equations [1b] and [1c] for the matrix and the fast-flow region, respectively. We also computed the approximate expansions, using equations [3], for the two regions. Then, we computed the relative error between the approximate approximations and the

QEI formulation for the two regions. These functions were computed to time dataset encompassing both transient and steady states.

The figure 2 shows that the first approximation, $I_{O(1)}$, is quickly imprecise, with a relative error increasing from 5% to more than 40% for the fracture (Figure 2d, green line), and from 5% to close to 15% for the matrix (Figure 2b, green line). Then, the second approximate expansion is much better, $I_{O(2)}$, with relative errors less than 2.5% for the matrix (Figure 2b, blue line) and less than 10% for the fracture (Figure 2d, blue line). For these two expansions, the relative errors increase with time. A validity time interval, $[0, t_{max}]$ may be defined for any given tolerance, e.g. 5%, or 1%. For steady state expansion, $I_{+\infty}$, the opposite trend is observed, with a large drop in relative errors (Figure 2b-d, red dashed line). Logically, the steady state expansion addresses to the cases of long times. By analogy with that mentioned above for the transient approximate expansions, a validity time interval, $[t_{max}, +\infty)$ can be defined for any given tolerance, e.g. 5%, or 1%. The validity times are proposed for 1% tolerance and listed in Table 1 (see below in the following section).

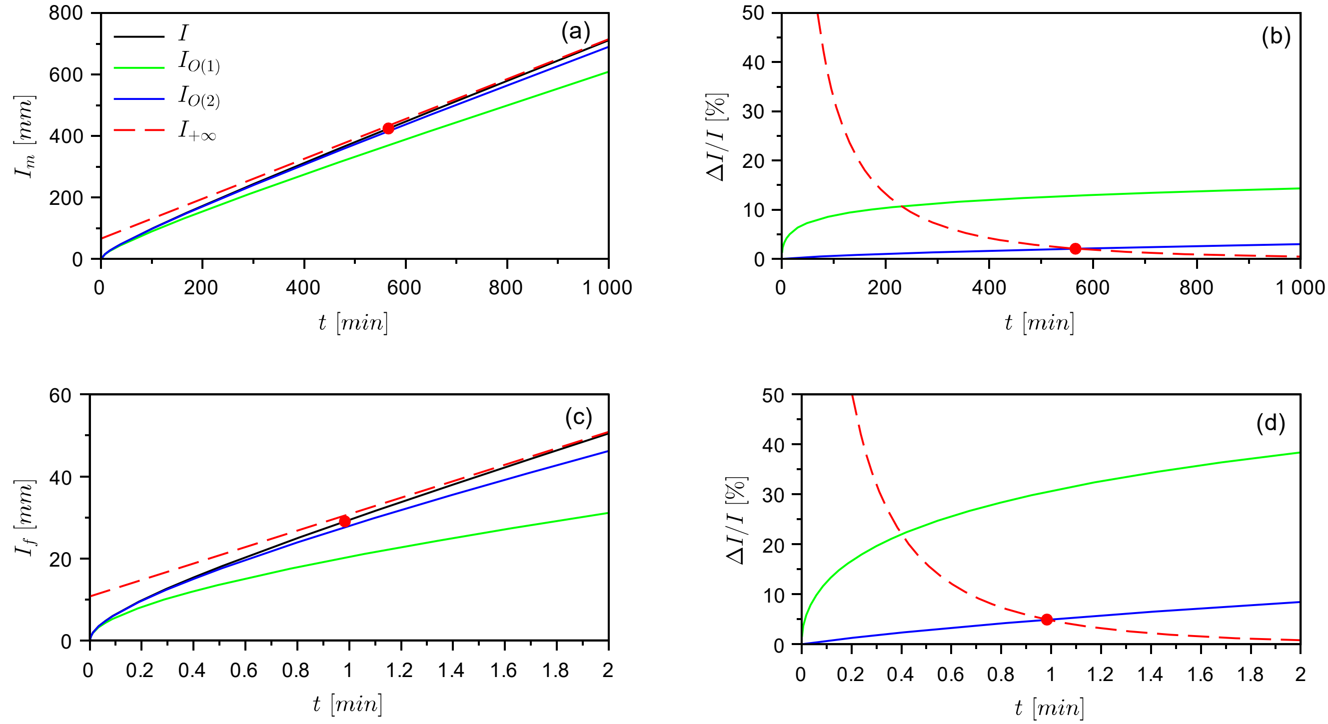

Figure 2: Separate cumulative infiltrations in the matrix and fast-flow regions: QEI formulation, $I(t)$, along with the transient, $I_{O(1)}(t)$ and $I_{O(2)}(t)$, and steady state, , $I_{+\infty}(t)$, expansions (left) and related relative errors (right). The red point corresponds to the intersection between the steady state and transient expansion relative errors.

Between the two validity times, the use of either the transient or the steady state expansion leads to errors higher than the chosen tolerance. That may involve quite large intervals, for instance [190.5,767.5] (min) for the matrix. In this time interval, we can build a shifting approximate expansion as follows. An intersection point can be identified between the two relative errors related to the transient and steady state expansions (Figure 2b and d, red point). For the case of the matrix, it corresponds to time ~ 566 min and to a relative error of ~ 2%. For the fast-flow region, it corresponds to 0.98 min and to a relative error of 5%.

The transient state expansion is considered valid below that transition point, the steady state expansion is considered afterwards, leading to the following definition of the shifting approximate expansion:

$$I_{3D\_SA}(t) = \begin{cases} I_{O(2)}(t) = S\sqrt{t} + \left(\frac{2-\beta}{3}\Delta K + K_0 + \frac{\gamma S^2}{r_d \Delta\theta}\right) t & if \quad t \le t_{tr} \\ I_{+\infty}(t) \; = \; K_s t - \frac{ln(\beta)}{2(1-\beta)}\frac{S^2}{\Delta K} & if \quad t \ge t_{tr} \end{cases} \qquad [8]$$

With the transition time, t_{tr}, corresponding to the intersection point:

$$t_{tr} \quad / \quad \frac{I(t_{tr}) - I_{O(2)}(t_{tr})}{I(t_{tr})} = \frac{I_{+\infty}(t_{tr}) - I(t_{tr})}{I(t_{tr})} \qquad [9a]$$

$$t_{tr} \quad / \quad I(t_{tr}) = \frac{I_{O(2)}(t_{tr}) + I_{+\infty}(t_{tr})}{2} \qquad [9b]$$

The two definitions [9a] and [9b] are strictly equivalent. This strategy has already been proposed by Fernandez-Galvez et al. (2019) and the function $I_{3D\_SA}(t)$ is referred to as the shifting approximate expansion. By construction, the relative error between $I_{3D\_SA}(t)$ and the target, $I(t)$, corresponds to that of the transient state expansion for $t \le t_{tr}$ (Figure 2b-c, blue line for $t \le t_{tr}$) and that of the steady state expansion (Figure 2b-c, red line for $t \ge t_{tr}$). By construction, the maximum error of $I_{3D\_SA}(t)$ corresponds to any of the two relative errors at time $t = t_{tr}$. The maximum errors and transitions times of $I_{3D\_SA}(t)$ are tabulated in Table 1.

4.2 Proposed approximations for the cumulative infiltration into the 2K system

As a second step, the proposed approximate expansions, defined by equations [4], or equivalently [5] were applied to the case of the dual permeability system described in the material and method section and compared to the bulk cumulative infiltrations computed with the QEI-2K model (Figure 3). As for the separate infiltration into the two regions, we considered the case of a Beerkan run, i.e. zero water pressure head at surface. The time dataset was adapted to encompass both the transient and the steady states.

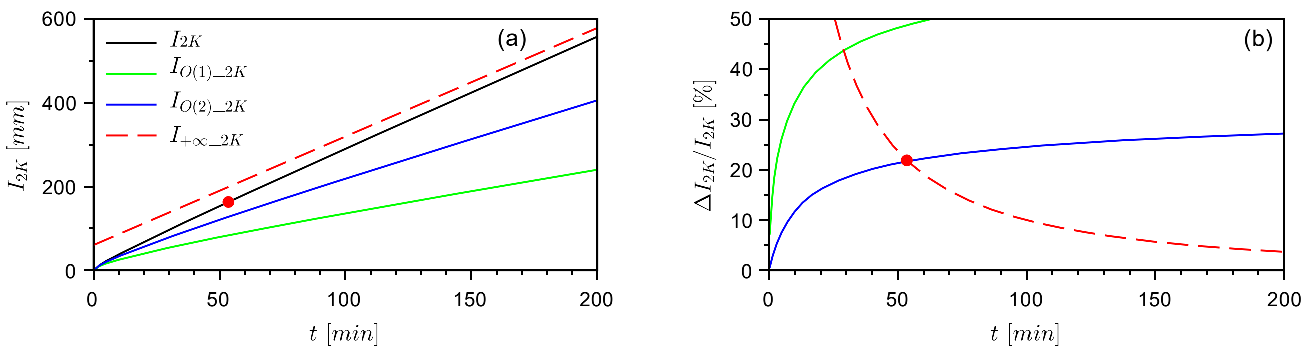

Figure 3: cumulative infiltrations into the dual permeability soil: QEI-2K formulation, $I_{2K}(t)$, along with the transient, $I_{O(1)\_2K}(t)$ and $I_{O(2)\_2K}(t)$, and steady state expansions, $I_{+\infty\_2K}(t)$ (left) and related relative errors (right). The red point corresponds to the intersection between the steady state and transient expansion relative errors

We found the same trends as for the matrix and fast-flow regions alone (Figure 3). The transient and steady state expansions, $I_{O(2)\_2K}(t)$ and $I_{+\infty\_2K}(t)$, approximate to a certain extent the bulk cumulative infiltration, $I_{2K}(t)$. However, the approximations are less accurate (Figure 3a versus Figure 2a or c) and the relative errors are much higher (Figure 3b versus Figure 2b or d). For instance, the relative errors of the transient expansion quickly increase up to 20% (Figure 3b, blue line) versus less than 5% for the fast-flow region (Figure 2d, blue line) and 1% for the matrix (Figure 2b, blue line). Similarly, the steady state expansion decreases much less than for the case of single permeability system (Figure 3, red line, versus Figure 2, red lines).

As for the matrix and the fast-flow regions alone, the validity times were estimated for transient and steady state approximations. We considered a tolerance of 1%, as for the single permeability systems. The corresponding validity time intervals are tabulated in Table 1. We can see the interval endpoints take intermediate values between the case of the matrix and the fast-flow regions. The time limit for the transient state related to 2K system is between that of the fast-flow region - very small- and that of the matrix region -very large (Table 1, column "Transient state"). Similarly, the time limit for the steady state and 2K system is between that of the fast flow region, quite small, and that of the matrix, much larger (Table 1, column "Steady state"). We also notice that the time limits of 2K systems are much closer to the limits of region that exhibit the smallest validity time intervals. For instance, for the transient state, the time limit of 2K system is closer to the endpoint of interval related the fast-flow region (Table 1). For the steady state expansion, the time limit of 2K is closer to that of the matrix region (Table 1). In other words, the validity time intervals of the 2K models are constrained by the accuracy of each of the approximate expansions of the two regions. When time is out of the validity interval for one of the regions, the whole approximation is spoiled and the accuracy is impacted. Consequently, the most restrictive constraint predominates.

Table 1: Validity time intervals for the approximate expansions: validity time endpoints for the transient and steady state expansions (columns "Transient state" and "Steady state"), and transition time, t_{tr}, with maximum errors for shifting approximate expansion (column Shifting Approx.). Times are indicated in minutes for the matrix ("matrix"), fast-flow region ("fast-flow") and the dual permeability system ("2K").

Transient state $I_{O(2)}(t)$			Steady state $I_{+\infty}(t)$			Shifting Approx. $I_{SA}(t)$		
matrix	fast-flow	2K	matrix	fast-flow	2K	matrix	fast-flow	2K
0	0	0	767.5	1.86	417.8	566	0.983	53.5
190.5	0.152	0.25	+∞	+∞	+∞	2.1%	4.9%	21.9%

As for the separate infiltrations into the single permeability systems, we considered the shifting approximate expansion for the dual permeability system, shifting from the transient to the steady state expansions:

$$I_{3D\_SA\_2K}(t) = \begin{cases} I_{O(2)\_2K}(t) = \left(\left(1-w_f\right) S_m + w_f S_f\right)\sqrt{t} + \left(\frac{2-\beta}{3}\Delta K_{2K} + K_{0,2K} + \frac{\gamma}{r_d}\left(\left(1-w_f\right)\frac{S_m^2}{\Delta\theta_m} + w_f\frac{S_f^2}{\Delta\theta_f}\right)\right) t & if \quad t \le t_{tr\_2K} \\ I_{+\infty\_2K}(t) = K_{s,2K} t - \frac{ln(\beta)}{2(1-\beta)}\left(\left(1-w_f\right)\frac{S_m^2}{\Delta K_m} + w_f\frac{S_f^2}{\Delta K_f}\right) & if \quad t \ge t_{tr\_2K} \end{cases} \quad [10]$$

With the transition time, t_{tr_2K}, corresponding to the intersection point:

$$t_{tr\_2K} \quad / \quad \frac{I_{2K}(t_{tr\_2K}) - I_{O(2)\_2K}(t_{tr\_2K})}{I_{2K}(t_{tr\_2K})} = \frac{I_{+\infty\_2K}(t_{tr\_2K}) - I_{2K}(t_{tr\_2K})}{I_{2K}(t_{tr\_2K})} \quad [11a]$$

$$t_{tr\_2K} \quad / \quad I_{2K}\left(t_{tr\_2K}\right) = \frac{I_{O(2)\_2K}(t_{tr\_2K}) + I_{+\infty\_2K}(t_{tr\_2K})}{2} \quad [11b]$$

The decrease in accuracy of each approximate expansion, as mentioned above impacts the accuracy of the bulk shifting approximate expansion. In that case, the transition time is around 53.5 min versus 1 min and 566 min for the fast flow and the matrix regions, respectively (Table 1, column "Shifting Approx."). Again, as already observed for the validity times, the transition time for the 2K soils takes intermediate values between the fast-flow regions, with very short transition time, and the matrix with a much longer transition time. Besides, the relative error of the shifting approximation is around 22% versus 5% and ~2% for the fast-flow and matrix regions. Such an increase in the relative errors reflects the degradation of the quality of the shifting approximate expansions. In other words, the shifting approximation is impacted by the less accurate transient and steady states expansions.

5. Conclusions

In this study, we investigated the development of approximate expansions for the quasi-exact implicit model developed by Lassabatere et al. (2014) for cumulative infiltration into dual permeability soils. The proposed approximate expansions make use of those developed by Haverkamp et al. (1994) for the case of single permeability systems. These expansions are easy to compute and avoid numerical indetermination due to the use of implicit equations. These expansions proved to be efficient for the approximation of the bulk cumulative infiltrations. However, their use should be restricted to the respective validity time intervals. Regarding the transient state expansion, the validity time interval of the 2K expansion resumes to that of the fast-flow region, thus restricted to less than a minute. Regarding the steady state expansion, the validity time interval of the 2K expansion resumes to that of the matrix region, requiring times over hundreds of minutes for a correct use of it. Briefly, the use of these approximate expansions requires the approximate expansions related to the two regions to be valid. Consequently, the fast-flow region imposes a very short time for the transient expansion. In opposite, the matrix imposes a very long time for the steady state expansion. That proves the difficulty to build precise approximate expansions for dual permeability systems. Consequently, the strategy based on the shifting approximate expansion produces a poor approximation of the bulk cumulative infiltration, whereas it remains efficient for single permeability systems. Further investigations are required for the determination of approximate expansions for bulk infiltrations into dual permeability systems.

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